

PLANE CURVE DIAGRAMS AND GEOMETRICAL APPLICATIONS

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ABSTRACT. We look at plane curve diagrams (f, α) , which are given by a plane curve multigerms $\alpha : (\mathbb{R}, S) \rightarrow \mathbb{R}^2$ and a function on it $f : (\mathbb{R}, S) \rightarrow \mathbb{R}$. We obtain a classification of all such diagrams where α has \mathcal{A}_e -codimension ≤ 2 and f has finite order. Then we define an equivalence between plane curves which we call \mathcal{A}_h -equivalence and which is determined by the class of the diagram (h_α, α) . Here, h_α denotes the height function of α with respect to its normal vector. This is an equivalence which not only takes into account the topology of the singularity of α , but also its flat geometry. Finally, we apply our results in order to obtain a classification of all the plane projections of a generic space curve γ embedded in \mathbb{R}^3 .

1. INTRODUCTION

Given a C^∞ embedded space curve $\gamma : I \rightarrow \mathbb{R}^3$, with $I \subset \mathbb{R}$ an open interval, we can look at the orthogonal projections of γ into planes. For a generic projection, it follows that the projected plane curve is a regular curve with only transverse double points (A_1 or *node* singularities), which correspond to pairs of points $\gamma(t)$ and $\gamma(s)$ such that the direction of projection is parallel to the secant line passing through these two points. If the projection is not generic, the plane curve can present more degenerated singularities. For instance, we will get a cusp if the direction of projection is parallel to some tangent line of γ .

David showed in [2] that for a *projection-generic* space curve $\gamma : I \rightarrow \mathbb{R}^3$, the only singularities that can appear in any orthogonal projection of γ are the following ten \mathcal{A} -classes: $A_0, A_1, \dots, A_5, D_4, D_5, D_6$, and \tilde{E}_7 . The notation here was introduced by Wall in [6] and it refers to the Arnold notation for the implicit equation of the image of the plane curve. Note that David considered not only orthogonal projections, but also radial projections. His list is then longer, since the family of radial projections has 3 parameters, while the family of orthogonal projections has only 2 (in fact, there are three more singularities, A_6, A_7 and D_8 , which appear only in radial projections).

Each one of these ten singularities corresponds to a geometric phenomenon of the space curve γ (zero torsion point, cross tangent, trisecant line, etc.) Here we consider not only local, but also multilocal singularities and all the singularities given in this list (except \tilde{E}_7) have \mathcal{A}_e -codimension ≤ 2 (\tilde{E}_7 is a non simple singularity of \mathcal{A}_e -codimension 3). Moreover, David showed that the set of projection-generic curves is residual in $C^\infty(I, \mathbb{R}^3)$ with the Whitney C^∞ topology, a result that had been previously announced by Wall in [6]. Recently, Wall has also proved in [7] the openness of this set.

Since we consider \mathcal{A} -equivalence (i.e., diffeomorphisms in the source and in the target) to classify plane curves, it is obvious that we loss some geometrical information. For

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instance, the ramphoid cusp or A_4 singularity of the plane projection corresponds to a direction of projection parallel to the tangent line at a flattening (i.e., a zero torsion point) of γ . In [2, 6], this singularity is drawn like in Figure 1.(a), obtained with the parametrization $t \mapsto (t^2, t^4 + t^5)$.



FIGURE 1. (a) $(t^2, t^4 + t^5)$ and (b) (t^2, t^5) .

In fact, this is the picture we get when we take a bent wire having a flattening and we look at it through one eye in the tangent direction. However, when classifying plane curves, the A_4 singularity is usually identified by the normal form $t \mapsto (t^2, t^5)$ which graphically looks like Figure 1.(b). The two plane curves are \mathcal{A} -equivalent, but Figure 1.(b) should correspond to the projection of a degenerate flattening of a space curve, which is not generic.

We find another interesting example when looking at the geometrical interpretation of the appearance, as a projection, of the A_3 and A_5 singularities. In Figure 2.(a) we present the A_3 singularity which appears under projection along a secant joining the tangency points in a bitangent (non-osculating) plane. In Figure 2.(b), we give a different presentation of the A_3 singularity, which appears when the bitangent plane osculates at one of the points. In [1], it is stated erroneously that this geometrical phenomenon gives an A_5 singularity. Also, the geometrical interpretation given in [2, 6] for the appearance of the A_5 singularity in terms of the space curve is not satisfactory. In Section 4, we give the correct geometrical interpretation (see Figure 2.(c)).

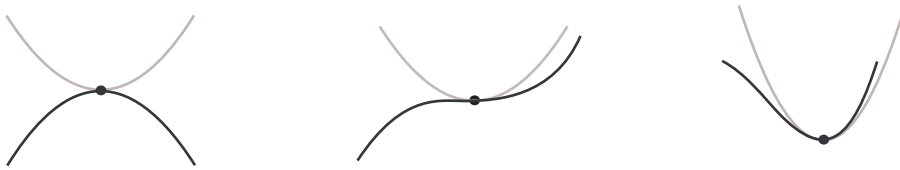


FIGURE 2. (a) A_3 (bitangent plane), (b) A_3 (bitangent osculating plane) and (c) A_5 .

In this paper, we propose a revision of the David classification by using a sharper equivalence which takes into account not only the \mathcal{A} -class, but also the flat geometry of the plane curve. Since the flat geometry of a regular curve is described by looking at the singularities of the height function, we define an equivalence which simultaneously looks at the singularity of the plane curve and its height function. Given a plane curve germ $\alpha : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ we denote by $h_\alpha : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ the germ of the height function of α in the “normal” direction (here we assume that α has isolated singularity and the “normal” direction is the limit of the normal directions when we are close to 0). We say

that two plane curves α, β are \mathcal{A}_h -equivalent if the plane curve diagrams $(h_\alpha, \alpha), (h_\beta, \beta)$ are equivalent. With this definition, we see easily that the two curves given in figures 1.(a) and 1.(b) are not \mathcal{A}_h -equivalent. This \mathcal{A}_h -equivalence can be extended to plane curve multigerms by choosing a different normal direction for each branch. Again, since the inflection points are preserved under \mathcal{A}_h -equivalence, the curve singularities of figures 2.(a) and 2.(b) are not \mathcal{A}_h -equivalent.

Under \mathcal{A}_h -equivalence, each one of the ten \mathcal{A} -classes of David split into several \mathcal{A}_h -classes. We will compute them and show in Theorem 4.4 that for a generic embedded space curve $\gamma : I \rightarrow \mathbb{R}^3$, there are only seventeen \mathcal{A}_h singularities that can appear in any orthogonal projection of γ (see Figure 3). Moreover, we correlate each of these \mathcal{A}_h -classes with geometric properties of γ .

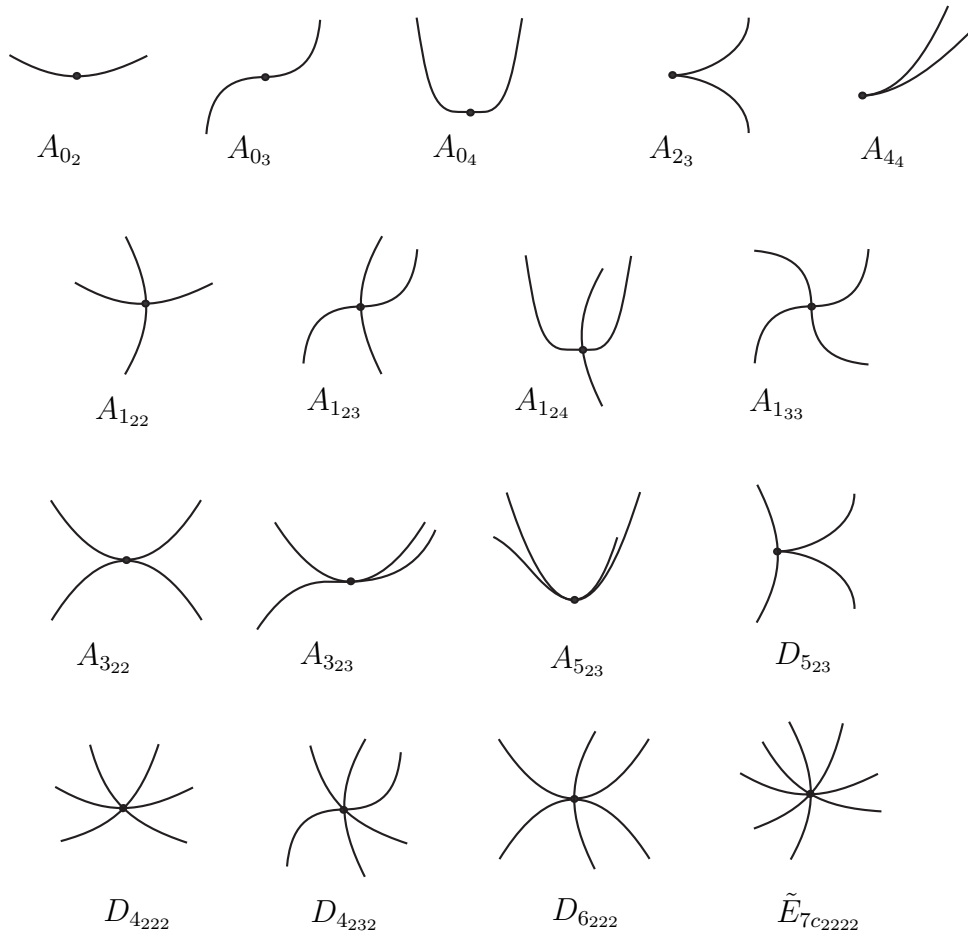


FIGURE 3. Singularities of the orthogonal projection of a generic space curve.

In Section 2, we study plane curve diagrams (f, α) where α is a plane curve singularity and f is a function. We classify all the plane curve diagrams such that α has \mathcal{A}_e -codimension ≤ 2 and f has finite order. Although equivalence of divergent diagrams has previously been studied (see [3],[4],[5]), it seems that the classification of plane curve diagrams (which is one of the simplest cases) has not been treated previously in the literature.

2. PLANE CURVE DIAGRAMS

Definition 2.1. A *divergent diagram* is a pair of C^∞ map germs (f, g) , where $f : (M, x) \rightarrow (N, y)$ and $g : (M, x) \rightarrow (P, z)$

$$(N, y) \xleftarrow{f} (M, x) \xrightarrow{g} (P, z) .$$

We say that two divergent diagrams (f_1, g_1) and (f_2, g_2) are *equivalent* if there are C^∞ diffeomorphism germs h, H, K such that the diagram

$$\begin{array}{ccccc} (N_1, y_1) & \xleftarrow{f_1} & (M_1, x_1) & \xrightarrow{g_1} & (P_1, z_1) \\ \downarrow h & & \downarrow H & & \downarrow K \\ (N_2, y_2) & \xleftarrow{f_2} & (M_2, x_2) & \xrightarrow{g_2} & (P_2, z_2) \end{array}$$

is commutative.

The definition can be extended to multigerms by considering a diagram of the form

$$(N, T) \xleftarrow{f} (M, S) \xrightarrow{g} (P, z) ,$$

where $S \subset M$ and $T \subset N$ are finite sets. The definition for equivalence can be easily extended to the multigerm case.

Let us introduce some useful notation. Given $r \geq 1$, we write

$$(\mathbb{R}_r, 0_r) := \prod_{i=1}^r (\mathbb{R}, 0) .$$

We identify a multigerm of map $f : (\mathbb{R}_r, 0_r) \rightarrow X$ with the r -tuple (f_1, \dots, f_r) of map germs $f_i : (\mathbb{R}, 0) \rightarrow X$ obtained by restriction of f to each branch.

We denote by \mathcal{R} the group of diffeomorphism germs $H : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ and by \mathcal{R}^r the group of diffeomorphism multigerms $H : (\mathbb{R}_r, 0_r) \rightarrow (\mathbb{R}_r, 0_r)$, so that we can identify $\mathcal{R}^r = \mathcal{R} \times \dots \times \mathcal{R}$, with \mathcal{R} repeated r times. We have a right group action of \mathcal{R}^r on the set of multigerms $f : (\mathbb{R}_r, 0_r) \rightarrow X$.

We also denote by \mathcal{L}^r the same group \mathcal{R}^r , but acting on the left on the set of multigerms $f : (X, S) \rightarrow (\mathbb{R}_r, 0_r)$. Finally, we will denote by \mathcal{L}_2 the group of diffeomorphism germs from $(\mathbb{R}^2, 0)$ to $(\mathbb{R}^2, 0)$, which will act on the left.

Definition 2.2. A *plane curve diagram* is a divergent diagram (f, α) of the form

$$(\mathbb{R}_r, 0_r) \xleftarrow{f} (\mathbb{R}_r, 0_r) \xrightarrow{\alpha} (\mathbb{R}^2, 0) .$$

We say that two plane curve diagrams (f, α) and (g, β) are *equivalent* if there are $H \in \mathcal{R}^r$, $h \in \mathcal{L}^r$ and $K \in \mathcal{L}_2$ such that the diagram

$$\begin{array}{ccccc} (\mathbb{R}_r, 0_r) & \xleftarrow{f} & (\mathbb{R}_r, 0_r) & \xrightarrow{\alpha} & (\mathbb{R}^2, 0) \\ \downarrow h & & \downarrow H & & \downarrow K \\ (\mathbb{R}_r, 0_r) & \xleftarrow{g} & (\mathbb{R}_r, 0_r) & \xrightarrow{\beta} & (\mathbb{R}^2, 0) \end{array}$$

is commutative. According to the notation introduced above, this means that there are diffeomorphism germs $K : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$, $H_i : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ and $h_i : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$,

with $i = 1, \dots, r$, such that

$$K \circ \alpha_i = \beta_i \circ H_i, \quad h_i \circ f_i = g_i \circ H_i.$$

On one hand we consider a plane curve multigerms $\alpha : (\mathbb{R}_r, 0_r) \rightarrow (\mathbb{R}^2, 0)$ and on the other hand we consider a function multigerms $f : (\mathbb{R}_r, 0_r) \rightarrow (\mathbb{R}_r, 0_r)$. Moreover, it is obvious that if (f, α) and (g, β) are equivalent, then α, β are \mathcal{A} -equivalent as plane curve multigerms (that is, equivalent under the action of $\mathcal{R}^r \times \mathcal{L}_2$) and f, g are \mathcal{A} -equivalent as function multigerms (that is, equivalent under the action of $\mathcal{R}^r \times \mathcal{L}^r$).

Definition 2.3. Let $\alpha : (\mathbb{R}_r, 0_r) \rightarrow (\mathbb{R}^2, 0)$ be a plane curve multigerms. We denote by \mathcal{R}^α the subgroup of \mathcal{R}^r of diffeomorphisms which “preserve” α . That is, it is given by

$$\mathcal{R}^\alpha = \{H \in \mathcal{R}^r : \exists K \in \mathcal{L}_2, K \circ \alpha = \alpha \circ H\}.$$

The next lemma is the fundamental tool for the classification of plane curve diagrams. Its proof follows easily from the definitions.

Lemma 2.4. (1) *Assume that α, β are \mathcal{A} -equivalent. Then there exists $H \in \mathcal{R}^r$ such that (f, α) is equivalent to $(f \circ H^{-1}, \beta)$.*
 (2) *Two plane curve diagrams (f, α) and (g, α) are equivalent if and only if f, g are $\mathcal{R}^\alpha \times \mathcal{L}^r$ -equivalent.*

We use this lemma in the following way: in order to classify a plane curve diagram (f, α) we first classify the plane curve α under \mathcal{A} -equivalence and obtain a normal form β . Then (f, α) is equivalent to some diagram (\tilde{f}, β) by (1). Now, we fix the curve β and by (2), it is enough to classify the function \tilde{f} with relation to the subgroup $\mathcal{R}^\beta \times \mathcal{L}^r \subset \mathcal{A}$ which preserves the curve β .

The purpose of this section is to prove the following classification theorem, which includes all the plane curve diagrams (f, α) , where α is a plane curve multigerms with \mathcal{A}_e -codimension ≤ 2 and f is any finitely determined function multigerms.

Theorem 2.5. *Let (f, α) be a plane curve diagram such that α has \mathcal{A}_e -codimension ≤ 2 and f has finite order. Then (f, α) is equivalent to one of the diagrams given in the following tables:*

Type	Germes
A_0	$t^m \leftarrow t \rightarrow (t, 0)$
A_2	$t^m \leftarrow t \rightarrow (t^2, t^3)$
A_4	$t^m + t^{m+1} \leftarrow t \rightarrow (t^2, t^5)$ $t^m \leftarrow t \rightarrow (t^2, t^5)$

Type	Bigermes
A_1	$t^m \leftarrow t \rightarrow (t, 0) ; t^n \leftarrow t \rightarrow (0, t)$
A_3	$t^m \leftarrow t \rightarrow (t, 0) ; t^n \leftarrow t \rightarrow (t, t^2)$
A_5	$t^m \leftarrow t \rightarrow (t, 0) ; t^n + t^{n+1} \leftarrow t \rightarrow (t, t^3)$ $t^m \leftarrow t \rightarrow (t, 0) ; t^n \leftarrow t \rightarrow (t, t^3)$
D_5	$t^m \leftarrow t \rightarrow (t, 0) ; t^n + t^{n+1} \leftarrow t \rightarrow (t^3, t^2)$ $t^m \leftarrow t \rightarrow (t, 0) ; t^n \leftarrow t \rightarrow (t^3, t^2)$

Type	Trigerms
D_4	$t^l \leftarrow t \rightarrow (t, 0) ; t^m \leftarrow t \rightarrow (0, t) ; t^n \leftarrow t \rightarrow (t, t)$
D_6	$t^l \leftarrow t \rightarrow (t, 0) ; t^m \leftarrow t \rightarrow (0, t) ; t^n + t^{n+1} \leftarrow t \rightarrow (t, t^2)$ $t^l \leftarrow t \rightarrow (t, 0) ; t^m \leftarrow t \rightarrow (0, t) ; t^n \leftarrow t \rightarrow (t, t^2)$

Type	Quadrigermes
\tilde{E}_7	$(t^k, (t, 0)) ; (t^l; (0, t)) ; (t^m + t^{m+1}, (t, t)) ; (t^n + ct^{n+1}, (t, \lambda t))$ $(t^k, (t, 0)) ; (t^l; (0, t)) ; (t^m, (t, t)) ; (t^n + t^{n+1}, (t, \lambda t))$ $(t^k, (t, 0)) ; (t^l; (0, t)) ; (t^m, (t, t)) ; (t^n, (t, \lambda t))$

Note that when α is a plane curve multigerms of \mathcal{A}_e -codimension ≤ 1 , then there is just one class for each \mathcal{A} -class of f (which is determined by the order sequence). However, for plane curve singularities of \mathcal{A}_e -codimension 2 or 3, each \mathcal{A} -class of f splits into two or three respectively distinct equivalence classes of diagrams.

2.1. The local case. Here we consider plane curve diagrams (f, α) , where α is a plane curve monogerm. That is, we consider diagrams of the form

$$(\mathbb{R}, 0) \xleftarrow{f} (\mathbb{R}, 0) \xrightarrow{\alpha} (\mathbb{R}^2, 0).$$

We will give a complete classification of plane curve diagrams (f, α) , where α is either regular or a plane curve singularity of type A_{2k} , with $k \geq 1$, and f is a function germ of finite order m . Note that this includes all the plane curve germs of \mathcal{A}_e -codimension ≤ 2 . We begin with the regular case.

Proposition 2.6. *Let (f, α) be a plane curve diagram such that α is regular and f has finite order m . Then (f, α) is equivalent to the diagram*

$$t^m \leftarrow t \rightarrow (t, 0).$$

Proof. Let β be the curve germ $\beta(t) = (t, 0)$. We observe that, in this case, $\mathcal{R}^\beta = \mathcal{R}$. In fact, if $H \in \mathcal{R}$, then we consider the diffeomorphism $K(x, y) = (H(x), y)$, which verifies $K \circ \beta = \beta \circ H$. Hence, $H \in \mathcal{R}^\beta$.

Now we use Lemma 2.4. Since α is regular, it is \mathcal{A} -equivalent to β and hence, (f, α) is equivalent to $(f \circ H^{-1}, \beta)$ for some $H \in \mathcal{R}$. Moreover, $f \circ H^{-1}$ has order m and therefore it is \mathcal{A} -equivalent to $g(t) = t^m$. But this implies that $(f \circ H^{-1}, \beta)$ is equivalent to (g, β) . \square

Assume now that $\alpha : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ is singular, that is, $\alpha'(0) = 0$. If $\alpha''(0) \neq 0$ and α is \mathcal{A} -finitely determined, then α is \mathcal{A} -equivalent to the A_{2k} singularity, which is given by the map germ $t \mapsto (t^2, t^{2k+1})$, with $k \geq 1$.

For instance, if $k = 1$, we have an *ordinary cusp* A_2 given by $t \mapsto (t^2, t^3)$. For $k = 2$, the A_4 singularity given by $t \mapsto (t^2, t^5)$ is called the *ramphoid cusp*.

Proposition 2.7. *Let (f, α) a plane curve diagram such that α has an ordinary cusp and f has finite order m . Then (f, α) is equivalent to the diagram*

$$t^m \leftarrow t \rightarrow (t^2, t^3).$$

Proof. The proof is analogous to the previous proposition. We only need to prove again that $\mathcal{R}^\beta = \mathcal{R}$, where $\beta(t) = (t^2, t^3)$. In fact, given $H \in \mathcal{R}$, by using the preparation theorem we can write

$$H(t)^2 = t^2 a_1(t^2) + t^3 a_2(t^2), \quad H(t)^3 = t^3 b_1(t^2) + t^4 b_2(t^2),$$

for some function germs a_1, a_2, b_1, b_2 , with $a_1(0), b_1(0) \neq 0$. Thus, $K(x, y) = (x a_1(x) + y a_2(x), y b_1(x) + x^2 b_2(x))$ defines a diffeomorphism germ K which verifies $K \circ \beta = \beta \circ H$. \square

In order to give a complete classification for A_{2k} diagrams, $k \geq 2$, we first need to characterize the group \mathcal{R}^β where $\beta(t) = (t^2, t^{2k+1})$. Then, we will obtain a \mathcal{R}^β -classification of finite order function germs $f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$.

Lemma 2.8. *Let $\beta : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ be the curve germ $\beta(t) = (t^2, t^{2k+1})$, $k \geq 2$. Then*

$$\mathcal{R}^\beta = \{H \in \mathcal{R} : H''(0) = H^{(4)}(0) = \dots = H^{(2k-2)}(0) = 0\}.$$

Proof. If $H \in \mathcal{R}^\beta$, then there exist a diffeomorphism germ K such that $K \circ \beta = \beta \circ H$. Assume that $K = (K_1, K_2)$ and write

$$\begin{aligned} H(t) &= a_1 t + a_2 t^2 + \dots, \text{ with } a_1 \neq 0 \\ K_1(x, y) &= b_{10} x + b_{01} y + b_{20} x^2 + b_{11} xy + b_{02} y^2 + \dots \end{aligned}$$

Since $K_1(t^2, t^{2k+1}) = H(t)^2$, we obtain

$$K_1(t^2, t^{2k+1}) = b_{10} t^2 + b_{20} t^4 + \dots + b_{k0} t^{2k} + b_{01} t^{2k+1} + \dots$$

By comparing coefficients we see that $a_2 = \dots = a_{2k-2} = 0$, which means $H''(0) = \dots = H^{(2k-2)}(0) = 0$.

Conversely, if $H \in \mathcal{R}$, we can write

$$\begin{aligned} H(t)^2 &= t^2 a_1(t^2) + t^3 a_2(t^2) \\ H(t)^{2k+1} &= t^{2k+1} b_1(t^2) + t^{2k+2} b_2(t^2) \end{aligned}$$

for some function germs a_1, a_2, b_1, b_2 , with $a_1(0), b_1(0) \neq 0$. By hypothesis, $H''(0) = \dots = H^{(2k-2)}(0) = 0$. Thus, $H(t)^2$ has no terms of order 3, 5, ..., $2k-1$. It implies $\text{ord}(a_2) \geq k-1$ and therefore, we can write $a_2(t) = t^{k-1} \tilde{a}_2(t)$, for some function germ \tilde{a}_2 . Hence,

$$\begin{aligned} H(t)^2 &= t^2 a_1(t^2) + t^{2k+1} \tilde{a}_2(t^2) \\ H(t)^{2k+1} &= t^{2k+1} b_1(t^2) + t^{2k+2} b_2(t^2). \end{aligned}$$

We define the diffeomorphism germ K by

$$K(x, y) = (x a_1(x) + y \tilde{a}_2(x), y b_1(x) + x^{k+1} b_2(x)),$$

so that $K \circ \beta = \beta \circ H$. Therefore $H \in \mathcal{R}^\beta$. □

Proposition 2.9. *Let (f, α) be a plane curve diagram such that α is of type A_{2k} , $k \geq 2$, and f has finite order m . Then, (f, α) is equivalent to either*

$$t^m \leftarrow t \rightarrow (t^2, t^{2k+1}),$$

or

$$t^m + t^{m+2r-1} + d_1 t^{m+2r+1} + \dots + d_{k-r-1} t^{m+2k-3} \leftarrow t \rightarrow (t^2, t^{2k+1}),$$

for some $1 \leq r \leq k-1$ and $d_1, \dots, d_{k-r-1} \in \mathbb{R}$.

Proof. We will show first that f is R^β -equivalent to

$$t^m + d'_1 t^{m+1} + \dots + d'_{k-1} t^{m+2k-3},$$

where β is as in Lemma 2.8. Suppose $f(t) = c_m t^m + c_{m+1} t^{m+1} + \dots$, with $c_m \neq 0$. To simplify we can also assume that $c_m > 0$, being the case $c_m < 0$ analogous.

Observe that for any choice of $d'_1, \dots, d'_{k-1} \in \mathbb{R}$, f is \mathcal{R} -equivalent to $t^m + d'_1 t^{m+1} + \dots + d'_{k-1} t^{m+2k-3}$. Consequently there is $H \in \mathcal{R}$ such that

$$(1) \quad f(t) = H(t)^m + d'_1 H(t)^{m+1} + \dots + d'_{k-1} H(t)^{m+2k-3}.$$

We see that it is possible to choose d'_1, \dots, d'_{k-1} in such a way that $H \in \mathcal{R}^\beta$, that is,

$$H''(0) = \dots = H^{(2k-2)}(0) = 0.$$

Suppose that $H(t) = a_1 t + a_2 t^2 + \dots$, with $a_1 \neq 0$. The coefficient of t^m in $H(t)^m$ is a_1^m , while for $r \geq 1$, the coefficient of t^{m+r} is of the form

$$m a_1^{m-1} a_{r+1} + p(a_1, \dots, a_r),$$

being $p(a_1, \dots, a_r)$ a certain polynomial in the coefficients a_1, \dots, a_r . Using this fact and comparing the coefficients in (1), we get

$$\begin{aligned} c_{m+1} &= ma_1^{m-1}a_2 + d'_1 a_1^{m+1}, \\ c_{m+3} &= ma_1^{m-1}a_4 + p_0^1(a_1, a_2, a_3) + d'_1 p_1^1(a_1, a_2, a_3) + d'_2 a_1^{m+3}, \\ &\vdots \\ c_{m+2k-3} &= ma_1^{m-1}a_{2k-2} + p_0^{k-2}(a_1, \dots, a_{2k-3}) + \sum_{i=1}^{k-2} d'_i p_i^{k-2}(a_1, \dots, a_{2k-3}) + \\ &\quad d'_{k-1} a_1^{m+2k-3}. \end{aligned}$$

Since $a_1 \neq 0$, obviously we can choose d'_1, \dots, d'_{k-1} such that $a_2 = a_4 = \dots = a_{2k-2} = 0$ and hence, $H \in \mathcal{R}^\beta$.

By Lemma 2.4, we have that (f, α) is equivalent to

$$t^m + d'_1 t^{m+1} + d'_2 t^{m+3} + \dots + d'_{k-1} t^{m+2k-3} \leftarrow t \rightarrow (t^2, t^{2k+1}),$$

for some $d'_1, \dots, d'_{k-1} \in \mathbb{R}$. If $d'_1, \dots, d'_{k-1} = 0$, we are in the first case of the statement. Assume they are not zero and let $r \in \{1, \dots, k-1\}$ be such that $d'_1 = \dots = d'_{r-1} = 0$ and $d'_r \neq 0$. We take linear changes of coordinates $H(t) = bt$ and $h(u) = \lambda u$ which transform the above diagram into

$$t^m + t^{m+2r-1} + d_1 t^{m+2r+1} + \dots + d_{k-r-1} t^{m+2k-3} \leftarrow t \rightarrow (t^2, t^{2k+1}).$$

In fact, it is enough to take $b = 2^{r-1} \sqrt{d'_r}$ and $\lambda = b^m$. \square

Corollary 2.10. *Assume $m > 2k - 3$. Then, each choice of $r \in \{1, \dots, k-1\}$ and the coefficients $d_1, \dots, d_{k-r-1} \in \mathbb{R}$ in the plane curve diagram*

$$t^m + t^{m+2r-1} + d_1 t^{m+2r+1} + \dots + d_{k-r-1} t^{m+2k-3} \leftarrow t \rightarrow (t^2, t^{2k+1}),$$

corresponds to a distinct equivalence class.

Proof. Let us consider

$$\begin{aligned} f(t) &= t^m + t^{m+2r-1} + d_1 t^{m+2r+1} + \dots + d_{k-r-1} t^{m+2k-3}, \\ g(t) &= t^m + t^{m+2r'-1} + d'_1 t^{m+2r'+1} + \dots + d'_{k-r'-1} t^{m+2k-3}, \end{aligned}$$

with $1 \leq r, r' \leq k-1$ and $d_1, \dots, d_{k-r-1}, d'_1, \dots, d'_{k-r'-1} \in \mathbb{R}$. Suppose that f and g are $\mathcal{R}^\beta \times \mathcal{L}$ -equivalent by means of (H, h) , where $\beta(t) = (t^2, t^{2k+1})$. We write $h(t) = \lambda_1 t + \lambda_2 t^2 + \dots$ with $\lambda_1 \neq 0$. Since $m > 2k - 3$, it follows that $2m > m + 2k - 3$ and therefore,

$$h(f(t)) = \lambda_1 (t^m + t^{m+2r-1} + d_1 t^{m+2r+1} + \dots + d_{k-r-1} t^{m+2k-3}) + \dots$$

being this function germ \mathcal{R}^β -equivalent to g . Thus from proof of the previous lemma we have that $r = r'$ and $d_i = d'_i$ for all $i = 1, \dots, k-r-1$. \square

When $m \leq 2k - 3$, sometimes it is possible to use the \mathcal{L}_1 -equivalence to eliminate other terms in the function f . Unfortunately, it is quite complicated to give a formula for the general case, since many particular cases may appear. As a sample, we see here some examples in simple cases:

(1) Assume $m = 2k - 3$. Then (f, α) is equivalent to either

$$t^{2k-3} \leftarrow t \rightarrow (t^2, t^{2k+1}),$$

or

$$t^{2k-3} + t^{2k+2r-4} + c_1 t^{2k+2r-2} + \cdots + c_{k-r-2} t^{4k-8} \leftarrow t \rightarrow (t^2, t^{2k+1}),$$

with $1 \leq r \leq k - 2$ and $c_1, \dots, c_{k-r-2} \in \mathbb{R}$. Moreover, each choice of r and the coefficients c_1, \dots, c_{k-r-2} corresponds to a distinct equivalence class.

(2) Assume $m = 2$. Then (f, α) is equivalent to either

$$t^2 \leftarrow t \rightarrow (t^2, t^{2k+1}),$$

or

$$t^2 + t^{2r+1} \leftarrow t \rightarrow (t^2, t^{2k+1}),$$

with $1 \leq r \leq k - 1$. Moreover, each choice of r corresponds to a distinct equivalence class.

Finally, we give an explicit classification of ramphoidal cusp diagrams ($k = 2$).

Corollary 2.11. *Let (f, α) be a plane curve diagram such that α is a ramphoid cusp and f has finite order m . Then (f, α) is equivalent to either*

$$t^m + t^{m+1} \leftarrow t \rightarrow (t^2, t^5)$$

or

$$t^m \leftarrow t \rightarrow (t^2, t^5).$$

Remark 2.12. It follows from the above discussion that the two diagrams

$$t^m + t^{m+1} \leftarrow t \rightarrow (t^2, t^5) \quad \text{and} \quad t^m \leftarrow t \rightarrow (t^2, t^5)$$

are not equivalent. Moreover, assume that $\alpha(t) = (a_2 t^2 + a_3 t^3 + \cdots, b_5 t^5 + b_6 t^6 + \cdots)$, with $a_2, b_5 \neq 0$, and $f(t) = c_m t^m + c_{m+1} t^{m+1} + \cdots$ with $c_m \neq 0$. Then (f, α) is equivalent to

$$\begin{cases} t^m + t^{m+1} \leftarrow t \rightarrow (t^2, t^5), & \text{if } 2a_2 c_{m+1} - m a_3 c_m \neq 0, \\ t^m \leftarrow t \rightarrow (t^2, t^5), & \text{if } 2a_2 c_{m+1} - m a_3 c_m = 0. \end{cases}$$

2.2. The multilocal case. Here we consider plane curve diagrams (f, α) , where α is a plane curve multigerms. According to the notation introduced in the beginning of this section, we identify a plane curve multigerms α with a r -tuple $(\alpha_1, \dots, \alpha_r)$ where each α_i is a plane curve germ. In the same way, a function multigerms f is given by a r -tuple (f_1, \dots, f_r) of function germs f_i . We say that f has orders (m_1, \dots, m_r) if each f_i has order m_i . Finally, a plane curve diagram (f, α) will be denoted by a r -tuple $(f_1, \alpha_1); \dots; (f_r, \alpha_r)$ where each (f_i, α_i) is a plane curve diagram monogerm.

For a plane curve bigerm of \mathcal{A}_e -codimension ≤ 2 , we have four possibilities, namely the A_1, A_3, A_5 and D_5 singularities.

Proposition 2.13. *Let (f, α) be a plane curve diagram such that α is a plane curve bigerm of type A_1 and f is a function bigerm of finite orders (m, n) . Then (f, α) is equivalent to*

$$(t^m, (t, 0)) ; (t^n, (0, t)).$$

Proof. It follows from the plane curve classification that $(\alpha_1; \alpha_2)$ is \mathcal{A} -equivalent to $(\beta_1; \beta_2)$, where $\beta_1(t) = (t, 0)$ and $\beta_2(t) = (0, t)$. By Lemma 2.4, $(f_1, \alpha_1); (f_2, \alpha_2)$ is equivalent to $(f_1 \circ H_1^{-1}, \beta_1); (f_2 \circ H_2^{-1}, \beta_2)$ for some pair $(H_1, H_2) \in \mathcal{R} \times \mathcal{R}$.

Moreover $f_1 \circ H_1^{-1}$ and $f_2 \circ H_2^{-1}$ also have orders m and n respectively, hence $f_1 \circ H_1^{-1}$ and $f_2 \circ H_2^{-1}$ are \mathcal{A} -equivalent to t^m and t^n . We will show that, in this case, $\mathcal{R}^\beta = \mathcal{R} \times \mathcal{R}$, which gives the desired result.

In fact, given $(H_1, H_2) \in \mathcal{R} \times \mathcal{R}$, we can consider the diffeomorphism K defined in an obvious way by $K(x, y) = (H_1(x), H_2(y))$, so that $K \circ \beta_i = \beta_i \circ H_i$, $i = 1, 2$. Hence, $(H_1, H_2) \in \mathcal{R}^\beta$. \square

Proposition 2.14. *Let (f, α) be a plane curve diagram such that α is a plane curve bigerm of type A_3 and f is a function bigerm of finite orders (m, n) . Then (f, α) is equivalent to*

$$(t^m, (t, 0)) ; (t^n, (t, t^2)).$$

Proof. Let β be the plane curve bigerm with branches $\beta_1(t) = (t, 0)$ and $\beta_2(t) = (t, t^2)$. We show first that

$$\mathcal{R}^\beta = \{(H_1, H_2) \in \mathcal{R} \times \mathcal{R} : H_1'(0) = H_2'(0)\}.$$

In fact, if $(H_1, H_2) \in \mathcal{R}^\beta$, there exist a diffeomorphism $K(x, y) = (K_1(x, y), K_2(x, y))$ such that $K \circ \beta_i = \beta_i \circ H_i$, $i = 1, 2$. This means that

$$K_1(t, 0) = H_1(t), \quad K_1(t, t^2) = H_2(t),$$

and by taking derivatives at $t = 0$, we arrive to $H_1'(0) = H_2'(0)$.

Conversely, let $(H_1, H_2) \in \mathcal{R} \times \mathcal{R}$ be with $H_1'(0) = H_2'(0)$. We can write

$$H_2(t) - H_1(t) = t^2 Q_1(t), \quad H_2(t)^2 = t^2 Q_2(t),$$

for some function germs Q_1, Q_2 with $Q_2(0) \neq 0$. Therefore, we define K by

$$K(x, y) = \left(H_1(x) + yQ_1(x), yQ_2(x) \right),$$

which satisfies the desired equalities $K \circ \beta_i = \beta_i \circ H_i$, $i = 1, 2$.

In the second part of the proof, we show that if (f_1, f_2) have order (m, n) , then there are $(H_1, H_2) \in \mathcal{R}^\beta$ and $(h_1, h_2) \in \mathcal{L} \times \mathcal{L}$ such that

$$h_1 \circ f_1 \circ H_1^{-1}(t) = t^m \quad \text{and} \quad h_2 \circ f_2 \circ H_2^{-1}(t) = t^n.$$

This will conclude the proof by Lemma 2.4.

Since $\text{ord}(f_2) = n$, there exist a pair $(H_2, h_2) \in \mathcal{R} \times \mathcal{L}$ such that $h_2 \circ f_2 \circ H_2^{-1}(t) = t^n$. We consider $h_1 \in \mathcal{L}$, given by

$$h_1(u) = \lambda u, \quad \lambda = \frac{m! H_2'(0)^m}{f_1^{(m)}(0)}.$$

It follows that $\text{ord}(h_1 \circ f_1) = m$ and $(h_1 \circ f_1)^{(m)}(0) > 0$ when m is even, therefore there is $H_1 \in \mathcal{R}$ such that

$$(2) \quad h_1 \circ f_1 \circ H_1^{-1}(t) = t^m.$$

Note that when m is even, $-H_1$ also satisfies the previous equality.

By computing the derivative of order m at $t = 0$ on both sides of equality (2), we obtain $H_1'(0)^m = H_2'(0)^m$. Therefore, if m is odd this implies $H_1'(0) = H_2'(0)$, and then $(H_1, H_2) \in \mathcal{R}^\beta$.

\mathcal{R}^β . Otherwise, if m is even, we have $H'_1(0) = \pm H'_2(0)$, so that either $(H_1, H_2) \in \mathcal{R}^\beta$ or $(-H_1, H_2) \in \mathcal{R}^\beta$, as desired. \square

Proposition 2.15. *Let (f, α) be a plane curve diagram such that α is a plane curve bigerm of type A_5 and f is a function bigerm of finite orders (m, n) . Then (f, α) is equivalent to either*

$$(t^m, (t, 0)) ; (t^n + t^{n+1}, (t, t^3))$$

or

$$(t^m, (t, 0)) ; (t^n, (t, t^3)).$$

Proof. As in previous proposition, we will divide the proof in two parts. Let β be the plane curve bigerm given by $\beta_1(t) = (t, 0)$ and $\beta_2(t) = (t, t^3)$. We show first that

$$\mathcal{R}^\beta = \{(H_1, H_2) \in \mathcal{R} \times \mathcal{R} : H'_1(0) = H'_2(0) \text{ and } H''_1(0) = H''_2(0)\}.$$

In fact, if $(H_1, H_2) \in \mathcal{R}^\beta$, there exists a diffeomorphism $K = (K_1, K_2)$ such that $K \circ \beta_i = \beta_i \circ H_i$, $i = 1, 2$, that is,

$$K_1(t, 0) = H_1(t), \quad K_1(t, t^3) = H_2(t).$$

By taking derivatives at $t = 0$, we get $H'_1(0) = H'_2(0)$ and $H''_1(0) = H''_2(0)$.

Conversely, let $(H_1, H_2) \in \mathcal{R} \times \mathcal{R}$ be with $H'_1(0) = H'_2(0)$ and $H''_1(0) = H''_2(0)$. We write

$$H_2(t) - H_1(t) = t^3 Q_1(t), \quad H_2(t)^3 = t^3 Q_2(t),$$

for some function germs Q_1, Q_2 with $Q_2(0) \neq 0$. The diffeomorphism K which satisfies $K \circ \beta_i = \beta_i \circ H_i$, $i = 1, 2$, is given by

$$K(x, y) = \left(H_1(x) + yQ_1(x), yQ_2(x) \right).$$

In the second part of the proof, we show that if (f_1, f_2) have orders (m, n) , then there are $(H_1, H_2) \in \mathcal{R}^\beta$ and $(h_1, h_2) \in \mathcal{L} \times \mathcal{L}$ such that

$$(3) \quad h_1 \circ f_1 \circ H_1^{-1}(t) = t^m, \quad h_2 \circ f_2 \circ H_2^{-1}(t) = t^n + dt^{n+1},$$

for some $d \in \mathbb{R}$. Observe that given any $d \in \mathbb{R}$, there is a pair $(H_2, h_2) \in \mathcal{R} \times \mathcal{L}$ such that $h_2 \circ f_2 \circ H_2^{-1}(t) = t^n + dt^{n+1}$, since f_2 have order n . Now, as in the proof of the above proposition, we can choose $h_1 \in \mathcal{L}$ in such a way that there is $H_1 \in \mathcal{R}$ such that (H_1, h_1) satisfy the first equality in (3) and $H'_1(0) = H'_2(0)$. Moreover, in order to have $H''_1(0) = H''_2(0)$, we can take

$$d = \frac{h'_2(0)f_2^{(n+1)}(0) - \frac{n(n+1)!}{2}H'_2(0)^{n-1}H''_1(0)}{(n+1)!H'_1(0)^{n+1}}.$$

Finally, it follows from (3) that the diagram is equivalent to $(t^m, (t, 0)); (t^n + dt^{n+1}, (t, t^3))$. If $d = 0$ then, we are in the first case. Otherwise, if $d \neq 0$, we take linear changes of coordinates which transform it into $(t^m, (t, 0)); (t^n + t^{n+1}, (t, t^3))$. \square

Proposition 2.16. *Let (f, α) be a plane curve diagram such that α is a plane curve bigerm of type D_5 and f is a function bigerm of finite orders (m, n) . Then (f, α) is equivalent to either*

$$(t^m, (t, 0)) ; (t^n + t^{n+1}, (t^3, t^2))$$

or

$$(t^m, (t, 0)) ; (t^n, (t^3, t^2)).$$

Proof. As in previous propositions, the proof consists of two parts, but we just give the details for the first part. Let β be the plane curve bigerm defined by $\beta_1(t) = (0, t)$ and $\beta_2(t) = (t^2, t^3)$. Then

$$\mathcal{R}^\beta = \{(H_1, H_2) \in \mathcal{R} \times \mathcal{R} : H_1'(0) = H_2'(0)^3 \text{ and } H_2''(0) = 0\}.$$

In fact, if $(H_1, H_2) \in \mathcal{R}^\beta$, there exists a diffeomorphism germ K such that $K \circ \beta_i = \beta_i \circ H_i$, $i = 1, 2$. Assume that

$$H_i(t) = a_{i1}t + a_{i2}t^2 + a_{i3}t^3 + \dots, \quad i = 1, 2$$

$$K(x, y) = (b_{10}x + b_{01}y + b_{20}x^2 + \dots, c_{10}x + c_{01}y + c_{20}x^2 + \dots).$$

Since $K(0, t) = (0, H_1(t))$ and $K(t^2, t^3) = (H_2(t)^2, H_2(t)^3)$, it follows

$$\begin{aligned} b_{01}t + b_{02}t^2 + \dots &= 0, \\ c_{01}t + c_{02}t^2 + \dots &= a_{11}t + a_{12}t^2 + \dots, \\ b_{10}t^2 + b_{01}t^3 + \dots &= a_{21}^2t^2 + 2a_{21}a_{22}t^3 + \dots, \\ c_{10}t^2 + c_{01}t^3 + c_{20}t^4 + \dots &= a_{21}^3t^3 + 3a_{21}^2a_{22}t^4 + \dots. \end{aligned}$$

By comparing coefficients in the equalities above, we get $a_{22} = 0$ and $a_{11} = a_{21}^3$.

Conversely, let $(H_1, H_2) \in \mathcal{R} \times \mathcal{R}$ be with $H_1'(0) = H_2'(0)^3$ and $H_2''(0) = 0$. Then, we can write $H_2(t)^2 = t^2Q_1(t^2, t^3)$, where $Q_1(0, 0) \neq 0$ and $H_2(t)^3 - H_1(t^3) = t^2Q_2(t^2, t^3)$. Now it is enough to take K defined by

$$K(x, y) = \left(xQ_1(x, y), H_1(y) + xQ_2(x, y) \right).$$

□

For a plane curve trigerm of \mathcal{A}_e -codimension ≤ 2 , we have just two possibilities, the D_4 and D_6 singularities which have \mathcal{A}_e -codimension 1 and 2 respectively. Finally, for a plane curve quadrigerm, we only consider the \tilde{E}_7 singularity. As we said in the introduction, this is a non simple singularity of \mathcal{A}_e -codimension 3. However, we consider the stratum given by the union of all the 1-parameter family of \mathcal{A} -classes, which has \mathcal{A}_e -codimension 2. The proofs are quite similar to the proofs of the above propositions and the details are left to the reader.

3. \mathcal{A}_h -EQUIVALENCE OF PLANE CURVES

We introduce here an equivalence for \mathcal{A} -finitely determined plane curve multigerms. We call this equivalence \mathcal{A}_h -equivalence since it takes into account, not only the \mathcal{A} -class of the curve α , but also the information of the height function family on α . However, instead of considering the whole height function family, we will restrict ourselves to the height function associated to the normal direction, since all the other height functions are regular. To do this, we need to consider an extension of the normal vector for plane curve singularities.

Let $\alpha : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ be plane curve germ. We assume that α is \mathcal{A} -finitely determined. Then there is $k \geq 1$ such that

$$\alpha'(0) = \dots = \alpha^{(k-1)}(0) = 0, \quad \alpha^{(k)}(0) \neq 0.$$

We choose a vector $v_\alpha \in S^1$ orthogonal to $\alpha^{(k)}(0)$ (v_α is well defined up to the sign). We denote by $h_\alpha : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ the germ of the height function given by the inner product $h_\alpha(t) = \langle \alpha(t), v_\alpha \rangle$.

This definition can be extended easily to plane curve multigerms in the following way: Let $\alpha : (\mathbb{R}_r, 0_r) \rightarrow (\mathbb{R}^2, 0)$ be plane curve multigerms defined by $\alpha = (\alpha_1, \dots, \alpha_r)$. Then $h_\alpha : (\mathbb{R}_r, 0_r) \rightarrow (\mathbb{R}_r, 0_r)$ is the function multigerms given by $h_\alpha = (h_{\alpha_1}, \dots, h_{\alpha_r})$.

Definition 3.1. We say that two plane curve singularities $\alpha, \beta : (\mathbb{R}_r, 0_r) \rightarrow (\mathbb{R}^2, 0)$ are \mathcal{A}_h -equivalent if the plane curve diagrams $(h_\alpha, \alpha), (h_\beta, \beta)$ are equivalent. That is, there are diffeomorphisms $H \in \mathcal{R}^r$, $h \in \mathcal{L}^r$ and $K \in \mathcal{L}_2$ such that the diagram

$$\begin{array}{ccccc} (\mathbb{R}_r, 0_r) & \xleftarrow{h_\alpha} & (\mathbb{R}_r, 0_r) & \xrightarrow{\alpha} & (\mathbb{R}^2, 0) \\ \downarrow h & & \downarrow H & & \downarrow K \\ (\mathbb{R}_r, 0_r) & \xleftarrow{h_\beta} & (\mathbb{R}_r, 0_r) & \xrightarrow{\beta} & (\mathbb{R}^2, 0) \end{array}$$

is commutative.

It is obvious that if α, β are \mathcal{A}_h -equivalent, then they are \mathcal{A} -equivalent. The converse is not true, although the \mathcal{A}_h -equivalence is preserved by smooth change of coordinates in the source and affine change of coordinates in the target.

Lemma 3.2. Let $\alpha : (\mathbb{R}_r, 0_r) \rightarrow (\mathbb{R}^2, 0)$ be a plane curve multigerms. If $H \in \mathcal{R}^r$ and $K : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an affine transformation, then $K \circ \alpha \circ H$ is \mathcal{A}_h -equivalent to α .

Proof. In order to simplify the notation we assume that α is a monogerm ($r = 1$). The proof for a multigerms is analogous. We first observe that if $\alpha'(0) = \dots = \alpha^{(k-1)}(0) = 0$ and $\alpha^{(k)}(0) \neq 0$, then

$$(\alpha \circ H)'(0) = \dots = (\alpha \circ H)^{(k-1)}(0) = 0, \quad (\alpha \circ H)^{(k)}(0) = H'(0)^k \alpha^{(k)}(0) \neq 0.$$

Thus, we can choose $v_\alpha = v_{\alpha \circ H}$ so that $h_{\alpha \circ H} = h_\alpha \circ H$. It is obvious that the diagrams (h_α, α) and $(h_\alpha \circ H, \alpha \circ H)$ are equivalent.

Suppose now that $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ and $K(x, y) = (ax + by, cx + dy)$. Again we have

$$(K \circ \alpha)'(0) = \dots = (K \circ \alpha)^{(k-1)}(0) = 0, \quad (K \circ \alpha)^{(k)}(0) = K(\alpha^{(k)}(0)) \neq 0,$$

which allow us to choose

$$v_\alpha = \frac{(-\alpha_2^{(k)}(0), \alpha_1^{(k)}(0))}{\|\alpha^{(k)}(0)\|} \quad \text{and} \quad v_{K \circ \alpha} = \frac{(-c\alpha_1^{(k)}(0) - d\alpha_2^{(k)}(0), a\alpha_1^{(k)}(0) + b\alpha_2^{(k)}(0))}{\|(K \circ \alpha)^{(k)}(0)\|}.$$

It is not difficult to see that $h_{K \circ \alpha} = h \circ h_\alpha$, where

$$h(u) = \det(K) \frac{\|\alpha^{(k)}(0)\|}{\|(K \circ \alpha)^{(k)}(0)\|} u.$$

This shows again that the diagrams (h_α, α) and $(h_{K \circ \alpha}, K \circ \alpha)$ are equivalent, concluding the proof. \square

Let $\alpha : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ be a plane curve germ. Then we can use the above lemma to show that α is \mathcal{A}_h -equivalent to another plane curve germ $\beta = (\beta_1, \beta_2)$, with $\text{ord}(\beta_1) < \text{ord}(\beta_2)$. In particular, we can take $v_\beta = (0, 1)$ and $h_\beta = \beta_2$.

For a multigerms $\alpha : (\mathbb{R}_r, 0_r) \rightarrow (\mathbb{R}^2, 0)$, we can always choose three normal vectors v_1, v_2, v_3 and show that α is \mathcal{A}_h -equivalent to β , with $v_{\beta_i} = v_i$, $i = 1, 2, 3$. When $r = 2$, we usually fix the normal vectors as $v_1 = (0, 1)$ and $v_2 = (1, 0)$. When $r = 3$, we will use $v_3 = (-1, 1)/\sqrt{2}$ as the third normal vector. Note that for $r \geq 4$, we cannot do this anymore (because of cross-ratio).

Next theorem synthesizes the \mathcal{A}_h -classification of all the plane curve multigerms of \mathcal{A}_e -codimension ≤ 2 . The proof of the theorem will be an easy consequence of the results of the previous section.

Theorem 3.3. *Let α be a plane curve singularity with \mathcal{A}_e -codimension ≤ 2 . Then α is \mathcal{A}_h -equivalent to one of the multigerms given in the following tables:*

Type	Germ
A_0	$(t, t^k), k \geq 2$
A_2	(t^2, t^3)
A_4	$(t^2, t^4 + t^5)$ $(t^2, t^5 + t^6)$ (t^2, t^5)

Type	Bigerm
A_1	$(t, t^k); (s^m, s), k, m \geq 2$
A_3	$(t, -t^2); (s, s^k), k \geq 2$
A_5	$(t, t^2); (s, s^2 + s^3)$ $(t, -t^3); (s, s^k + s^{k+1}), k \geq 3$ $(t, -t^3); (s, s^k), k \geq 3$
D_5	$(t^2, t^3 + t^4); (s^k, s), k \geq 2$ $(t^2, t^3); (s^k, s), k \geq 2$

Type	Trigerm
D_4	$(t, t^k); (s^m, s); (u - u^n, u + u^n), k, m, n \geq 2$
D_6	$(t, -t^2); (s^k, s); (u, u^m + u^{m+1}), k, m \geq 2$ $(t, -t^2); (s^k, s); (u, u^m), k, m \geq 2$

Type	Quadrigerms
\tilde{E}_7	$(t, t^k + t^{k+1}); (s^l + cs^{l+1}, s); (u - u^m, u + u^m); (v - v^n, \lambda v + v^n), k, l, m, n \geq 2$ $(t, t^k); (s^l + s^{l+1}, s); (u - u^m, u + u^m); (v - v^n, \lambda v + v^n), k, l, m, n \geq 2$ $(t, t^k); (s^l, s); (u - u^m, u + u^m); (v - v^n, \lambda v + v^n), k, l, m, n \geq 2$

We begin with the regular case. Let $\alpha : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ be a regular plane curve germ. Then, there is a well defined curvature function κ_α . We denote by $k \geq 2$ the number defined by $k := \text{ord}(\kappa_\alpha) + 2$, that is,

$$\kappa_\alpha(0) = \dots = \kappa_\alpha^{(k-3)}(0) = 0, \quad \kappa_\alpha^{(k-2)}(0) \neq 0.$$

Proposition 3.4. *Let α be a regular plane curve germ. Then α is \mathcal{A}_h -equivalent to (t, t^k) .*

Proof. Since α is regular, by Lemma 3.2 we can write $\alpha(t) = (t, \alpha_2(t))$, with $\alpha_2'(0) = 0$. We observe now that if $\text{ord}(\kappa_\alpha) = k - 2$, then $\text{ord}(\alpha_2) = k$. On the other hand, $h_\alpha(t) = \alpha_2(t)$ and $h_\beta(t) = t^k$, where $\beta(t) = (t, t^k)$. Thus, by Proposition 2.6, (h_α, α) is equivalent to (h_β, β) . Therefore, α and β are \mathcal{A}_h -equivalent. \square

From this proposition we deduce that the order of κ_α is an \mathcal{A}_h -invariant of α (this is not true for \mathcal{A} -equivalence). In fact, it is a complete invariant: α and β are \mathcal{A}_h -equivalent if and only if $\text{ord}(\kappa_\alpha) = \text{ord}(\kappa_\beta)$. Moreover, if $\kappa_\alpha(0) = \kappa_\beta(0) = 0$, then α and β are \mathcal{A}_h -equivalent if and only if κ_α and κ_β are \mathcal{A} -equivalent.

The next cases in the classification correspond to plane curve germs with either an ordinary or ramphoid cusp.

Proposition 3.5. *Assume α has a cusp singularity A_{2k} , $k = 1, 2$.*

- (1) *If α has an ordinary cusp, then it is \mathcal{A}_h -equivalent to (t^2, t^3) .*
- (2) *If α has a ramphoid cusp, then it is \mathcal{A}_h -equivalent to either*

$$(t^2, t^4 + t^5); \quad (t^2, t^5 + t^6); \quad \text{or} \quad (t^2, t^5).$$

Proof. We can assume that

$$\alpha(t) = (a_2 t^2 + \dots, b_3 t^3 + b_4 t^4 + b_5 t^5 + \dots), \quad h_\alpha(t) = b_3 t^3 + b_4 t^4 + b_5 t^5 + \dots$$

with $a_2 \neq 0$ and $(b_3, b_5) \neq (0, 0)$. If α has an ordinary cusp, then $b_3 \neq 0$ and (h_α, α) is equivalent to $(t^3, (t^2, t^3))$, by Proposition 2.7.

Otherwise, $b_3 = 0$, $b_5 \neq 0$ and α has a ramphoid cusp. Then we use Corollary 2.11, which gives that (h_α, α) is equivalent to either

$$\begin{cases} (t^4, (t^2, t^4 + t^5)), & \text{if } b_4 \neq 0, \\ (t^5 + t^6, (t^2, t^5 + t^6)), & \text{if } b_4 = 0, b_6 \neq 0, \text{ or} \\ (t^5, (t^2, t^5)), & \text{if } b_4 = b_6 = 0. \end{cases}$$

□

Now we proceed to the bigerm singularities. The simplest case is the node singularity or type A_1 , where two regular branches meet transversely. In this case, we also consider the curvature κ_{α_i} of each branch α_i , $i = 1, 2$. We denote by $k, m \geq 2$ the numbers defined by $k := \text{ord}(\kappa_{\alpha_1}) + 2$ and $m := \text{ord}(\kappa_{\alpha_2}) + 2$.

Proposition 3.6. *Let α be a plane curve bigerm of type A_1 . Then it is \mathcal{A}_h -equivalent to*

$$(t, t^k); (s^m, s).$$

Proof. We first use Lemma 3.2. By changing the parameters of the curve and after a linear change of coordinates, we can assume that the curve bigerm is given by

$$(t, a_k t^k + \dots); (b_m s^m + \dots, s),$$

with $a_k, b_m \neq 0$. Therefore, by Proposition 2.13, the associated diagram $(h_{\alpha_1}, \alpha_1); (h_{\alpha_2}, \alpha_2)$ is equivalent to the diagram $(t^k, (t, t^k)); (s^m, (s^m, s))$. □

All the remaining cases follow easily by using a similar argument and the corresponding result which classifies the plane curve diagram. We will state the propositions for these cases, although the details of the proofs are left to the reader.

The A_3 singularity is a bigerm with two regular branches meeting tangentially with a first order contact. This means that the curvatures κ_{α_1} and κ_{α_2} are not equal at 0. We can assume without loss of generality that $\kappa_{\alpha_1}(0) \neq 0$ and we denote $k := \text{ord}(\kappa_{\alpha_2}) + 2 \geq 2$.

Proposition 3.7. *Let α be a plane curve bigerm of type A_3 . Then it is \mathcal{A}_h -equivalent to*

$$(t, -t^2); (s, s^k).$$

In a singularity of type A_5 we have two regular branches meeting tangentially with a second order contact. This means that the curvatures κ_{α_1} and κ_{α_2} are equal at 0, but with distinct first order derivatives. Thus, we have two possibilities: either $\kappa_{\alpha_1}(0) = \kappa_{\alpha_2}(0) \neq 0$ or $\kappa_{\alpha_1}(0) = \kappa_{\alpha_2}(0) = 0$, $\kappa'_{\alpha_1}(0) \neq 0$ and $k := \text{ord}(\kappa_{\alpha_2}) + 2 \geq 3$.

Proposition 3.8. *Let α be a plane curve bigerm of type A_5 . Then it is \mathcal{A}_h -equivalent to either*

$$(t, t^2); (s, s^2 + s^3), \quad (t, -t^3); (s, s^k + s^{k+1}), \quad \text{or} \quad (t, -t^3); (s, s^k).$$

The last case for bigerm singularities corresponds to the D_5 singularity. Now we have a singular branch with an ordinary cusp and a regular branch “transverse” to the cusp (that is, with non collinear normal vectors). If α_2 is the regular branch, we put as usual $k := \text{ord}(\kappa_{\alpha_2}) + 2 \geq 3$.

Proposition 3.9. *Let α be a plane curve bigerm of type D_5 . Then it is \mathcal{A}_h -equivalent to either*

$$(t^2, t^3 + t^4); (s^k, s), \quad \text{or} \quad (t^2, t^3); (s^k, s).$$

We consider now a trigerm singularity of type D_4 . We have three regular branches which are pairwise transverse. Let $k, m, n \geq 2$ be the numbers defined by $k := \text{ord}(\kappa_{\alpha_1}) + 2$, $m := \text{ord}(\kappa_{\alpha_2}) + 2$ and $n := \text{ord}(\kappa_{\alpha_3}) + 2$.

Proposition 3.10. *Let α be a plane curve trigerm of type D_4 . Then it is \mathcal{A}_h -equivalent to*

$$(t, t^k); (s^m, s); (u - u^n, u + u^n).$$

There is one more trigerm singularity, D_6 . In this case the three branches of the curve are regular, but two of them meet tangentially with first order contact and the third one is transverse. We assume that α_1, α_3 are tangent and α_2 is transverse. Moreover, we denote $k := \text{ord}(\kappa_{\alpha_2}) + 2$ and $m := \text{ord}(\kappa_{\alpha_3}) + 2$.

Proposition 3.11. *Let α be a plane curve trigerm of type D_6 . Then it is \mathcal{A}_h -equivalent to either*

$$(t, -t^2); (s^k, s); (u, u^m + u^{m+1}), \quad \text{or} \quad (t, -t^2); (s^k, s); (u, u^m).$$

Finally, the only remaining case corresponds to the \tilde{E}_7 singularity. We have four regular branches which are pairwise transverse. Let $k, l, m, n \geq 2$ be the numbers defined by $k := \text{ord}(\kappa_{\alpha_1}) + 2$, $l := \text{ord}(\kappa_{\alpha_2}) + 2$, $m := \text{ord}(\kappa_{\alpha_3}) + 2$ and $n := \text{ord}(\kappa_{\alpha_4}) + 2$.

Proposition 3.12. *Let α be a plane curve quadrigerms of type \tilde{E}_7 . Then it is \mathcal{A}_h -equivalent to either*

$$(t, t^k + t^{k+1}); (s^l + cs^{l+1}, s); (u - u^m, u + u^m); (v - v^n, \lambda v + v^n),$$

$$(t, t^k); (s^l + s^{l+1}, s); (u - u^m, u + u^m); (v - v^n, \lambda v + v^n)$$

or

$$(t, t^k); (s^l, s); (u - u^m, u + u^m); (v - v^n, \lambda v + v^n).$$

4. SINGULARITIES OF PROJECTIONS OF SPACE CURVES

Let $I \subset \mathbb{R}$ be an interval. The set of C^∞ space curves $\gamma : I \rightarrow \mathbb{R}^3$ is denoted by $C^\infty(I, \mathbb{R}^3)$. We consider in this set the Whitney C^∞ topology.

Given $\mathbf{v} \in S^2$, we denote by $p_{\mathbf{v}} : \mathbb{R}^3 \rightarrow \langle \mathbf{v} \rangle^\perp$ the orthogonal projection in the direction determined by \mathbf{v} :

$$p_{\mathbf{v}}(x) = x - \langle x, \mathbf{v} \rangle \mathbf{v}.$$

If $\gamma : I \rightarrow \mathbb{R}^3$ is a space curve, we can consider the plane curve $p_{\mathbf{v}} \circ \gamma$ obtained by taking the orthogonal projection into the plane $\langle \mathbf{v} \rangle^\perp$.

Theorem 4.1. [2] *There is a residual subset $\Omega \subset C^\infty(I, \mathbb{R}^3)$ such that if $\gamma \in \Omega$, then it is an embedding and for any $\mathbf{v} \in S^2$, $p_{\mathbf{v}} \circ \gamma$ has only singularities of type $A_0, \dots, A_5, D_4, D_5, D_6$ and \tilde{E}_7 .*

The curves in the subset $\Omega \subset C^\infty(I, \mathbb{R}^3)$ are called *projection-generic* by David in [2] (see also [6]).

Given a projection-generic curve γ , it follows from genericity that it is an embedding and moreover, for any $t \in I$, $\gamma'(t), \gamma''(t)$ are not collinear. In particular, it has a well defined Frenet frame. We denote by $\kappa(t), \tau(t)$ the curvature and torsion respectively of

γ at the point $\gamma(t)$. Moreover, we will denote by $T(t), O(t)$ the tangent line and the osculating plane respectively of γ at $\gamma(t)$.

Given a line ℓ in \mathbb{R}^3 which meets the curve γ in 2, 3 or 4 points, we will say that it is a secant, trisecant or quadrisecant line, respectively. A cross tangent is a secant line which is tangent to the curve at one of the points. Given a plane π in \mathbb{R}^3 which is tangent to the curve in 2 or 3 points, we will call it a bitangent or tritangent plane, respectively.

Definition 4.2. Let $\gamma : I \rightarrow \mathbb{R}^3$ be a C^∞ space curve. We will say that γ is *generic* if it is projection-generic in the above sense and moreover the following conditions hold:

- (1) Assume that the secant line ℓ to γ at two points $\gamma(t_i), i = 1, 2$ is contained in the osculating planes $\ell \subset O(t_i)$, for any $i = 1, 2$. Then, $\tau(t_i) \neq 0$, for any $i = 1, 2$.
- (2) Let ℓ be a cross tangent to γ at $\gamma(t_i)$, for any $i = 1, 2$, tangent in $\gamma(t_1)$. Then $\gamma^{(4)}(t_1) \notin O(t_1)$.
- (3) Let ℓ be a trisecant line to γ at three points $\gamma(t_i), i = 1, 2, 3$. If $\ell \subset O(t_1)$, then $\tau(t_1) \neq 0$ and $\ell \not\subset O(t_i)$, for any $i = 2, 3$.
- (4) Let ℓ be a trisecant line to γ at three points $\gamma(t_i), i = 1, 2, 3$. If ℓ is contained in a bitangent plane π to γ at two of these points, then π does not osculate at none of the three points and $\gamma^{(3)}(t_i)$ is not contained in the bitangent plane.
- (5) Let ℓ be a quadrisecant line to γ at four points $\gamma(t_i), i = 1, 2, 3, 4$. Then, $\ell \not\subset O(t_i)$ and $\ell \parallel \gamma^3(t_i)$ at most in two points.

By using standard transversality techniques, it is possible to show the following result.

Lemma 4.3. *The subset of generic space curves is also residual in $C^\infty(I, \mathbb{R}^3)$.*

The classification of all the singularities of projections of a generic space curve is summarized in the following theorem.

Theorem 4.4. *Let γ be a generic space curve. For any $\mathbf{v} \in S^2$, the only \mathcal{A}_h -singularities of the orthogonal projection $p_{\mathbf{v}} \circ \gamma$ are given in the following tables (see also Figure 3):*

\mathcal{A} -class	\mathcal{A}_h -class	Germ
A_0	A_{0_2}	(t, t^2)
	A_{0_3}	(t, t^3)
	A_{0_4}	(t, t^4)
	A_{2_3}	(t^2, t^3)
A_4	A_{4_4}	$(t^2, t^4 + t^5)$

\mathcal{A} -class	\mathcal{A}_h -class	Bigerm
A_1	$A_{1_{22}}$	$(t, t^2); (s^2, s)$
	$A_{1_{23}}$	$(t, t^2); (s^3, s)$
	$A_{1_{24}}$	$(t, t^2); (s^4, s)$
	$A_{1_{33}}$	$(t, t^3); (s^3, s)$
A_3	$A_{3_{22}}$	$(t, -t^2); (s, s^2)$
	$A_{3_{23}}$	$(t, -t^2); (s, s^3)$
A_5	$A_{5_{23}}$	$(t, t^2); (s, s^2 + s^3)$
D_5	$D_{5_{23}}$	$(t^2, t^3 + t^4); (s^2, s)$

\mathcal{A} -class	\mathcal{A}_h -class	Trigerm
D_4	$D_{4_{222}}$	$(t, t^2); (s^2, s); (u - u^2, u + u^2)$
	$D_{4_{232}}$	$(t, t^2); (s^3, s); (u - u^2, u + u^2)$
D_6	$D_{6_{222}}$	$(t, -t^2); (s^2, s); (u, u^2 + u^3)$

\mathcal{A} -class	\mathcal{A}_h -class	Quadriger
\tilde{E}_7	$\tilde{E}_{7_{2222}}$	$(t, t^2 + t^3); (s^2 + cs^3, s); (u - u^2, u + u^2); (v - v^2, \lambda v + v^2)$

Let γ be a generic curve in \mathbb{R}^3 and let $\mathbf{v} \in S^2$. All the \mathcal{A}_h -singularities of the orthogonal projection $p_{\mathbf{v}} \circ \gamma$ given in the above tables have a geometrical interpretation which is described in the following propositions.

Proposition 4.5. *Let γ be a generic space curve and let $\mathbf{v} \in S^2$ be such that \mathbf{v} does not lie in any secant line of γ passing through $\gamma(t_0)$. Then the germ of $p_{\mathbf{v}} \circ \gamma$ at t_0 has a singularity of type:*

$A_0 \Leftrightarrow \mathbf{v} \notin T(t_0)$. Moreover, it is \mathcal{A}_h -equivalent to:

$$\begin{cases} A_{0_2}, & \text{if } \mathbf{v} \notin O(t_0). \\ A_{0_3}, & \text{if } \mathbf{v} \in O(t_0) \text{ and } \tau(t_0) \neq 0. \\ A_{0_4}, & \text{if } \mathbf{v} \in O(t_0) \text{ and } \tau(t_0) = 0. \end{cases}$$

$A_2 \Leftrightarrow \mathbf{v} \in T(t_0)$ and $\tau(t_0) \neq 0$.

$A_{4_4} \Leftrightarrow \mathbf{v} \in T(t_0)$ and $\tau(t_0) = 0$.

Proof. Let us denote $\alpha = p_{\mathbf{v}} \circ \gamma$. It is obvious that α is regular if and only if $\mathbf{v} \notin T(t_0)$. Moreover, if it is regular, then it has a singularity of type A_{0_k} at t_0 , $k = 2, 3, 4$ if and only if $\alpha'(t_0), \dots, \alpha^{(k-1)}(t_0)$ are collinear, but $\alpha'(t_0), \dots, \alpha^{(k)}(t_0)$ are not. In terms of \mathbf{v} and γ , it is equivalent to the fact that $\mathbf{v}, \gamma'(t_0), \dots, \gamma^{(k-1)}(t_0)$ are collinear, but $\mathbf{v}, \gamma'(t_0), \dots, \gamma^{(k)}(t_0)$ are not. Now we have the following obvious equivalences:

- (1) $k = 2$ if and only if $\mathbf{v} \notin O(t_0)$;
- (2) $k = 3$ if and only if $\mathbf{v} \in O(t_0)$ and $\tau(t_0) \neq 0$;
- (3) $k = 4$ if and only if $\mathbf{v} \in O(t_0)$ and $\tau(t_0) = 0$.

In the second part, we see that α has a singularity A_2 at t_0 if and only if $\alpha'(t_0) = 0$ but the vectors $\alpha''(t_0), \alpha'''(t_0)$ are linearly independent. This means that $\mathbf{v}, \gamma'(t_0)$ are parallel, but $\gamma'(t_0), \gamma''(t_0), \gamma'''(t_0)$ are not collinear, that is, $\tau(t_0) \neq 0$.

Finally, α has a singularity A_4 at t_0 if and only if $\alpha'(t_0) = 0$, the vectors $\alpha''(t_0), \alpha'''(t_0)$ are collinear, but $\alpha''(t_0), \alpha^{(4)}(t_0)$ are not. This is equivalent to that $\mathbf{v}, \gamma'(t_0)$ are parallel, $\tau(t_0) = 0$ and $\tau'(t_0) \neq 0$. \square

Proposition 4.6. *Let γ be a generic space curve and let $\mathbf{v} \in S^2$ be such that \mathbf{v} lies in the secant line to γ passing through $\gamma(t_0)$ and $\gamma(s_0)$, which is not a trisecant. Then the bigerm of $p_{\mathbf{v}} \circ \gamma$ at (t_0, s_0) has a singularity of type:*

$A_1 \Leftrightarrow \mathbf{v}$ does not lie in a bitangent plane to γ at (t_0, s_0) . Moreover, it is \mathcal{A}_h -equivalent to:

$$\begin{cases} A_{1_{22}}, & \text{if } \mathbf{v} \notin O(t_0) \text{ and } \mathbf{v} \notin O(s_0). \\ A_{1_{23}}, & \text{if } \mathbf{v} \notin O(t_0), \text{ but } \mathbf{v} \in O(s_0) \text{ and } \tau(s_0) \neq 0. \\ A_{1_{24}}, & \text{if } \mathbf{v} \notin O(t_0), \text{ but } \mathbf{v} \in O(s_0) \text{ and } \tau(s_0) = 0. \\ A_{1_{33}}, & \text{if } \mathbf{v} \in O(t_0) \text{ and } \mathbf{v} \in O(s_0). \end{cases}$$

$A_3 \Leftrightarrow \mathbf{v}$ lies in a bitangent plane to γ at (t_0, s_0) and the secant line is not a cross tangent. Moreover, it is \mathcal{A}_h -equivalent to:

$$\begin{cases} A_{3_{22}}, & \text{if } \mathbf{v} \notin O(t_0) \text{ and } \mathbf{v} \notin O(s_0). \\ A_{3_{23}}, & \text{if } \mathbf{v} \notin O(t_0), \text{ but } \mathbf{v} \in O(s_0). \end{cases}$$

$A_5 \Leftrightarrow \mathbf{v}$ lies in a bitangent plane to γ at (t_0, s_0) and the secant line is not a cross tangent. Moreover, the bitangent plane contains the vector

$$\|\gamma'(s_0)\|^2 \sin(\theta_2)^2 \gamma''(t_0) + \|\gamma'(t_0)\|^2 \sin(\theta_1)^2 \gamma''(s_0),$$

where θ_1, θ_2 are the angles $\theta_1 = \angle(\gamma(t_0) - \gamma(s_0), \gamma'(t_0))$ and $\theta_2 = \angle(\gamma(t_0) - \gamma(s_0), \gamma'(s_0))$.

$D_5 \Leftrightarrow$ The secant line is a cross tangent.

Proof. We will show just the A_5 singularity which is perhaps the least obvious. The proof of the remaining cases is analogous to the proof of the above proposition.

By Proposition 3.8, we have that the projection $\alpha = p_{\mathbf{v}} \circ \gamma$ has a singularity A_5 if and only if the two branches of the curve have the same tangent and the same curvature (and distinct first order derivatives of the curvature). It is easy to see that this is equivalent to

$$\begin{aligned} \alpha(t_0) &= \alpha(s_0), \\ \alpha'(t_0) &= \lambda \alpha'(s_0), \\ \det(\alpha'(t_0), \alpha''(t_0)) &= \lambda^3 \det(\alpha'(s_0), \alpha''(s_0)), \quad \lambda \in \mathbb{R}. \end{aligned}$$

The two first equalities are equivalent to that \mathbf{v} is parallel to the secant line and that it belongs to the bitangent plane, respectively. Thus, the vectors $\gamma(t_0) - \gamma(s_0), \gamma'(t_0), \gamma'(s_0)$ are collinear and there exist A, B and C not all zero, such that

$$C(\gamma(t_0) - \gamma(s_0)) + A\gamma'(t_0) + B\gamma'(s_0) = 0.$$

Since \mathbf{v} is parallel to $\gamma(t_0) - \gamma(s_0)$, it follows that $A\alpha'(t_0) + B\alpha'(s_0) = 0$ and hence, $\lambda = -B/A$. Note that $A, B \neq 0$, since otherwise we would have a cross tangent. We consider now the triangle given in Figure 4, from which we obtain

$$\frac{\sin(\theta_1)}{|B\gamma'(s_0)|} = \frac{\sin(\theta_2)}{|A\gamma'(t_0)|}.$$

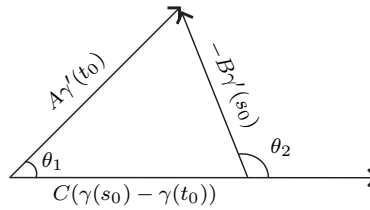


FIGURE 4

On the other hand, the condition $\det(\alpha'(s_0), \alpha''(t_0) - \lambda^2 \alpha''(s_0)) = 0$ is equivalent to the fact that the vector $\gamma''(t_0) - \lambda^2 \gamma''(s_0)$ belongs to the bitangent plane at (s_0, t_0) . After substituting the values of λ and A in the vector $\gamma''(t_0) - \lambda^2 \gamma''(s_0)$, we obtain that it is equivalent to that the vector

$$|\gamma'(s_0)|^2 \sin(\theta_2)^2 \gamma''(t_0) + |\gamma'(t_0)|^2 \sin(\theta_1)^2 \gamma''(s_0)$$

belongs to the bitangent plane at (s_0, t_0) . □

The remaining cases of the theorem are given in the two following propositions, which we state without proof.

Proposition 4.7. *Let γ be a generic space curve and let $\mathbf{v} \in S^2$ be such that \mathbf{v} lies in a line that is a trisecant to γ passing through $\gamma(t_0), \gamma(s_0)$ and $\gamma(u_0)$, which is not a quadriseccant. Then the trigerm of $p_{\mathbf{v}} \circ \gamma$ at (t_0, s_0, u_0) has a singularity of type:*

$D_4 \Leftrightarrow \mathbf{v}$ does not lie in a bitangent plane to γ at two of the three points. Moreover, it is \mathcal{A}_h -equivalent to:

$$\begin{cases} D_{4_{222}}, & \text{if } \mathbf{v} \notin O(t_0), \mathbf{v} \notin O(s_0) \text{ and } \mathbf{v} \notin O(u_0). \\ D_{4_{232}}, & \text{if } \mathbf{v} \notin O(t_0), \mathbf{v} \in O(s_0) \text{ and } \mathbf{v} \notin O(u_0). \end{cases}$$

$D_6 \Leftrightarrow \mathbf{v}$ lies in a bitangent plane to γ at two of the three points.

Proposition 4.8. *Let γ be a generic space curve and let $\mathbf{v} \in S^2$ be such that \mathbf{v} lies in a quadrisecant line to γ passing through $\gamma(t_0)$, $\gamma(s_0)$, $\gamma(u_0)$ and $\gamma(v_0)$. Then the quadrigerms of $p_{\mathbf{v}} \circ \gamma$ at (t_0, s_0, u_0, v_0) has a singularity of type \tilde{E}_7 .*

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