The Bohr radius of a Banach space

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Abstract. Let $1 \leq p, q < \infty$ and let X be a complex Banach space. For each $f(z) = \sum_{n=0}^{\infty} x_n z^n$ with $\|f\|_{H^\infty(\mathbb{D},X)} \leq 1$ we define $R_{p,q}(f,X) = \sup\{r \geq 0 : \|x_0\|^p + (\sum_{n=1}^{\infty} \|x_n\| r^n)^q \leq 1\}$ and denote the Bohr radius of X by $R_{p,q}(X) = \inf\{R_{p,q}(f,X) : \|f\|_{H^\infty(\mathbb{D},X)} \leq 1\}$. The aim of this note is to study for which spaces $X = L^s(\mu)$ or $X = \ell^s$ one has $R_{p,q}(X) > 0$.

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1. Introduction and preliminaries

In 1914 H. Bohr [3] showed that

$$\sum_{n=0}^{\infty} |a_n| (\frac{1}{3})^n \leq \|f\|_\infty, \tag{1.1}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ a bounded analytic function on the open unit disc. The value 1/3 is sharp.

A bit later some other proofs of such inequality were given (see [9, 10]). Also several authors have found some extensions (see [4, 5, 8, 11]).

Another basic inequality was discovered in [9, Corollary 2.7] much later and will play an special role for us, namely

$$|a_0|^2 + \sum_{n=1}^{\infty} |a_n| (\frac{1}{2})^n \leq 1, \tag{1.2}$$

whenever $\|f\|_{H^\infty} \leq 1$ and the value 1/2 is sharp in this case.

Later on some multidimensional analogues of Bohr's inequality in which the disc $\mathbb{D}$ is replaced by a domain $\Omega \subset \mathbb{C}^m$ were considered (see [1, 2]) and several applications of this multidimensional Bohr radius and connections concerning local Banach space theory have been recently achieved (see [7, 6]).

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Our point of view will be to keep $\mathbb{D}$ as domain for the functions but allow them to take values in a complex (possibly infinite dimensional) Banach space. Throughout this paper X stands for a complex Banach space and $H^p(\mathbb{D}, X)$, usual, for $1 \leq p \leq \infty$, denotes the Hardy spaces of X-valued holomorphic functions from the unit disc.

Definition 1.1. Given $f(z) = \sum_{n=0}^{\infty} x_n z^n$ with $\|f\|_{H^\infty(\mathbb{D},X)} \leq 1$ we denote

$$R(f, X) = \sup\{r \geq 0 : \sum_{n=0}^{\infty} \|x_n\| r^n \leq 1\}. \tag{1.3}$$

Let us now define the Bohr's radius of X by

$$R(X) = \inf\{R(f, X) : \|f\|_{H^\infty(\mathbb{D},X)} \leq 1\}. \tag{1.4}$$

That is to say

$$R(X) = \sup\{r \geq 0 : \sum_{n=0}^{\infty} \|x_n\| r^n \leq \|f\|_{H^\infty(\mathbb{D},X)}\}.$$

Since $\mathbb{C}$ is embedded into any complex Banach space we have, due to (1.1) that $R(X) \leq \frac{1}{3}$ for any Banach space. However the notion is not very useful even in the finite dimensional case for dimension greater than one. Let us denote $\mathbb{C}_p^m$ the space $\mathbb{C}^m$ endowed with the norm the $\|w\|_p = (\sum_{i=1}^m |w_i|^p)^{1/p}$, or $\|w\|_\infty = \sup_{i=1}^m |w_i|$.

Theorem 1.2. *Let $m \geq 2$ and $1 \leq p \leq \infty$. Then $R(\mathbb{C}_p^m) = 0$.*

Proof. It suffices to do the case $m = 2$. In the case $p = \infty$ one can easily find f with $\|f\|_\infty = 1$ and $R(f, \mathbb{C}_\infty^m) = 0$, (take for instance $f(z) = e_1 + e_2 z$). This shows that $R(\mathbb{C}_\infty^2) = 0$.

Assume $1 < p < \infty$. Let us now use that $\lim_{y\to\infty} y^{1/p} - (y-1)^{1/p} = 0$ to get, for each $\varepsilon > 0$, a value $0 < \gamma < 1$ such that

$$1 - (1-\gamma)^{1/p} < \varepsilon \gamma^{1/p}. \tag{1.5}$$

Now define

$$f(z) = ((1-\gamma)^{1/p}, \gamma^{1/p} z) = (1-\gamma)^{1/p} e_1 + \gamma^{1/p} e_2 z.$$

Clearly $\sup_{|z|<1} \|f(z)\|_p = 1$. On the other hand using (1.5) one has

$$\|x_0\|_p + \varepsilon \|x_1\|_p = (1-\gamma)^{1/p} + \varepsilon \gamma^{1/p} > 1.$$

This shows that $R(f, \mathbb{C}_p^m) \leq \varepsilon$. Hence $R(\mathbb{C}_p^m) = 0$.

Assume now $p = 1$. As above for each $\varepsilon > 0$ we can find $0 < \gamma < 1$ satisfying

$$1 - \sqrt{1-\gamma} < \varepsilon \sqrt{\gamma}. \tag{1.6}$$

and define

$$f(z) = \frac{\sqrt{1-\gamma}}{2}(1,1) + \frac{\sqrt{\gamma}}{2}(1,-1)z = \frac{1}{2}(\sqrt{1-\gamma} + \sqrt{\gamma} z, \sqrt{1-\gamma} - \sqrt{\gamma} z).$$

Observe that

$$\begin{aligned} \|f(z)\|_1 &= \frac{1}{2}\Big(|\sqrt{1-\gamma}+\sqrt{\gamma}z| + |\sqrt{1-\gamma}-\sqrt{\gamma}z|\Big) \\ &\le \frac{1}{\sqrt{2}}\Big(|\sqrt{1-\gamma}+\sqrt{\gamma}z|^2 + |\sqrt{1-\gamma}-\sqrt{\gamma}z|^2\Big)^{1/2} = 1. \end{aligned}$$

On the other hand, from(1.6),

$$\|x_0\|_1 + \varepsilon\|x_1\|_1 = \sqrt{1-\gamma} + \varepsilon\sqrt{\gamma} > 1.$$

This shows that $R(f, \mathbb{C}_1^m) \le \varepsilon$. Hence $R(\mathbb{C}_1^m) = 0$. □

However, following the observation in [9] and extending inequality (1.2), we are going to define another a modified Bohr radius which needs not be zero even for infinite dimensional Banach spaces.

Definition 1.3. Let $1 \le p, q < \infty$ and let X be a complex Banach space. Given $f(z) = \sum_{n=0}^\infty x_n z^n$ with $\|f\|_{H^\infty(\mathbb{D},X)} \le 1$ we denote

$$R_{p,q}(f, X) = \sup\{r \ge 0 : \|x_0\|^p + (\sum_{n=1}^\infty \|x_n\| r^n)^q \le 1\}. \tag{1.7}$$

We now define

$$R_{p,q}(X) = \inf\{R_{p,q}(f, X) : \|f\|_{H^\infty(\mathbb{D},X)} \le 1\}. \tag{1.8}$$

Of course $R_{1,1}(X) = R(X)$ and we have the following chain of inclusions:

$$R_{p_1,q_1}(X) \le R_{p_2,q_2}(X), \quad p_1 \le p_2, \quad q_1 \le q_2. \tag{1.9}$$

To compute the precise value of $R_{p,q}(\mathbb{C}_2^m)$ is difficult in general, even for $m = 1$. In [9, Cor. 2.7] it was shown that $R_{2,1}(\mathbb{C}) = \frac{1}{2}$. Let us adapt the same argument to cover the cases $1 \le p \le 2$.

Proposition 1.4. *If $1 \le p \le 2$ then $R_{p,1}(\mathbb{C}) = \frac{p}{2+p}$.*

Proof. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ belong to the unit ball of $H^\infty(\mathbb{D}, \mathbb{C})$. We recall the estimate, first observed by Wiener [3],

$$|a_n| \le 1 - |a_0|^2 \tag{1.10}$$

(see also [9] for a proof). From (1.10) one concludes that

$$|a_0|^p + \sum_{n=1}^\infty |a_n| r^n \le |a_0|^p + (1 - |a_0|^2)\frac{r}{1-r} \tag{1.11}$$

Since $1 \le p \le 2$ we estimate (1.11) by $|a_0|^p + \frac{2}{p}(1 - |a_0|^p)\frac{r}{1-r}$. Now $|a_0|^p + \frac{2}{p}(1 - |a_0|^p)\frac{r}{1-r} \le 1$ if and only if $r \le \frac{p}{2+p}$. This gives that $R_{p,1}(f, \mathbb{C}) \ge \frac{p}{2+p}$. Hence $R_{p,1}(\mathbb{C}) \ge \frac{p}{2+p}$

For the converse we use Moebius transformations $\phi_a(z) = \frac{z-a}{1-az}$ for $0 < a < 1$. Since $\phi_a(z) = -a + \frac{1-a^2}{a}\sum_{n=1}^\infty a^n z^n$ one obtains

$$a^p + \frac{1-a^2}{a}\sum_{n=1}^\infty a^n r^n = a^p + \frac{1-a^2}{a}\frac{ra}{1-ra}.$$

This shows that

$$R_{p,1}(\phi_a, \mathbb{C}) = \frac{\frac{1-a^p}{1-a}}{(1+a) + a(\frac{1-a^p}{1-a})}.$$

Taking limits as $a \to 1$ one gets $R_{p,1}(\mathbb{C}) \leq \frac{p}{2+p}$. □

2. The Bohr radius $R_{p,q}(X)$ for L^p-spaces.

Using the same example as in Theorem 1.2 one gets the following:

Proposition 2.1. *$R_{p,q}(\mathbb{C}^m_\infty) = 0$ for any $m \geq 2$ and $1 \leq p, q < \infty$.*

Theorem 2.2. *Let $m \geq 2$. Then $R_{p,p}(\mathbb{C}^m_2) > 0$ if and only if $p \geq 2$.*

Proof. Assume $p \geq 2$. From (1.9) it suffices to see that $R_{1,2}(\mathbb{C}^m_2) > 0$. Now given f in the unit ball of $H^\infty(\mathbb{D}, \mathbb{C}^m_2)$ one has, in particular, that $\|f\|^2_{H^2(\mathbb{D},\mathbb{C}^m_2)} = \sum_{n=0}^\infty \|x_n\|_2^2 \leq 1$.

Therefore

$$\begin{aligned} \|x_0\| + (\sum_{n=1}^\infty \|x_n\| r^n)^2 &\leq \|x_0\| + (\sum_{n=1}^\infty \|x_n\|^2)(\sum_{n=1}^\infty r^{2n}) \\ &\leq \|x_0\| + 2(1 - \|x_0\|)\frac{r^2}{1-r^2} \\ &\leq \max\{1, 2\frac{r^2}{1-r^2}\}. \end{aligned}$$

Now, since $2\frac{r^2}{1-r^2} = 1$ for $r = \frac{1}{\sqrt{3}}$, one obtains that for $R_{1,2}(\mathbb{C}^m_2) \geq \frac{1}{\sqrt{3}}$.

Conversely, assume now that $1 \leq p < 2$. Arguing as in Therorem 1.2, one has that for each $\varepsilon > 0$ we can find $0 < \gamma < 1$ such that

$$(1-\gamma)^{p/2} + \varepsilon^p \gamma^{p/2} > 1. \tag{2.1}$$

Now selecting $f(z) = \sqrt{1-\gamma}e_1 + \sqrt{\gamma}e_2 z$ and using (2.1) we get $R_{p,p}(f, \mathbb{C}^m_2) \leq \varepsilon$. This implies that $R_{p,p}(\mathbb{C}^m_2) = 0$. □

Let us now study the situation for L^p-spaces in the infinite dimensional case.

Theorem 2.3. *Let $1 \leq p, q, s < \infty$ and let (Ω, Σ, μ) be a measure space such that there exists a sequence of pairwise disjoint sets with $0 < \mu(A_n) < \infty$. Then $R_{p,q}(L^s(\mu)) = 0$ whenever $1 \leq q < s$.*

Proof. Let $0 < \beta < 1$ and $a = \frac{1-\beta}{2-\beta}$. Set $a_0 = \beta^{1/s}\mu(A_0)^{-1/s}$ and $a_n = a^{n/s}\mu(A_n)^{-1/s}$ for $n \geq 1$. Now define $\phi_n = a_n\chi_{A_n}$ and

$$F_\beta(z) = \sum_{n=0}^{\infty} \phi_n z^n.$$

Clearly F_β belongs to the unit ball of $H^\infty(\mathbb{D}, H)$, because

$$\|F_\beta(z)\|^s_{L^s(\mu)} = \sum_{n=0}^{\infty} \int_{A_n} |z|^{ns}|a_n|^s d\mu \leq \sum_{n=0}^{\infty} |a_n|^s \mu(A_n) = 1.$$

On the other hand

$$\begin{aligned} \|\phi_0\|^p_{L^s(\mu)} + (\sum_{n=1}^{\infty} \|\phi_n\|_{L^s(\mu)} r^n)^q &= a_0^p \mu(A_0)^{p/s} + (\sum_{n=1}^{\infty} a_n \mu(A_n)^{1/s} r^n)^q \\ &= \beta^{p/s} + (\sum_{n=1}^{\infty} (a^{1/s} r)^n)^q \\ &= \beta^{p/s} + (\frac{a^{1/s} r}{1 - a^{1/s} r})^q. \end{aligned}$$

Now $\beta^{p/s} + (\frac{a^{1/s}r}{1-a^{1/s}r})^q \leq 1$ if and only if

$$r \leq \frac{a^{-1/s}(1-\beta^{p/s})^{1/q}}{1 + (1-\beta^{p/s})^{1/q}} = \frac{(1-\beta^{p/s})^{1/q}}{(1-\beta)^{1/s}} \frac{(2-\beta)^{1/s}}{1 + (1-\beta^{p/s})^{1/q}}.$$

Since $1/q > 1/s$, taking limits as β goes to 1 one gets that $R_{p,q}(L^s(\mu)) = 0$.
□

Corollary 2.4. *Let $1 \leq p, s < \infty$ and $1 \leq q < s$. Then $R_{p,q}(\ell^s) = R_{p,q}(L^s(\mathbb{R})) = 0$.*

A look to the proof in Theorem 2.2 shows that actually for $X = L^2(\mu)$ one gets $R_{1,2}(L^2(\mu)) \geq \frac{1}{\sqrt{3}}$. Let us give some lower estimates of $R_{p,q}(L^s(\mu))$ for some values of $q \geq s$. As usual p' stands for the conjugate exponent satisfying $1/p + 1/p' = 1$.

Theorem 2.5. *Let $1 < s < \infty$, $q = \max\{s, s'\}$ and $1 \leq p \leq q$. Then*

$$R_{p,q}(L^s(\mu)) \geq \frac{p^{1/q}}{(q^{q'/q} + p^{q'/q})^{1/q'}}. \tag{2.2}$$

Proof. Let $X = L^s(\mu)$ and let $f(z) = \sum_{n=0}^{\infty} x_n z^n$ belong to the unit ball of $H^\infty(\mathbb{D}, X)$. It follows easily from complex interpolation (considering X_1 to be $L^1(\mu)$ or $L^\infty(\mu)$ and X_2 to be $L^2(\mu)$) that $(\sum_{n=0}^{\infty} \|x_n\|^q)^{1/q} \leq \|f\|_{H^{q'}(\mathbb{D}, L^s(\mu))}$. In particular we have $\sum_{n=1}^{\infty} \|x_n\|^q \leq 1 - \|x_0\|^q, n \geq 1$.

Therefore

$$\|x_0\|^p + (\sum_{n=1}^{\infty} \|x_n\| r^n)^q \leq \|x_0\|^p + (1 - \|x_0\|^q)(\frac{r^{q'}}{1 - r^{q'}})^{q/q'}.$$

Hence, for $p \leq q$, we estimate

$$\begin{aligned} \|x_0\|^p + (\sum_{n=1}^{\infty} \|x_n\| r^n)^q &\leq \|x_0\|^p + \frac{q}{p}(1 - \|x_0\|^p)\frac{r^q}{(1 - r^{q'})^{q/q'}} \\ &\leq \max\{1, \frac{q}{p}\frac{r^q}{(1 - r^{q'})^{q/q'}}\}. \end{aligned}$$

Note that $\frac{q}{p}\frac{r^q}{(1-r^{q'})^{q/q'}} = 1$ gives the value $r = \frac{(\frac{p}{q})^{1/q}}{(1+(\frac{p}{q})^{q'/q})^{1/q'}}$.

□

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