Codimension 2 submanifolds with flat normal bundle in euclidean space

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Abstract

Given an immersed submanifold $M^n \subset \mathbb{R}^{n+2}$, we characterize the vanishing of the normal curvature $R_D$ at a point $p \in M$ in terms of the behaviour of the asymptotic directions and the curvature locus at $p$. We also consider submanifolds with vanishing normal curvature everywhere, or in other words, with flat normal bundle. This property is equivalent to the existence of a locally defined parallel normal vector field $\nu$. We use this fact in order to relate the affine properties of codimension 2 submanifolds with flat normal bundle with the conformal properties of hypersurfaces in Euclidean space.

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1 Introduction

Several authors have studied the geometrical properties of surfaces immersed in $\mathbb{R}^4$ from different viewpoints (see for instance [5], [8], [6], [12], [14]). The main tool used in most cases is the analysis of the second fundamental form, from which several geometrical concepts arise: curvature ellipses, shape operators associated to normal fields and associated principal configurations, asymptotic direction fields, normal curvature, etc. An interesting fact is the equivalence among the following properties at a given point of the surface:

a) Vanishing normal curvature.

b) Degeneracy of the curvature ellipse.

c) Critical point of some principal configuration.

d) Singularity of corank 2 for some squared distance function.

e) Existence of two orthogonal asymptotic directions.

In the present paper we investigate the generalization of these equivalences to submanifolds of codimension 2 in Euclidean space. We introduce the concept of curvature locus at each point as the natural generalization
of the curvature ellipse at a point of a surface immersed in $\mathbb{R}^n$. We see that this is a convex subset in the normal space of the submanifold at the given point. This subset becomes a convex polygon at points at which the normal curvature of the submanifold vanishes (Corollary 3.8). We analyze with detail the case of codimension 2 submanifolds with vanishing normal curvature. A consequence of well known properties of such submanifolds is the fact that all their shape operators share an orthogonal basis of principal directions. In some sense, we may say that, similarly to what happens with the hypersurfaces, they have a unique principal configuration. We explore here this fact in order to show that they admit an orthogonal basis of asymptotic directions at each point. Such directions were introduced in [7] in terms of contacts of the submanifolds with hyperplanes. It was shown in [7] that any $n$-manifold immersed in $\mathbb{R}^{n+2}$ admits at most $n$ asymptotic directions at each one of its points. Moreover, a sufficient condition for the existence of some asymptotic direction at a point is the local convexity. We show in Section 4 that vanishing normal curvature and strict local convexity are both non coincident sufficient conditions for the existence of the maximal number of asymptotic directions. Moreover, we observe that vanishing normal curvature is equivalent to the existence of a basis of mutually orthogonal asymptotic directions at each point (Theorem 3.2).

An important particular case of codimension 2 submanifolds with vanishing normal curvature consists of the semiumbilical submanifolds, that is, submanifolds all whose points are $\nu$-umbilics, for some normal field $\nu$. We show that in this case, the curvature locus becomes a segment, whose direction is orthogonal to $\nu$. We point out that, although this property characterizes the vanishing of the normal curvature in the particular case of surfaces immersed in 4-space, this is not the case in higher dimensions.

In Section 5 we use the fact that codimension 2 submanifolds with vanishing normal curvature have some parallel normal field in order to show that the study of the contacts of such submanifolds with hyperplanes (leading to affinely invariant properties) is equivalent to the study of the contacts of hypersurfaces with hyperspheres. In other words, there is an equivalence between affinely invariant properties of $n$-manifolds with vanishing normal curvature in $\mathbb{R}^{n+2}$ and the conformally invariant properties of hypersurfaces in $\mathbb{R}^{n+1}$. This property has been explored by the first author in [10] in order to extend results concerning the Carathéodory’s and Loewner’s Conjecture for closed surfaces with flat normal bundle in 4-space, and the “4 vertices theorem” for closed curves with flat normal bundle in $\mathbb{R}^3$.

A special class of semiumbilical submanifolds is provided by those contained in a hypersphere ([1]). We analyze in Section 6 some necessary and sufficient conditions for the hypersphericity of a semiumbilical submanifold of codimension 2 in terms of the properties of the curvature locus and the asymptotic directions. We also analyze the conformal flatness of semiumbilical submanifolds in this context.
A detailed study of the implications of vanishing normal curvature and semiumbility of codimension 2 submanifolds in the singularities of their families of distance squared functions will be made in a forthcoming paper.

2 Vanishing normal curvature and principal configurations

Let $M$ be a smooth immersed $n$-dimensional manifold in $\mathbb{R}^{n+k}$. We consider in $M$ the riemannian metric induced by the Euclidean metric of $\mathbb{R}^{n+k}$. Given a point $p \in M$, we have a decomposition

$$\mathbb{R}^{n+k} = T_pM \oplus N_pM,$$

where $N_pM = T_pM^\perp$ and the corresponding orthogonal projections $\top : \mathbb{R}^{n+k} \to T_pM$ and $\perp : \mathbb{R}^{n+k} \to N_pM$.

We denote by $\nabla'$ the covariant derivative in $\mathbb{R}^{n+k}$. For vector fields $X, Y$ tangent along $M$ in a neighbourhood of $p$, we have the Gauss formula,

$$\nabla' X Y = \nabla X Y + s(X, Y),$$

where

1. $\nabla$ is the induced covariant derivative in $M$ given by $\nabla X Y = \top(\nabla_X Y)$.
2. $s$ is the second fundamental form, that is, $s : T_pM \times T_pM \to N_pM$ is the bilinear symmetric map defined by $s(X, Y) = \perp(\nabla_X Y)$.

We denote the curvature tensor of $\nabla$ by $R : T_pM \times T_pM \times T_pM \to T_pM$, which is defined as

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z.$$

Since the curvature of $\nabla'$ in $\mathbb{R}^{n+k}$ is identically zero, we can compute $R$ by using the Gauss equation:

$$(R(X, Y)Z, W) = (s(X, W), s(Y, Z)) - (s(X, Z), s(Y, W)).$$

Analogously, if now $\nu$ is a normal vector field along $M$ in a neighbourhood of $p$, we have the Weingarten equation:

$$\nabla'_X \nu = -A_\nu(X) + DX\nu,$$

where

1. $D$ is the covariant derivative of the normal bundle of $M$, given by $DX\nu = \perp(\nabla'_X \nu)$. 

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2. For each normal vector $\nu \in T_pM^\perp$, $A_\nu$ is the shape operator, that is, it is the self-adjoint linear map $A_\nu : T_pM \to T_pM$ defined by $A_\nu(X) = -\nabla_X\nu$. It is related to the second fundamental form by

$$\langle A_\nu(X), Y \rangle = \langle s(X, Y), \nu \rangle,$$

for any $X, Y \in T_pM$.

Finally, we can also consider the curvature tensor of the normal bundle connection, $R_D : T_pM \times T_pM \times N_pM \to N_pM$, given by

$$R_D(X, Y)\nu = D_X(D_Y\nu) - D_Y(D_X\nu) - D_{[X,Y]}\nu.$$

Since $\mathbb{R}^{n+k}$ has vanishing curvature, the Ricci equation gives

$$R_D(X, Y)\nu = s(A_\nu(Y), X) - s(A_\nu(X), Y),$$

for any $X, Y \in T_pM$ and $\nu \in N_pM$.

**Definition 2.1.** Let $\nu \in N_pM$ be a normal vector. We consider the self-adjoint linear map $A_\nu : T_pM \to T_pM$. The eigenvalues $\mu_1, \ldots, \mu_n$ of $A_\nu$ are called $\nu$-principal curvatures of $M$ at $p$.

We say that $X \in T_pM$ is a $\nu$-principal direction if it is an eigenvector of $A_\nu$. It is well known that for any $\nu \in N_pM$ we can choose an orthonormal frame $\{X_1, \ldots, X_n\}$ of $T_pM$ made of $\nu$-principal directions, that is, such that

$$A_\nu(X_i) = \mu_i X_i, \quad i = 1, \ldots, n.$$

A point $p \in M$ is said to be $\nu$-umbilic if there is $\mu \in \mathbb{R}$ such that $A_\nu = \mu \text{Id}$, or equivalently, if the $\nu$-principal curvatures are equal. We say that $p$ is umbilic if it is $\nu$-umbilic for any $\nu \in N_pM$. We say that $p$ is semiumbilic if it is $\nu$-umbilic for some non-zero $\nu \in N_pM$.

**Theorem 2.2.** Let $M^n \subset \mathbb{R}^{n+k}$ be an immersed submanifold and let $p \in M$. They are equivalent:

1. $R_D(p) = 0$.
2. $A_\nu \circ A_\xi = A_\xi \circ A_\nu$, for any $\nu, \xi \in N_pM$.
3. There is an orthonormal frame $\{X_1, \ldots, X_n\}$ of $T_pM$ made of $\nu$-principal directions, for any $\nu \in N_pM$.

**Proof.** See [13].
3 Binormal and asymptotic directions

In this section, we restrict ourselves to the codimension 2 case. That is, we consider $M^n \subset \mathbb{R}^{n+2}$, an immersed submanifold. In this codimension, it is very natural to look at the binormal and asymptotic directions, since we have generically a finite number of such directions. These concepts have been studied by Mochida, Ruas and the second author in [6] for surfaces (i.e., $n = 2$) and in [7] for general $n$.

Definition 3.1. We say that $\nu \in N_pM$ is a binormal vector if $\det A_\nu = 0$, where $A_\nu : T_pM \to T_pM$. We say that $X \in T_pM$ is an asymptotic direction if there is a non-zero binormal vector $\nu \in N_pM$ such that $X \in \ker A_\nu$.

We remark that in [6, 7] the binormal and asymptotic directions have been introduced in a different way. Given a non zero normal vector $\nu \in N_pM$, it is called binormal if the height function $h_\nu : M \to \mathbb{R}$, defined by $h_\nu(x) = \langle x, \nu \rangle$, has a degenerated (i.e., non Morse) singularity at $p$. The asymptotic directions are the tangent vectors in $T_pM$ which belong to the kernel of the Hessian of $h_\nu$, where $\nu$ is binormal. It is easy to see that both concepts coincide with the corresponding ones given in Definition 3.1.

To compute them, we take $\{\nu_1, \nu_2\}$ a frame of $N_pM$ and $\{X_1, \ldots, X_n\}$ a frame of $T_pM$. Assume that $A_{\nu_1}(X_i) = \sum_j a_{ij}X_j$, $A_{\nu_2}(X_i) = \sum_j b_{ij}X_j$.

Given a normal vector $\nu = p\nu_1 + q\nu_2$, we have

$$\det A_\nu = \det(pA_{\nu_1} + qA_{\nu_2}) = \det([pa_{ij} + qb_{ij}]) = H^n(p, q),$$

where $H^n(p, q)$ is a homogeneous polynomial of degree $n$ in the variables $p, q$. Thus, if $H^n(p, q) \neq 0$, we may have up to $n$ binormal directions. Moreover, if $\dim \ker A_\nu = r$, then we have that $(p, q)$ is a root of $H^n(p, q)$ of multiplicity $r$. In such case we say that the binormal $p\nu_1 + q\nu_2$ has multiplicity $r$. It follows that there may be at most $n$ linearly independent asymptotic directions.

Given a surface $M^2 \subset \mathbb{R}^4$, it was shown in [12] that $p$ is semiumbilic if and only if there are two asymptotic directions which are orthogonal. We see that this can be generalized to higher dimensions, although we have to substitute the semiumbliity with the condition that $R_D(p) = 0$ (we refer to next section for the relationship between these two concepts).

Theorem 3.2. Let $M^n \subset \mathbb{R}^{n+2}$ be an immersed submanifold and let $p \in M$. Then $R_D(p) = 0$ if and only if there is an orthonormal frame $\{X_1, \ldots, X_n\}$ of $T_pM$ made of asymptotic directions.
Proof. Assume that $R_D(p) = 0$. By theorem 2.2, there is an orthonormal frame $\{X_1, \ldots, X_n\}$ of $T_pM$ made of $\nu$-principal directions, for any $\nu \in N_pM$. If $\{\nu_1, \nu_2\}$ is a orthonormal frame of $N_pM$, we have

$$A_{\nu_1}(X_i) = \lambda_i X_i, \quad A_{\nu_2}(X_i) = \mu_i X_i,$$

for $i = 1, \ldots, n$. This gives that

$$H^n(p, q) = (p\lambda_1 + q\mu_1) \ldots (p\lambda_n + q\mu_n).$$

Let $i = 1, \ldots, n$ and let $(p_i, q_i) \neq (0, 0)$ such that $p_i\lambda_i + q_i\mu_i = 0$. Then, $\nu = p_i \nu_1 + q_i \nu_2$ is a non-zero binormal vector and

$$A_{\nu}(X_i) = p_i A_{\nu_1}(X_i) + q_i A_{\nu_2}(X_i) = (p_i \lambda_i + q_i \mu_i) X_i = 0,$$

so that $X_i$ is an asymptotic direction.

Let us see the converse. Consider an orthonormal frame $\{X_1, \ldots, X_n\}$ of $T_pM$ made of asymptotic directions and let $\{\nu_1, \nu_2\}$ be a orthonormal frame of $N_pM$. We assume that

$$A_{\nu_1}(X_i) = \sum_j a_{ij} X_j, \quad A_{\nu_2}(X_i) = \sum_j b_{ij} X_j.$$

For each $i = 1, \ldots, n$, there is $(p_i, q_i) \neq (0, 0)$ such that $\nu = p_i \nu_1 + q_i \nu_2$ is a non-zero binormal vector and

$$0 = A_{\nu}(X_i) = p_i A_{\nu_1}(X_i) + q_i A_{\nu_2}(X_i) = \sum_j (p_i a_{ij} + q_i b_{ij}) X_j.$$

This implies $p_i a_{ij} + q_i b_{ij} = 0$, for any $i, j$.

Let $i, j = 1, \ldots, n$ and assume that $X_i$, $X_j$ are asymptotic directions associated to different binormal vectors, that is, such that

$$\begin{vmatrix} p_i & p_j \\ q_i & q_j \end{vmatrix} \neq 0.$$

Since we have $p_i a_{ij} + q_i b_{ij} = 0$ and $p_j a_{ij} + q_j b_{ij} = 0$, we obtain $a_{ij} = b_{ij} = 0$.

Otherwise, if $X_i$, $X_j$ are asymptotic directions associated to the same binormal vector, we get

$$\begin{vmatrix} p_i & p_j \\ q_i & q_j \end{vmatrix} = 0.$$

If for instance, $q_i \neq 0$, we have $p_i = \lambda q_i$ and $p_j = \lambda q_j$, for some $\lambda \in \mathbb{R}$. Since $p_i a_{ij} + q_i b_{ij} = 0$, this gives $b_{ij} = -\lambda a_{ij}$. In general, we will have that either $b_{ij} = \mu a_{ij}$ or $a_{ij} = \lambda b_{ij}$, for some $\lambda, \mu \in \mathbb{R}$. 


Therefore, we have showed that the matrices $A = (a_{ij})$ and $B = (b_{ij})$ have the form:

$$
A = \begin{pmatrix}
A_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_m
\end{pmatrix}, \quad B = \begin{pmatrix}
B_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & B_m
\end{pmatrix},
$$

where $A_i, B_i$ are submatrices of $A, B$ respectively such that $A_i = \lambda_i C_i$ and $B_i = \mu_i C_i$, where $\lambda_i, \mu_i \in \mathbb{R}$, $C_i$ is a matrix and $m$ is the number of distinct binormal vectors. Hence,

$$
AB = \begin{pmatrix}
\lambda_1 \mu_1 C_1^2 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_m \mu_m C_m^2
\end{pmatrix} = BA,
$$

which implies that $R_D(p) = 0$ by theorem 2.2. □

**Remark 3.3.** It follows from the proof of the above theorem that if $R_D(p) = 0$ and $\{X_1, \ldots, X_n\}$ is an orthonormal frame of $T_p M$ made of $\nu$-principal directions, for any $\nu \in N_p M$, then $\{X_1, \ldots, X_n\}$ are asymptotic directions. The converse is not true in general.

**Corollary 3.4.** Let $M^n \subset \mathbb{R}^{n+2}$ be an immersed submanifold and assume that $R_D(p) = 0$. Then, there are $n$ binormal directions (counting their multiplicities) at $p$.

## 4 Curvature locus

We introduce now the concept of curvature locus of an immersed submanifold $M^n \subset \mathbb{R}^{n+2}$. When $n = 2$, this is nothing but the curvature ellipse at a point $p \in M$ and it contains all the second order information of the immersion.

**Definition 4.1.** Given $p \in M$, we denote $S_p = \{X \in T_p M, \langle X, X \rangle = 1\}$. We define the curvature locus, $\Delta_p$, as the image set of the map $\eta : S_p \to N_p M$ given by $\eta(X) = s(X, X)$.

Let $\{\nu_1, \nu_2\}$ and $\{X_1, \ldots, X_n\}$ be orthonormal frames of $N_p M$ and $T_p M$ respectively. Assume that

$$
A_{\nu_1}(X_i) = \sum_j a_{ij} X_j, \quad A_{\nu_2}(X_i) = \sum_j b_{ij} X_j.
$$

Given $X \in S_p$, we set $X = x_1 X_1 + \cdots + x_n X_n$ with $x_1^2 + \cdots + x_n^2 = 1$. Then,

$$
\eta(X) = s(X, X) = \langle A_{\nu_1}(X), X \rangle \nu_1 + \langle A_{\nu_2}(X), X \rangle \nu_2
$$

$$
= \left( \sum_{i,j} a_{ij} x_i x_j \right) \nu_1 + \left( \sum_{i,j} b_{ij} x_i x_j \right) \nu_2.
$$
Therefore, \( \eta \) is the restriction to the sphere \( S_p \) of a homogeneous polynomial map of degree 2 (i.e., a quadratic map). If \( n = 2 \), then \( \Delta_p \) is an ellipse in the normal plane \( N_pM \). In general, \( \Delta_p \) is a plane projection of a \((n - 1)\)-dimensional Veronese manifold.

The following lemma is an easy exercise for quadratic maps in the plane.

**Lemma 4.2.** Let \( h = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a quadratic map. The ellipse \( h(S^1) \) degenerates to a segment \( PQ \) if and only if \( P = h(X) \), \( Q = h(Y) \), where \( \{X, Y\} \) is an orthonormal frame of \( \mathbb{R}^2 \) which diagonalize the two quadratic forms \( h_1, h_2 \) simultaneously.

**Theorem 4.3.** Let \( M^n \subset \mathbb{R}^{n+2} \) be an immersed submanifold and let \( p \in M \). Then \( R_D(p) = 0 \) if and only if there are \( X_1, \ldots, X_n \in S_p \) such that for any \( i \neq j \), \( X_i, X_j \) generate a 2-plane \( \pi_{ij} \subset T_pM \) and \( \eta(S_p \cap \pi_{ij}) \) is the segment \( \overline{\eta(X_i)\eta(X_j)} \).

**Proof.** Assume that \( R_D(p) = 0 \) and let \( \{X_1, \ldots, X_n\} \) be an orthonormal frame of \( T_pM \) made of \( \nu \)-principal directions, for any \( \nu \in N_pM \). Let also \( \{\nu_1, \nu_2\} \) be an orthonormal frame of \( N_pM \). Since \( \{X_1, \ldots, X_n\} \) diagonalize simultaneously the two quadratic forms of \( \eta \), the same is true for its restriction to the plane \( \pi_{ij} \) and the result follows from the above lemma.

Conversely, assume that there are \( X_1, \ldots, X_n \in S_p \) such that for any \( i \neq j \), \( X_i, X_j \) generate a 2-plane \( \pi_{ij} \subset T_pM \) and \( \eta(S_p \cap \pi_{ij}) = \overline{P_iP_j} \), where \( P_i = \eta(X_i) \). The above lemma implies that \( X_1, \ldots, X_n \) is an orthonormal frame of \( T_pM \) which diagonalize simultaneously the two quadratic forms of \( \eta \). Thus, \( R_D(p) = 0 \) by Theorem 2.2. \( \square \)

**Corollary 4.4.** Let \( M^n \subset \mathbb{R}^{n+2} \) be an immersed submanifold and \( p \in M \) a point such that \( R_D(p) = 0 \). Let \( \{\nu_1, \nu_2\} \) and \( \{X_1, \ldots, X_n\} \) be orthonormal frames of \( N_pM \) and \( T_pM \) respectively, such that

\[
A_{\nu_1}(X_i) = \lambda_i X_i, \quad A_{\nu_2}(X_i) = \mu_i X_i.
\]

Then \( \Delta_p \) is the convex envelope of \( P_1, \ldots, P_n \), where \( P_i = \eta(X_i) = \lambda_i\nu_1 + \mu_i\nu_2 \).

**Proof.** The assertion follows easily from the above computations. Just note that given \( X \in S_p \), we set \( X = x_1X_1 + \cdots + x_nX_n \) with \( x_1^2 + \cdots + x_n^2 = 1 \). Then,

\[
\eta(X) = \left( \sum_i \lambda_i x_i^2 \right) \nu_1 + \left( \sum_i \mu_i x_i^2 \right) \nu_2 = \sum_i x_i^2 P_i.
\]

\( \square \)
Example 4.5. We see that the converse of the above corollary is not true in general. Let us consider $g : \mathbb{R}^5 \to \mathbb{R}^7$ given by

$$g(x, y, z, t, u) = (x, y, z, t, u, 2x^2 - 2z^2 + u^2, -x^2 + 2y^2 - z^2 + t^2 + tu).$$

A simple computation shows that the curvature locus $\Delta_p$ at $p = 0$ is the triangle with vertices (2,-1), (0,2) and (-2,-1), but $R_D(p) \neq 0$. In fact, the restriction to the $(t, u)$-plane gives a non degenerate ellipse contained in the triangle $\Delta_p$ (see Figure 2).

Corollary 4.6. Let $M^n \subset \mathbb{R}^{n+2}$ be an immersed submanifold. If $\Delta_p$ is a convex $n$-gon, then $R_D(p) = 0$.

Corollary 4.7. Let $M^3 \subset \mathbb{R}^5$ be an immersed submanifold. Then $R_D(p) = 0$ if and only if $\Delta_p$ is a polygon.
Remark 4.8. Suppose that $R_D(p) = 0$ and let $\{X_i\}_{i=1}^n$ be the common $\nu$-principal directions. Given any normal direction $\nu$ at $p$, the principal curvatures of $A_\nu$ are given by $\kappa_i^\nu = \langle \nu, \eta(X_i) \rangle$, $i = 1, \ldots, n$. It follows that $\nu$ points in a binormal direction at $p$ if and only if $\nu = \eta(X_i)$. Therefore, we have that the normal vectors $\{\eta(X_i)\}_{i=1}^n$ define the binormal directions at $p$. We observe that we may have $\eta(X_i) = \pm \eta(X_j)$ with $i \neq j$, in such case the binormal $\eta(X_i)$ has multiplicity at least 2.

On the other hand, it follows that the normal vectors orthogonal to the different segments $\eta(X_i) - \eta(X_j)$ have an associated principal curvature of multiplicity at least 2.

Proposition 4.9. Let $M^n \subset \mathbb{R}^{n+2}$ be an immersed submanifold. Let $p \in M$. Suppose that $R_D(p) = 0$, then $p$ is either umbilic for at least a normal direction, or there exist at least 3 and at most $\frac{1}{2}n(n-1)$ normal directions defining shape operators with some curvature of multiplicity at least 2 at $p$.

Proof. Since $R_D(p) = 0$ the curvature locus $\Delta_p$ is either a point, a segment or an $r$-gon, for some $r \geq 3$. If $\Delta_p$ is a point, then $\eta(X_i) = \eta(X_j)$, $\forall i, j = 1, \ldots, n$, where $\{X_i\}_{i=1}^n$ are the common $\nu$-principal directions at $p$. In this case it follows from the remark 4.8 that for any normal direction $\nu \in N_pM$, $\kappa_i^\nu = \langle \nu, \eta(X_i) \rangle = \langle \nu, \eta(X_j) \rangle = \kappa_j^\nu$, $\forall i, j = 1, \ldots, n$. Therefore $p$ is $\nu$-umbilic for any normal direction $\nu$. In case $\Delta_p$ is a segment, we have that $\eta(X_i) - \eta(X_j) = \eta(X_i) - \eta(X_k)$, $\forall i, j = 1, \ldots, n$. Then the direction $\langle \eta(X_i) - \eta(X_j) \rangle^\perp$ defines a normal vector $\nu$, such that $\kappa_i^\nu = \kappa_j^\nu$, $\forall i, j = 1, \ldots, n$. So $p$ is $\nu$-umbilic (i.e., semiumbilic). Finally, if $\Delta_p$ is an $r$-gon, for some $r \geq 3$, the orthogonal directions to either the sides or the diagonals of $\Delta_p$ provide the required normal vectors. Since there are at least 3 and at most $\frac{1}{2}n(n-1)$ of such directions we get the required result.

Definition 4.10. A normal vector $\nu$ is said to be quasiumbilic at $p \in M$ if one of its associated principal curvatures has multiplicity $\geq n - 1$ at $p$.

Corollary 4.11. Let $M^n \subset \mathbb{R}^{n+2}, n \geq 3$ be an immersed submanifold, suppose that $R_D(p) = 0$. Then if $p$ is a (non semiumbilic) quasiumbilic point, $\Delta_p$ is a triangle.

Corollary 4.12. Let $M^3 \subset \mathbb{R}^5$ be an immersed submanifold. Then $R_D(p) = 0$ if and only if either $p$ is umbilic for at least a normal direction, or quasiumbilic for 3 different normal directions.

An $n$-dimensional submanifolds $M$ of $\mathbb{R}^{n+2}$ is said to be quasiumbilical with respect to a normal field $\nu$ if $A_\nu$ has some eigenvalue with multiplicity $\geq n - 1$ at every point. The manifold $M$ is said to be totally quasiumbilical provided it admits two mutually orthogonal quasiumbilical normal sections.

Proposition 4.13. Let $M^n \subset \mathbb{R}^{n+2}, n \geq 3$ be a totally quasiumbilical submanifold with $R_D \equiv 0$. Then the curvature locus at any non semiumbilic point of $M$ is a right-angled triangle.
Proof. If \( R_D \equiv 0 \) over \( M \) we have that \( \Delta_p \) is a polygon at every point of \( M \).

Suppose now that \( M \) is totally quasiumbilical. Then there exist normal fields \( \nu_1 \) and \( \nu_2 \) which are quasiumbilical all over \( M \). But this implies that, for each \( p \in M \), there is an \((n - 2)\)-subspace of \( T_p M \) made of principal directions for any normal field on \( M \). In other words, it is possible to find \( n - 2 \) mutually orthogonal common principal directions \( \{X_i\}_{i=1}^{n-2} \) for all the normal fields on \( M \) such that \( \eta(X_i) = \eta(X_j), \forall i, j = 1, \ldots, n - 2 \). Consequently \( \Delta_p \) is either a point (when \( p \) is umbilic), a segment (when \( p \) is semiumbilic), or a triangle. Since the fields \( \nu_1 \) and \( \nu_2 \) must be orthogonal to some side of the triangle and they are mutually orthogonal, we have that the triangle must have a right angle.

Corollary 4.14. Let \( M^3 \subset \mathbb{R}^5 \) be an immersed submanifold. Suppose that \( M \) does not have semiumbilic, nor umbilic points. Then \( M \) is totally quasi-umbilic and has flat normal bundle if and only if the curvature locus at each point is a right-angled triangle.

5 Semiumbilical and strictly locally convex submanifolds

Given an immersed surface \( M^2 \subset \mathbb{R}^4 \), we have the following equivalent statements for a point \( p \in M \) (see [12]):

1. \( p \) is semiumbilic,
2. the curvature ellipse at \( p \) degenerates to a segment,
3. \( R_D(p) = 0 \).

We will see that the equivalence between (1) and (2) remains true when \( n \geq 3 \), but they are not equivalent to (3) in general. We will also discuss about the relationship of these notions with the local convexity of the submanifold.

Proposition 5.1. Let \( M^n \subset \mathbb{R}^{n+2} \) be an immersed submanifold. If \( p \in M \) is semi-umbilic, then \( R_D(p) = 0 \).

Proof. Let \( \nu_1 \in N_p M \) be a unit normal vector such that \( p \) is \( \nu_1 \)-umbilic. This means that \( A_{\nu_1} = \lambda \text{Id} \), for some \( \lambda \in \mathbb{R} \). Now we complete \( \nu_1 \) to an orthonormal frame \( \{\nu_1, \nu_2\} \) of \( N_p M \). Hence,

\[
A_{\nu_1} \circ A_{\nu_2} = \lambda A_{\nu_2} = A_{\nu_2} \circ A_{\nu_1},
\]

and thus \( R_D(p) = 0 \) by Theorem 2.2.

Corollary 5.2. Let \( M^n \subset \mathbb{R}^{n+2} \) be an immersed submanifold and let \( p \in M \).
1. \( p \) is semiumbilic if and only if \( \Delta_p \) degenerates to a segment. Moreover, if \( L_p \) is the linear subspace spanned by \( \Delta_p \) and \( \nu \) is a normal vector such that \( p \) is \( \nu \)-umbilic, then \( \nu \perp L_p \).

2. \( p \) is umbilic if and only if \( \Delta_p \) degenerates to a point.

**Proof.** By the above proposition and Theorem 4.3, we can assume that \( R_D(p) = 0 \). Let \( \{\nu_1, \nu_2\} \) and \( \{X_1, \ldots, X_n\} \) be orthonormal frames of \( N_pM \) and \( T_pM \) respectively, such that

\[
A_{\nu_1}(X_i) = \lambda_i X_i, \quad A_{\nu_2}(X_i) = \mu_i X_i.
\]

Moreover, \( \Delta_p \) is the convex envelope of \( P_1, \ldots, P_n \) in \( N_pM \), where \( P_i = (\lambda_i, \mu_i) \).

By definition, \( p \) is semiumbilic if and only if there is a non-zero normal vector \( \nu \in N_pM \) such that \( p \) is \( \nu \)-umbilic. This is equivalent to the fact that there exist \( a, b, \alpha \in \mathbb{R} \) such that \( a^2 + b^2 > 0 \) and

\[
aA_{\nu_1} + bA_{\nu_2} = \alpha \text{ Id}.
\]

But this happens if and only if the following matrix has rank \( \leq 2 \)

\[
\begin{pmatrix}
\lambda_1 & \cdots & \lambda_n \\
\mu_1 & \cdots & \mu_n \\
1 & \cdots & 1
\end{pmatrix},
\]

which is equivalent to the collinearity of the points \( P_1, \ldots, P_n \). In fact, if \( p \) is \( \nu \)-umbilic, \( P_1, \ldots, P_n \) are contained in the line \( L_p \) whose equation is \( ax + by = \alpha \), which is perpendicular to \( \nu \).

To see the second part, it follows from the definition that \( p \) is umbilic if and only if \( \lambda_1 = \cdots = \lambda_n \) and \( \mu_1 = \cdots = \mu_n \), which is equivalent to \( P_1 = \cdots = P_n \). \( \square \)

In the last part of this section, we will see that the semiumbilicity is related to the local convexity of \( M \) in \( \mathbb{R}^{n+2} \).

**Definition 5.3.** We say that \( M \) is **locally convex** at \( p \) if there is a hyperplane \( \pi \) of \( \mathbb{R}^{n+2} \) such that \( p \in \pi \) and \( \pi \) supports \( M \) in a neighbourhood of \( p \). If \( \pi \) locally supports \( M \) at \( p \), it is obvious that \( \pi \) is tangent to \( M \) at \( p \). If it has a non-degenerate contact (i.e., of Morse type), then we say that \( M \) is **strictly locally convex** at \( p \). This happens if and only if there is \( \nu \in N_pM \) such that all the \( \nu \)-principal curvatures are positive (or all of them are negative). In particular, this gives the following immediate consequence.

**Proposition 5.4.** Let \( M^n \subset \mathbb{R}^{n+2} \) be an immersed submanifold. If \( p \in M \) is semiumbilic and has non-zero \( \nu \)-principal curvature (where \( p \) is \( \nu \)-umbilic), then \( M \) is strictly locally convex at \( p \).
**Proposition 5.5.** Let $M^n \subset \mathbb{R}^{n+2}$ be an immersed submanifold. If $M$ strictly locally convex at $p$, then there are $n$ binormal directions at $p$ (counting multiplicity).

**Proof.** Let $\{\nu_1, \nu_2\}$ and $\{X_1, \ldots , X_n\}$ be orthonormal frames of $N_pM$ and $T_pM$ respectively, such that all the $\nu_1$-principal curvatures are positive and $\{X_1, \ldots , X_n\}$ are $\nu_1$-principal directions. This means that

$$A_{\nu_1}(X_i) = \lambda_i X_i, \quad A_{\nu_2}(X_i) = \sum_j b_{ij} X_j,$$

with $\lambda_i > 0$ and $b_{ij} = b_{ji}$. We now change the frame in $T_pM$ to $\overline{X}_i = X_i/\sqrt{\lambda_i}$, so that

$$A_{\nu_1}(\overline{X}_i) = \overline{X}_i, \quad A_{\nu_2}(\overline{X}_i) = \sum_j \overline{b}_{ij} \overline{X}_j,$$

with $\overline{b}_{ij} = \overline{b}_{ji}$. Given a non-zero normal vector $\nu = pv_1 + q\nu_2 \in N_pM$, it is binormal if and only if $\det(A_\nu) = \det(pA_{\nu_1} + qA_{\nu_2}) = 0$. But by using the above frame, this is equivalent in coordinates to

$$\det(pI + q\overline{B}) = 0,$$

where $I$ is the identity matrix and $\overline{B} = [\overline{b}_{ij}]$.

Note that if $\nu$ is binormal, then $q \neq 0$ (otherwise, $q = 0$ would also imply $p = 0$) and $-p/q$ is an eigenvalue of $\overline{B}$. Conversely, if $t$ is an eigenvalue of $\overline{B}$, then $\nu = -t\nu_1 + \nu_2$ is binormal. Moreover, since $\overline{B}$ is a symmetric matrix, it has $n$ eigenvalues (counting multiplicity). \qed
Remark 5.6. We have the following implications:

\[ RD = 0 \]

\[ \text{semiumbilic (non flat)} \]

\[ \exists n \text{ binormals.} \]

\[ \text{strictly locally convex} \]

When \( n = 2 \), we have that

\[ \text{semiumbilic} \iff RD = 0, \]

\[ \text{strictly locally convex} \iff \exists 2 \text{ binormals.} \]

However, when \( n \geq 3 \), these equivalences are not true in general. For instance, let \( M \) be the 3-manifold embedded in \( \mathbb{R}^5 \), which is parametrized by the map \( g : \mathbb{R}^3 \rightarrow \mathbb{R}^5 \) defined as

\[ g(x, y, z) = (x, y, z, x^2 - z^2, y^2 - z^2). \]

It is not difficult to see that \( M \) is not strictly locally convex at \( p = 0 \), although it has vanishing normal curvature at this point.

Another difference with the case \( n = 2 \) occurs with the codimension 2 submanifolds which are a product of two hypersurfaces. If \( M = C_1 \times C_2 \subset \mathbb{R}^4 \) is a product of two plane curves, then \( M \) is always semiumbilic (and hence has vanishing normal curvature). This is not true in general when \( n \geq 3 \).

Proposition 5.7. Let \( M^n \subset \mathbb{R}^{n+1} \) and \( P^m \subset \mathbb{R}^{m+1} \) be immersed hypersurfaces and consider \( M \times P \subset \mathbb{R}^{n+m+2} \). Then \( M \times P \) always satisfies \( RD = 0 \). However, \((m, p) \in M \times P \) is semiumbilic if and only if either

1. both \( m, p \) are umbilic in \( M, P \) respectively, or
2. one of \( m, p \) is a flat umbilic in \( M, P \) respectively.

Proof. Fix \( p = (p_1, p_2) \in M \times P \). Let \( \nu_1 \) be a unit normal vector of \( M \) in \( \mathbb{R}^{n+1} \) at \( p_1 \) and take \( \{X_1, \ldots, X_n\} \) an orthonormal frame of \( T_{p_1}M \) made of principal directions with principal curvatures \( \lambda_1, \ldots, \lambda_n \).

Analogously, we consider \( \nu_2 \) a unit normal vector of \( P \) in \( \mathbb{R}^{m+1} \) at \( p_2 \) and \( \{Y_1, \ldots, Y_m\} \) an orthonormal frame of \( T_{p_2}P \) made of principal directions with principal curvatures \( \mu_1, \ldots, \mu_m \).

We identify \( N_p(M \times P) \equiv N_{p_1}M \oplus N_{p_2}P \) and \( T_p(M \times P) \equiv T_{p_1}M \oplus T_{p_2}P \). Then \( \{\nu_1, \nu_2\} \) and \( \{X_1, \ldots, X_n, Y_1, \ldots, Y_m\} \) are orthonormal frames of \( N_p(M \times P) \) and \( T_p(M \times P) \) respectively, which verify

\[ A_{\nu_1}(X_i) = \lambda_i X_i, \quad A_{\nu_1}(Y_j) = 0, \quad A_{\nu_2}(X_i) = 0, \quad A_{\nu_2}(Y_j) = \mu_j Y_j. \]
This shows that \( R_D \) at \( p \) by Theorem 2.2.

To see the second part, we use Corollary 5.2. In fact, \( p \) is semiumbilic if and only if \( P_1, \ldots, P_n, Q_1, \ldots, Q_m \) are collinear, where \( P_i = (\lambda_i, 0) \) and \( Q_j = (0, \mu_j) \). But this can only happen in one the following cases:

1. \( P_1 = \cdots = P_n \) and \( Q_1 = \cdots = Q_m \),
2. \( P_1 = \cdots = P_n = 0 \), or
3. \( Q_1 = \cdots = Q_m = 0 \).

\[ \square \]

6 Parallel normal vector fields and parallel submanifolds

In this section we consider an immersed submanifold \( M \) which locally admits a non-degenerate parallel normal vector field \( \nu \). This kind of submanifolds have been introduced in [10] where it is shown that the contact of \( M \) with hyperplanes is the same as the contact of the parallel hyperspherical submanifold \( M^\nu \) with the translated hyperplane. In the codimension 2 case, the existence of such a vector field will be equivalent to the flatness of the normal bundle and the non flatness in some sense of the tangent bundle.

**Definition 6.1.** We say that a normal vector field \( \nu \) is parallel if \( D_X \nu = 0 \) for any \( X \in T_pM \) and any \( p \in M \).

The proof of the following lemma can be found in [11] in a more general version for a codimension \( k \) submanifold \( M^n \subset \mathbb{R}^{n+k} \).

**Lemma 6.2.** Let \( M^n \subset \mathbb{R}^{n+2} \) be an immersed submanifold. They are equivalent:

1. \( R_D = 0 \) in \( M \).
2. There is a locally defined non-zero parallel normal vector field \( \nu \).
3. There are locally defined parallel normal vector fields \( \nu_1, \nu_2 \) which provide an orthonormal frame of \( N_pM \) at any point \( p \in M \).

We note that if a normal vector field is parallel, then it has constant length. Moreover, if \( M^n \subset \mathbb{R}^{n+2} \) and \( \nu \) is parallel then \( \nu^\perp \) is also parallel.

**Definition 6.3.** We say that a normal vector field \( \nu \) is non-degenerate if \( \det A_{\nu_p} \neq 0 \), for any \( p \in M \), that is if \( \nu_p \) is not binormal at any \( p \in M \).

We introduce now the Gaussian curvature of a riemannian manifold \( M \), a concept which generalizes in some sense the Gaussian curvature of a hypersurface \( M^n \subset \mathbb{R}^{n+1} \).
Definition 6.4. Let $p \in M$ and let $R : T_pM \times T_pM \times T_pM \to T_pM$ be the curvature tensor of $M$. Given $X, Y \in T_pM$, we denote by $R(X, Y) \in \Lambda^2_pM$ the 2-form given by
\[
R(X, Y)(Z, W) = \langle R(X, Y)Z, W \rangle.
\]
We also denote by $T^*_pM \to T_pM$, $\alpha \mapsto \alpha^*$ the isomorphism induced by the metric. Then we have an endomorphism $\Phi : \Lambda^2_pM \to \Lambda^2_pM$ defined by
\[
\Phi(\alpha \wedge \beta) = R(\alpha^*, \beta^*),
\]
where this map is extended linearly to all the 2-forms of $M$ at $p$. We call the determinant of this endomorphism the Gaussian curvature of $M$ at $p$,
\[
K = \det \Phi.
\]

Lemma 6.5. Let $M^n \subset \mathbb{R}^{n+2}$ be an immersed submanifold and assume that $R_D = 0$ at $p \in M$. We take $\{\nu_1, \nu_2\}$ and $\{X_1, \ldots, X_n\}$ orthonormal frames of $N_pM$ and $T_pM$ respectively such that
\[
A_{\nu_1}(X_i) = \lambda_i X_i, \quad A_{\nu_2}(X_i) = \mu_i X_i.
\]
Then,
\[
K = (-1)^{n(n-1)/2} \prod_{i<j} (\lambda_i \lambda_j + \mu_i \mu_j).
\]

Proof. We consider a basis of $\Lambda^2_pM$ by taking $X_i^* \wedge X_j^*$, with $i < j$. Here, $X \mapsto X^*$ denotes the isomorphism induced by the metric from $T_pM$ to $T^*_pM$. We will see that
\[
\Phi(X_i^* \wedge X_j^*) = -(\lambda_i \lambda_j + \mu_i \mu_j) X_i^* \wedge X_j^*,
\]
which implies that $X_i^* \wedge X_j^*$ are in fact eigenvectors of $\Phi$ with eigenvalues $-(\lambda_i \lambda_j + \mu_i \mu_j)$. This will conclude the proof.

On one hand, given $k < l$ we have
\[
X_i^* \wedge X_j^*(X_k, X_l) = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk},
\]
where $\delta_{ij}$ is the Kronecker delta.

On the other hand, we use the Gauss Equation
\[
\langle R(X, Y)Z, W \rangle = \langle s(X, W), s(Y, Z) \rangle - \langle s(X, Z), s(Y, W) \rangle,
\]
which gives
\[
\Phi(X_i^* \wedge X_j^*)(X_k, X_l) = \langle R(X_i, X_j)X_k, X_l \rangle \nonumber
\]
\[
= \langle s(X_i, X_l), s(X_j, X_k) \rangle - \langle s(X_i, X_k), s(X_j, X_l) \rangle \nonumber
\]
\[
= (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) (\lambda_i \lambda_j + \mu_i \mu_j) \nonumber
\]
\[
= -(\lambda_i \lambda_j + \mu_i \mu_j) X_i^* \wedge X_j^*(X_k, X_l).
\]
Remark 6.6. Let $M^n \subset \mathbb{R}^{n+1}$ be a hypersurface with principal curvatures $\lambda_1, \ldots, \lambda_n$. We can consider $M^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ in such a way that it is an immersed submanifold with vanishing normal curvature. We put $\mu_i = 0$ and the above lemma gives that
\[
K = (-1)^{\frac{n(n-1)}{2}} \prod_{i<j}(\lambda_i \lambda_j) = (-1)^{\frac{n(n-1)}{2}} (\lambda_1 \ldots \lambda_n)^2.
\]
We recall that the product $\lambda_1 \ldots \lambda_n$ is the classical Gaussian curvature of the hypersurface.

Proposition 6.7. Let $M^n \subset \mathbb{R}^{n+2}$ be an immersed submanifold. If $R_D = 0$ and $K \neq 0$ in $M$, then there is a non-degenerate parallel normal vector field defined locally in a neighbourhood of each point $p \in M$.

Proof. Let $\{\nu_1, \nu_2\}$ be locally defined parallel normal vector fields which provide an orthonormal frame of $N_p M$ at any point $p \in M$. As usual, we also take $\{X_1, \ldots, X_n\}$ an orthonormal frame of $T_p M$ such that
\[
A_{\nu_1}(X_i) = \lambda_i X_i, \quad A_{\nu_2}(X_i) = \mu_i X_i,
\]
and by the above lemma,
\[
K = (-1)^{\frac{n(n-1)}{2}} \prod_{i<j}(\lambda_i \lambda_j + \mu_i \mu_j).
\]
The hypothesis that $K \neq 0$ implies that either $\lambda_1 \ldots \lambda_n \neq 0$ or $\mu_1 \ldots \mu_n \neq 0$. Thus, either $\nu_1$ or $\nu_2$ are non-degenerate.

Definition 6.8. Given a unit normal vector field $\nu \in \mathfrak{X}(M)^1$, we can consider the Gauss map $G_\nu : M \to S^{n+1}$. For each point $p \in M$, $G(p)$ is the unit normal vector $\nu_p$ translated to the origin of $\mathbb{R}^{n+2}$.

Lemma 6.9. Let $M^n \subset \mathbb{R}^{n+2}$ be an immersed submanifold and let $\nu \in \mathfrak{X}(M)^1$ be a unit normal vector field. We denote by $D_p G_\nu : T_p M \to T_{G(p)} S^{n+1}$ the differential of $G_\nu$ at $p \in M$.

1. $\nu$ is parallel if and only if $D_p G_\nu(T_p M) \subset T_p M$, for each $p \in M$.
2. $\nu$ is parallel and non-degenerate if and only if $D_p G_\nu(T_p M) = T_p M$, for each $p \in M$.

Proof. Given $X \in T_p M$, we use Weingarten Equation:
\[
D_p G_\nu(X) = \nabla_X \nu = -A_\nu(X) + D_X \nu.
\]
Hence, $D_p G_\nu(X) \in T_p M$ if and only if $D_X \nu = 0$. This proves the first part.

For the second part, if $\nu$ is parallel, then $D_p G_\nu(X) = -A_\nu(X)$, for any $X \in T_p M$. Therefore, $D_p G_\nu(T_p M) = T_p M$ if and only if $A_\nu(T_p M) = T_p M$ if and only if $\det A_\nu \neq 0$. \qed
Definition 6.10. Let $\nu \in \mathfrak{X}(M)^\perp$ be a non-degenerate parallel unit normal vector field. By the above lemma, the Gauss map $G_\nu : M \to S^{n+1}$ is an immersion. We will denote its image by $M^\nu = G_\nu(M)$ and call it parallel submanifold to $M$ with respect to $\nu$. It follows that $M^\nu$ is an immersed submanifold in the sphere $S^{n+1}$ which verifies:

$$T_{p'}M^\nu = T_pM, \quad N_{p'}M^\nu = N_pM,$$

for any $p \in M$.

Lemma 6.11. Let $M^n \subset \mathbb{R}^{n+2}$ be an immersed submanifold and let $\nu \in \mathfrak{X}(M)^\perp$ be a non-degenerate parallel unit normal vector field. We denote by $A_\nu^\xi$ the shape operator of the parallel submanifold $M^\nu$. Then,

$$A_\xi^\nu = -A^\nu_\xi \circ A^{-1}_\nu,$$

for any $\xi \in N_pM$ and any $p \in M$.

Proof. We denote by $s^\nu(X,Y)$ the second fundamental form of $M^\nu$. We compare it with the second fundamental form $s(X,Y)$ of $M$. Let $X,Y$ be tangent vectors in $T_pM = T_pM^\nu$. Remember that

$$s^\nu(X,Y) = (\nabla'_X \tilde{Y})^\perp, \quad s(X,Y) = (\nabla'_X \bar{Y})^\perp,$$

where $\tilde{Y}, \bar{Y}$ are local extensions of $Y$ in $M^\nu, M$ respectively in a neighbourhood of $\nu, p$ respectively.

We can write

$$\tilde{Y} = \sum_{i=1}^{n+2} f_i \frac{\partial}{\partial x_i},$$

where $f_i$ are functions locally defined in a neighbourhood of $\nu_p$ in $M^\nu$ and $x_1, \ldots, x_{n+2}$ denote the coordinates of $\mathbb{R}^{n+2}$. Then,

$$\bar{Y} = \sum_{i=1}^{n+2} (f_i \circ G_\nu) \frac{\partial}{\partial x_i},$$

defines a local extension of $Y$ in $M$. By using these extension, we get

$$s(X,Y) = (\nabla'_X \bar{Y})^\perp$$

$$= \left( \sum_{i=1}^{n+2} X(f_i \circ G_\nu) \frac{\partial}{\partial x_i} \right)^\perp$$

$$= \left( \sum_{i=1}^{n+2} D_pG_\nu(X)(f_i) \frac{\partial}{\partial x_i} \right)^\perp$$

$$= (\nabla_{D_pG_\nu(X)} \tilde{Y})^\perp$$

$$= s^\nu(D_pG_\nu(X), Y)$$

$$= -s^\nu(A_\nu(X), Y).$$

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This shows that \( s(X, Y) = -s^\nu(A_\nu(X), Y) \) or equivalently, that
\[
s^\nu(X, Y) = -s^\nu(A^{-1}_\nu(X), Y),
\]
for any \( X, Y \in T_p M \).

Finally, given \( \xi \in N_p M \),
\[
\langle A^\nu_\xi(X), Y \rangle = \langle s^\nu(X, Y), \xi \rangle = -(s^\nu(A^{-1}_\nu(X), Y), \xi) = -\langle A_\xi(A^{-1}_\nu(X)), Y \rangle,
\]
hence \( A^\nu_\xi = -A_\xi \circ A^{-1}_\nu \). \(\square\)

We finish this section with the following theorem which has been proved by the first author in [10] in the case \( n = 2 \). An interesting consequence of it was the extension of the Carathéodory and Loewner conjectures for totally semiumbilic surfaces in \( \mathbb{R}^3 \) with non-zero Gaussian curvature.

**Theorem 6.12.** Let \( M^n \subset \mathbb{R}^{n+2} \) be an immersed submanifold and let \( \nu \in \mathcal{X}(M)^1 \) be a non-degenerate parallel unit normal vector field. Then the binormal and asymptotic directions of \( M \) at \( p \) coincide with the binormal and asymptotic directions of \( M^\nu \) at \( \nu_p \).

**Proof.** Let \( p \in M \) and let \( \{X_1, \ldots, X_n\} \) be an orthonormal frame of \( T_p M \) made of \( \xi \)-principal directions, for any \( \xi \in N_p M \). We also take \( \{\nu_1, \nu_2\} \) an orthonormal frame of \( N_p M \) such that \( \nu_1 = \nu_p \). Then,
\[
A_{\nu_1}(X_i) = \lambda_i X_i, \quad A_{\nu_2}(X_i) = \mu_i X_i,
\]
with \( \lambda_i \neq 0 \). We know that \( \{X_1, \ldots, X_n\} \) are the asymptotic directions of \( M \) and the corresponding binormal vectors are given by
\[
\xi_i = \frac{\mu_i}{\lambda_i} \nu_1 - \nu_2, \quad i = 1, \ldots, n.
\]

In the parallel submanifold \( M^\nu \) at \( \nu_p \) we consider the same frames \( \{X_1, \ldots, X_n\} \) and \( \{\nu_1, \nu_2\} \). By the above lemma, we obtain
\[
A^\nu_{\nu_1}(X_i) = -A_{\nu_1} \circ A^{-1}_{\nu_1}(X_i) = -X_i,
\]
\[
A^\nu_{\nu_2}(X_i) = -A_{\nu_2} \circ A^{-1}_{\nu_2}(X_i) = -\frac{\mu_i}{\lambda_i} X_i.
\]
Therefore, we get that the asymptotic directions of \( M^\nu \) are again \( \{X_1, \ldots, X_n\} \) with binormal vectors \( \{\xi_1, \ldots, \xi_n\} \). \(\square\)

We denote by \( \pi : S^{n+1} \setminus \{q\} \to \mathbb{R}^{n+1} \) the stereographical projection with respect to some point \( q \in S^{n+1} \). This is a diffeomorphism which is also a conformal map. Assume that \( M^n \subset \mathbb{R}^{n+2} \) is an immersed submanifold and let \( \nu \in \mathcal{X}(M)^1 \) be a non-degenerate parallel unit normal vector field. We can consider the immersion \( \pi \circ G_\nu : M \to \mathbb{R}^{n+1} \) and we denote its image by \( \bar{M}^\nu \), which is an immersed hypersurface in \( \mathbb{R}^{n+1} \). Moreover, given \( p \in M \) and \( X \in T_p M \) we also denote \( \bar{p} = \pi(\nu_p) \in \bar{M}^\nu \) and \( \bar{X} = D_p(\pi \circ G_\nu)(X) \in T_{\bar{p}} \bar{M}^\nu \).
Corollary 6.13. Let $M^n \subset \mathbb{R}^{n+2}$ be an immersed submanifold and let $\nu \in \mathcal{X}(M)^\perp$ be a non-degenerate parallel unit normal vector field. Then $X \in T_p M$ is an asymptotic direction of $M$ at $p$ if and only $\tilde{X}$ is a principal direction of $M^\nu$ at $\tilde{p}$.

Proof. This is a consequence of Theorem 6.12 and the fact that the stereographic projection is a conformal map and hence it transforms asymptotic directions of a submanifold $N \subset S^{n+1}$ into principal directions of its image $\pi(N) \subset \mathbb{R}^{n+1}$ (see [7] for details).

7 Semiumbilicity, hypersphericity and conformal flatness

Given an $n$-manifold $M$ lying in a hypersphere of $\mathbb{R}^{n+2}$, the radius vector is an umbilic normal field over $M$, so $M$ is semiumbilical and hence has vanishing normal curvature. On the other hand, submanifolds with vanishing normal curvature, or even semiumbilical submanifolds, do not need to lie in a hypersphere in general. The connection between semiumbilicity and hypersphericity for surfaces in 4-space was investigated in [12], where necessary and sufficient conditions for semiumbilical surfaces to be hyperspherical in terms of the binormals and the curvature ellipses were obtained. We shall use here some results due to Chen and Yano ([1], [2], [3]) and Chen and Vegrauel ([4]), together with our previous results, in order to characterize the hypersphericity and the conformal flatness in terms of the curvature locus at each point.

We first observe that Chen and Yano [1] proved that an $n$-manifold $M$ immersed in $\mathbb{R}^{n+2}$ is hyperspherical if and only if $M$ is $\nu$-umbilic for some parallel normal field $\nu$. Therefore we can assert:

Corollary 7.1. Let $M$ be an $n$-manifold $M$ immersed in $\mathbb{R}^{n+2}$, $n \geq 2$ with isolated umbilic points. Then $M$ is hyperspherical if and only if the curvature locus at every point of $M$ is a segment defining a parallel field off the umbilic set of $M$.

Proof. We have seen that the curvature locus is a segment at a point $p \in M$ if and only if $p$ is $\nu$-umbilic for some field $\nu$. Moreover, $\nu(p)$ is orthogonal to the direction defined by the segment in the normal plane of $M$ at $p$. The assertion follows now easily over the open and dense submanifold determined by the complement of the umbilic points from Chen and Yano’s result and is extended to the whole $M$ by continuity.

In the case of a submanifold $M$ with vanishing normal curvature, if $\{X_i\}_{i=1}^n$ represents the orthonormal basis of common principal directions on $M$, then the condition above is equivalent to asking that the normal fields
\( \eta(X_i) - \eta(X_j) \) be parallel and pairwise linearly dependent, for all \( i \neq j \). Therefore, we have:

**Corollary 7.2.** Let \( M \) be a semiumbilical \( n \)-manifold in \( \mathbb{R}^{n+2}, n \geq 2 \) and let \( \{X_i\}_{i=1}^n \) be the orthonormal basis of common principal directions on \( M \). Then \( M \) is hyperspherical if and only if

\[
D_X(\eta(X_i)) = D_X(\eta(X_j)), \quad \forall X \in T_pM, \forall i \neq j, \forall p \in M .
\]

We recall now some facts concerning the relations between quasiumbility and conformal flatness. Chen and Yano [2] proved that any totally quasiumbilical \( n \)-manifold with \( n \geq 4 \) is conformally flat. On the other hand, Chen and Verstraelen [4] proved that any \( n \)-manifold \( M \) which is conformally flat and has flat normal connection in \( \mathbb{R}^{n+p} \) is totally quasiumbilical, provided \( n > 4 \) and \( p < n - 2 \). Therefore we can state:

If \( M \) is an \( n \)-manifold with vanishing normal curvature in \( \mathbb{R}^{n+2}, n > 4 \), then

\( M \) is conformally flat \( \Leftrightarrow \) \( M \) is totally quasiumbilical.

Moreover, Chen and Yano [3] proved that if \( M \) is \( v \)-umbilic for some non parallel normal field \( v \), then \( M \) is totally quasiumbilical. Here, saying that \( v \) is non parallel means that for each \( p \in M \), there is a tangent vector \( X \in T_pM \) such that \( D_Xv \neq 0 \). It then follows that totally semiumbilical \( n \)-manifolds immersed in \( \mathbb{R}^{n+2}, n > 4 \), whose curvature segments determine non parallel normal fields, are conformally flat. Consequently, we can state:

**Corollary 7.3.** Let \( M \) be an \( n \)-manifold immersed in \( \mathbb{R}^{n+2}, n > 4 \) and suppose that the curvature locus degenerates to a segment at every point of \( M \). Consider a normal field \( v \) determined by the curvature segment (locally defined) at each point. If \( v \) is non parallel, then \( M \) is conformally flat.

Summarizing, we have:

**Corollary 7.4.** Let \( M \) be a semiumbilical \( n \)-manifold immersed in \( \mathbb{R}^{n+2}, n > 4 \) with a finite subset of umbilic points \( \{p_1, \ldots, p_k\} \). Let \( \{X_i\}_{i=1}^n \) be the orthonormal basis of common principal directions on \( M \). Denote

\[
\nu_{ij} = \eta(X_i) - \eta(X_j), i \neq j .
\]

The vectors \( \{\nu_{ij}\}_{i,j=1}^n, i \neq j \) do not vanish simultaneously at each \( p \not\in \{p_1, \ldots, p_k\} \) and define a normal direction \( \nu \) over \( M - \{p_1, \ldots, p_k\} \). Then we have:

a) If \( \nu \) is a parallel field, then \( M \) is hyperspherical.

b) If \( \nu \) is a non parallel field, then \( M \) is conformally flat.

**References**


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