A note on fractional integral operators defined by weights and non-doubling measures

Oscar Blasco, Vicente Casanova, Joaquín Motos*

Abstract

Given a metric measure space (X, d, μ), a weight w defined on $(0, \infty)$ and a kernel $k_w(x, y)$ satisfying the standard fractional integral type estimates, we study the boundedness of the operators $K_w f(x) = \int_X k_w(x, y) f(y) d\mu(y)$ and $\tilde{K}_w f(x) = \int_X \big(k_w(x, y) - k_w(x_0, y)\big) f(y) d\mu(y)$ on Lebesgue spaces $L^p(\mu)$ and generalized Lipschitz spaces Lip_ϕ, respectively, for certain range of the parameters depending on the n-dimension of μ and some indices associated to the weight w.

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1 Introduction.

It is well known that a basic assumption in the classical Calderón-Zygmung theory in $\mathbb{R}^n$ is the doubling property of the underlying measure space, i.e. $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in \mathbb{R}^n$ and $r > 0$. However, it has been recently shown that many results of the theory still hold for general metric spaces X assuming only that $\mu(B(x, r)) \leq Cr^n$ for all $x \in X$ and $r > 0$. The reader is referred to [6, 7, 15] for results on vector-valued inequalities and weights and to [8, 13, 21, 22] for results on classical spaces such as H^1 and BMO in the setting of non-doubling measures.

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The aim of this note is to analyze the boundedness of the fractional integral-type operators defined on non-doubling measure spaces acting on Lebesgue spaces and generalized Lipschitz spaces. This study was initiated in the work of J. García-Cuerva and A.E. Gatto (see [4, 5]) for the classical fractional integral operators and Lipschitz spaces. In this paper we are able to extend some of their results, including weights more general than the potential ones, and to see that a similar theory can be applied to operators defined with kernels more general than the fractional integral ones.

The action of the fractional integral operator on Hölder spaces goes back to the work of Hardy-Littlewood in [9]. Since then, many different extensions have been considered. Similar results for power weights were proved in [16, 17] and later, extended to other classes of weights, including power-logarithmic type ones, in [14]. On a different direction some development of the theory in the setting of generalized Lipschitz spaces and spaces of homogeneous type was initiated in [11, 12].

Throughout the paper (X, d, μ) will be a metric measure space, that is a metric space (X, d) equipped with a Borel measure μ such that

$$\mu(B(x,r)) \leq Cr^n \tag{1}$$

for every ball $B(x,r) = \{y \in X : d(x,y) < r\}$, where $n > 0$ is some fixed constant and C is independent of of x and r. We shall deal, for simplicity, only with the case $diam(X) = \infty$.

For us a weight w on an interval $I \subset (0,\infty)$ will always be a continuous function $w : I \to (0,\infty)$. We shall use weights defined on $(0,\infty)$ but we shall relate them with the known theory for weights defined on $(0,1]$. Given $w : (0,\infty) \to (0,\infty)$ we denote by $w_0(t) = w(t)$ and $w_\infty(t) = w(1/t)$ for $0 < t \leq 1$.

We consider the indices $m(w)$, $M(w)$, $m_\infty(w)$ and $M_\infty(w)$ introduced by N.G. Samko in the case of weights defined on the finite interval $(0,1]$ (see [18]) or by N.G. et al. in the case $[1,\infty)$ (see [19]) and we shall also work in the class of weights $\tilde{W}$ such that there exists $a, b \in \mathbb{R}$ such that $t^a w(t)$ is almost increasing in $(0,1]$, $t^b w(t)$ is almost decreasing in $[1,\infty)$ and $-\infty < M(w), m_\infty(w) < +\infty$.

In the paper we shall consider $\mathcal{B}(X) \times \mathcal{B}(X)$-measurable functions $k_w : X \times X \to \mathbb{C}$ that satisfy the following conditions:

$$|k_w(x,y)| \leq C\frac{w(d(x,y))}{d(x,y)^n}, \qquad x, y \in X, x \neq y \tag{2}$$

and there exists $\varepsilon > 0$ such that

$$|k_w(x,z) - k_w(y,z)| \leq C\Big(\frac{d(x,y)}{d(x,z)}\Big)^{\varepsilon}\frac{w(d(x,y))}{d(x,z)^n}, \qquad d(x,z) \geq 2d(x,y) > 0. \tag{3}$$

For such kernels we define the operators

$$K_w f(x) = \int_X k_w(x,y)f(y)d\mu(y)$$

and

$$\tilde{K}_w f(x) = \int_X \big(k_w(x,y) - k(x_0,y)\big)f(y)d\mu(y)$$

and study their boundedness on Lebesgue spaces and generalized Lipschitz spaces.

Our considerations are inspired by those developed in the case $w(t) = t^{\alpha}$ corresponding to the classical fractional integrals. However we will explore the connections between the weight w and the measure μ that still allow the operators K_w and $\tilde{K}_w$ to be well defined for functions in $L^p(\mu)$ and will find the dependence between their boundedness on some spaces and the indices the weight w. We shall find a Hardy-Littlewood-Sobolev type inequality for K_w in our setting in Theorem 3.3. We will study the boundedness of $\tilde{K}_w$ from $L^p(\mu)$ into Lip_{ϕ} for $\phi(t) = t^{-n/p}w(t)$ in Theorem 4.3 and from Lip_{ϕ} into Lip_{ψ}, where ψ depends on ϕ and w in some special fashion, in Theorem 4.5. Our results recover those obtained in [4] for the fractional integral operator (corresponding to $w(t) = t^{\alpha}$) and classical Lipschitz classes (corresponding to $\phi(t) = t^{\beta}$).

The paper is divided into three sections. In the first one we prove the basic lemmas on weights to be used in the paper. Section 3 is devoted to get conditions on the weights for the operator K_w to be defined on $L^p(\mu)$ for some values on p. Section 4 contains the results on $\tilde{K}_w$ and its boundedness on the generalized Lipschitz classes.

As usual $A \approx B$ means that $K^{-1}A \leq B \leq KA$ for some $K > 1$, C denotes a constant that may vary from line to line and p' stands for the conjugate exponent, $1/p + 1/p' = 1$.

2 Admisible weights.

In what follows we shall use the following indices introduced by N.G. Samko for weights defined on $(0,1]$ (see [18, Def. 2.3]) or by N.G. Samko et al. for weights defined on $[1,\infty)$ (see [19, Pag. 566]). We write

$$m(w)=\sup_{x>1}\frac{\log(\underline{\lim}_{h\to 0}\frac{w(xh)}{w(h)})}{\log x},\, M(w)=\inf_{x>1}\frac{\log(\overline{\lim}_{h\to 0}\frac{w(xh)}{w(h)})}{\log x}, \tag{4}$$

$$m_\infty(w)=\sup_{x>1}\frac{\log(\underline{\lim}_{h\to \infty}\frac{w(xh)}{w(h)})}{\log x},\, M_\infty(w)=\inf_{x>1}\frac{\log(\overline{\lim}_{h\to \infty}\frac{w(xh)}{w(h)})}{\log x}. \tag{5}$$

Definition 2.1 *We shall say that a weight on* $(0,\infty)$ *belongs to the class* $\tilde{W}$ *if there exist* $a,b\in\mathbb{R}$ *such that* $t^a w(t)$ *is almost increasing in* $(0,1]$ *(i.e. there exists* $C\geq 1$ *such that* $t^a w(t)\leq Cs^a w(s)$ *for* $0<t\leq s\leq 1$*),* $t^b w(t)$ *is almost decreasing in* $[1,\infty)$ *(i.e. there exists* $C\geq 1$ *such that* $s^b w(s)\leq Ct^b w(t)$ *for* $1\leq t\leq s<\infty$*) and* $-\infty<M(w),m_\infty(w)<+\infty$*.*

For a weight $w\in\tilde{W}$*, we use the notation* $m_w=\min\{m(w),m_\infty(w)\}$ *and* $M_w=\max\{M(w),M_\infty(w)\}$*.*

Definition 2.2 *Given* $-\infty<\sigma_1,\sigma_2<\infty$*, we say that a weight* w *on* $(0,\infty)$ *belongs to* $\Delta(\sigma_1,\sigma_2)$ *if* $t^{\sigma_1}w(t)$ *is almost increasing in* $(0,\infty)$ *and* $t^{\sigma_2}w(t)$ *is almost decreasing in* $(0,\infty)$*.*

Remark 2.1 *Observe that if* $w\in\Delta(\sigma_1,\sigma_2)$ *then there exists* $C\geq 1$ *such that, for* $0<s<\infty$*,*

$$C^{-1}x^{-\sigma_2}w(s)\leq w(xs)\leq Cx^{-\sigma_1}w(s),\quad 0<x\leq 1, \tag{6}$$

$$C^{-1}x^{-\sigma_1}w(s)\leq w(xs)\leq Cx^{-\sigma_2}w(s),\quad 1\leq x. \tag{7}$$

Hence it follows immediately that if $w\in\Delta(\sigma_1,\sigma_2)$ *then* $\sigma_2\leq\sigma_1$*.*

Our first objective is to show that the class $\tilde{W}$ can be described as $\tilde{W}=\cup_{\sigma_1,\sigma_2}\Delta(\sigma_1,\sigma_2)$.

To such a purpose, let us first recall some classical weights considered by Zygmund, Bari and Stechkin (see [1]) which play an important role in extending results valid for $w(t)=t^\alpha$ to more general weights and that will be connected with our class of weights.

Let $-\infty<\beta,\gamma<\infty$ and let w be a weight on $(0,1]$. w is said to belong to $\mathcal{Z}^\beta([0,1])$ if there exists $C>0$ such that

$$\int_0^h\frac{w(t)}{t^{1+\beta}}dt\leq C\frac{w(h)}{h^\beta},\quad h<1. \tag{8}$$

w is said to belong to $\mathcal{Z}_\gamma([0,1])$ if there exists $C>0$ such that

$$\int_h^1 \frac{w(t)}{t^{1+\gamma}}dt \le C\frac{w(h)}{h^\gamma}, \quad h\le 1. \tag{9}$$

w is said to belong to $\tilde{W}_0([0,1])$ if there exists $a\in\mathbb{R}$ such that

$$t^a u(t) \text{is almost increasing.} \tag{10}$$

The class of weights in $\mathcal{Z}^\beta([0,1])\cap\mathcal{Z}_\gamma([0,1])\cap\tilde{W}_0([0,1])$ is called the generalized Zygmund-Bari-Stechkin class in [10]. These classes of weights have been used by many authors and under different names (see [2, 3] for the notation d_ϵ and b_δ and references therein).

We have the following connection between the Zygmund-Bari-Steckin classes and the former indices (see [18, Pg 125], [10, Thm 3.1 and Thm 3.2], [19, Thm 2.4]).

Theorem 2.3 *Let $w\in\tilde{W}_0([0,1])$ and $-\infty<\beta,\gamma<\infty$. The following are equivalent.*

(a) $w\in\mathcal{Z}^\beta([0,1])$ (resp. $w\in\mathcal{Z}_\gamma([0,1])$).

(b) $m(w)>\beta$ (resp. $M(w)<\gamma$).

(c) For all $m(w)>\delta>\beta$ one has $\frac{w(t)}{t^\delta}$ is almost increasing in $(0,1]$ (resp. for all $M(w)<\delta<\gamma$ such that $\frac{w(t)}{t^\delta}$ is almost decreasing in $(0,1]$).

By using this theorem it is easily seen that $m(w)\le M(w)$ when $w\in\tilde{W}_0([0,1])$.

Next we prove our claim: $\tilde{W}=\cup_{\sigma_1,\sigma_2}\Delta(\sigma_1,\sigma_2)$.

Theorem 2.4 *Let w be a weight on $(0,\infty)$. The following are equivalent.*

(i) $w\in\cup_{\sigma_1,\sigma_2}\Delta(\sigma_1,\sigma_2)$.

(ii) $w\in\tilde{W}$.

(iii) There exist $u,v\in\tilde{W}_0([0,1])$ such that $u(1)=v(1)$, $M(u),M(v)\in\mathbb{R}$ and

$$w(t)=\begin{cases} u(t), & 0<t\le 1;\\ v(1/t), & 1\le t<\infty.\end{cases}$$

PROOF.

(i) $\Longrightarrow$ (ii). Assume $w \in \Delta(\sigma_1, \sigma_2)$. From (6) and (7) it follows that

$$-\sigma_1 \le m(w), M(w), m_\infty(w), M_\infty(w) \le -\sigma_2.$$

(ii) $\Longrightarrow$ (iii). Let $w \in \tilde{W}$ and define

$$u(t) = w(t), t \in (0,1], \quad v(t) = w(\frac{1}{t}), t \in (0,1].$$

Clearly $u, v \in \tilde{W}_0([0,1])$ and $M(u) = M(w)$. On the other hand

$$\begin{aligned} m_\infty(w) &= \sup_{x>1} \frac{-\log(\overline{\lim}_{h\to\infty} \frac{w(h)}{w(xh)})}{\log x} \\ &= -\inf_{x>1} \frac{\log(\overline{\lim}_{h\to\infty} \frac{v(1/h)}{v(1/xh)})}{\log x} \\ &= -\inf_{x>1} \frac{\log(\overline{\lim}_{h\to 0^+} \frac{v(h)}{v(h/x)})}{\log x} \\ &= -\inf_{x>1} \frac{\log(\overline{\lim}_{h\to 0^+} \frac{v(xh)}{v(h)})}{\log x} = -M(v). \end{aligned}$$

This concludes the implication.

(iii) $\Longrightarrow$ (i). Let $a_1, b_1 \in \mathbb{R}$ such that $t^{a_1}u(t)$ and $t^{b_1}v(t)$ are almost increasing in $(0,1]$. From Theorem 2.3 we also have that $t^{a_2}u(t)$ and $t^{b_2}v(t)$ are almost decreasing in $(0,1]$ for some $a_2, b_2 \in \mathbb{R}$. This gives that

$$\begin{aligned} t_1^{a_1} w(t_1) &\le C t_2^{a_1} w(t_2), \quad t_1 < t_2 \le 1, \\ t_2^{a_2} w(t_2) &\le C t_1^{a_2} w(t_1), \quad t_1 < t_2 \le 1, \\ t_1^{-b_2} w(t_1) &\le C t_2^{-b_2} w(t_2), \quad 1 \le t_1 < t_2, \\ t_2^{-b_1} w(t_2) &\le C t_1^{-b_1} w(t_1), \quad 1 \le t_1 < t_2, \end{aligned}$$

for some constant $C \ge 1$. In particular, combining the previous estimates, we also have

$$\begin{aligned} t_1^{a_1} w(t_1) &\le C t_2^{-b_2} w(t_2), \quad t_1 < 1 < t_2, \\ t_2^{-b_1} w(t_2) &\le C t_1^{a_2} w(t_1), \quad t_1 < 1 < t_2. \end{aligned}$$

Multiplying by t^γ we may assume that $a = a_1 = b_1 > 0$ and $a_2 = b_2 = -a < 0$. This allows to see that $t^a w(t)$ is almost increasing and $t^{-a} w(t)$ is almost decreasing. Hence $w \in \Delta(a, -a)$. $\square$

Example 2.1 *Let $\alpha, \beta \in \mathbb{R}$ and define*

$$w^{\alpha,\beta}(t) = \begin{cases} t^{\alpha} \log^{\beta}(\frac{e}{t}), & 0 < t \leq 1; \\ t^{\alpha} \log^{-\beta}(et), & 1 \leq t < \infty. \end{cases}$$

Note that $m(w^{\alpha,\beta}) = M(w^{\alpha,\beta}) = m_\infty(w^{\alpha,\beta}) = M_\infty(w^{\alpha,\beta}) = \alpha$.

Of course for $\beta = 0$ we have $w^{\alpha,0} \in \Delta(-\alpha, -\alpha)$. Let us see that $w^{\alpha,\beta} \in \Delta(\sigma_1, -\alpha)$ for any $\sigma_1 > -\alpha$ and $\beta > 0$ and $w^{\alpha,\beta} \in \Delta(-\alpha, \sigma_2)$ for any $\sigma_2 < -\alpha$ and $\beta < 0$.

Due to the facts that $w \in \Delta(\sigma_1, \sigma_2)$ implies $w^{-1} \in \Delta(-\sigma_2, -\sigma_1)$ and $w^\gamma \in \Delta(\gamma\sigma_1, \gamma\sigma_2)$ for $\gamma > 0$, it suffices to see that $w^{\delta,1} \in \Delta(\delta_1, -\delta)$ for all $\delta_1 > -\delta$.

This now follows since $\log(\frac{e}{t})\chi_{(0,1]}(t) + \log^{-1}(et)\chi_{(1,\infty)}(t))$ is decreasing and $t^\epsilon\Big(\log(\frac{e}{t})\chi_{(0,1]}(t) + \log^{-1}(et)\chi_{(1,\infty)}(t))\Big)$ is almost increasing for $\epsilon > 0$.

In particular we have that $w^{\alpha,\beta} \in \Delta(\sigma_1, \sigma_2)$ whenever $\sigma_2 < -\alpha < \sigma_1$ and $\beta \in \mathbb{R}$.

Let us mention the following useful result given in terms of the indices previously defined.

Theorem 2.5 *Let $w \in \tilde{W}$ and $\beta < m_w \leq M_w < \gamma$. Then $w \in \Delta(-\beta_1, -\gamma_1)$ for any $\beta < \beta_1 < m_w$ and $M_w < \gamma_1 < \gamma$.*

Proof. Using Theorem 2.3 applied to w_0 and w_∞, since $m(w_0) = m(w) > \alpha$ and $M(w_\infty) = -m_\infty(w) < -\alpha$, we have $t^{-\beta_1}w(t)$ and $t^{\beta_1}w_\infty(t)$ are almost increasing and decreasing in $(0, 1]$ respectively. This shows that $t^{-\beta_1}w(t)$ is almost increasing in $(0, \infty)$.

Similarly we get the corresponding result for γ_1. ■

We shall start by proving a couple of basic lemmas that will be used in the sequel.

Lemma 2.6 *Let $w \in \tilde{W}$ and $\varepsilon \in \mathbb{R}$. Then there exists $C > 0$ such that, for all $x \in X$ and $r > 0$,*

$$\int_{B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n-\varepsilon}} d\mu(y) \leq C \int_0^r t^\varepsilon w(t) \frac{dt}{t}. \tag{11}$$

PROOF. Assume $w \in \Delta(\sigma_1, \sigma_2)$. Define, for $j = 0, 1, ...,$

$$B_j = \{y \in B(x,r) : 2^{-(j+1)}r \leq d(x,y) < 2^{-j}r\}.$$

Note that (6) gives that

$$C^{-1}w(2^{-j}r) \leq w(d(x,y)) \leq Cw(2^{-j}r), y \in B_j. \tag{12}$$

Observe that $\cup_j B_j = B(x,r)\setminus\{x\}$ and $\mu(\{x\}) = 0$. Now, using condition (1), we have

$$\begin{aligned}
\int_{B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n-\varepsilon}} d\mu(y) &= \sum_{j=0}^{\infty} \int_{B_j} \frac{w(d(x,y))}{d(x,y)^{n-\varepsilon}} d\mu(y) \\
&\approx \sum_{j=0}^{\infty} w(2^{-j}r)(2^{-j}r)^{\varepsilon-n} \int_{B_j} d\mu(y) \\
&\leq C\sum_{j=0}^{\infty} w(2^{-j}r)(2^{-j}r)^{\varepsilon-n}\mu(B(x,2^{-j}r)) \\
&\leq C\sum_{j=0}^{\infty} (2^{-j}r)^{\varepsilon} w(2^{-j}r) \\
&\leq C\sum_{j=0}^{\infty} (2^{-j}r)^{\varepsilon} \int_{2^{-(j+1)}r}^{2^{-j}r} w(t)\frac{dt}{t} \\
&\leq C\sum_{j=0}^{\infty} \int_{2^{-(j+1)}r}^{2^{-j}r} t^{\varepsilon} w(t)\frac{dt}{t} \\
&= C\int_0^r t^{\varepsilon} w(t)\frac{dt}{t}.
\end{aligned}$$

□

Corollary 2.7 *Let $w \in \tilde{W}$ and $-\varepsilon < m_w$. Then there exists $C > 0$ such that, for all $x \in X$ and $r > 0$,*

$$\int_{B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n-\varepsilon}} d\mu(y) \leq Cr^{\varepsilon} w(r). \tag{13}$$

PROOF. From Proposition 2.5 one obtains $w \in \Delta(\sigma_1, \sigma_2)$ for some $\varepsilon > \sigma_1$. Invoking Lemma 2.6 and using (6) we have

$$\int_0^r t^{\varepsilon} w(t)\frac{dt}{t} = r^{\varepsilon}\int_0^1 s^{\varepsilon} w(rs)\frac{ds}{s} \leq Cr^{\varepsilon} w(r)\int_0^1 s^{\varepsilon-\sigma_1}\frac{ds}{s} \leq Cr^{\varepsilon} w(r).$$

□

Remark 2.2 *If $\gamma > 0$ and $\beta \in \mathbb{R}$ then (see [4, Lemma 2.1] for $\beta = 0$)*

$$\int_{B(x,r)} \frac{\big(1 + |\log(d(x,y)|\big)^{\beta}}{d(x,y)^{n-\gamma}} d\mu(y) \le Cr^{\gamma}(1 + |\log r|)^{\beta}, \quad 0 < r < \infty. \tag{14}$$

To obtain (14) for $0 < r \le 1$ apply Corollary 2.7 for $\varepsilon = 0$ to $w(t) = w^{\gamma,\beta}(t)$ which belongs to $\Delta(\sigma_1, \sigma_2)$ whenever $-\sigma_1 < \gamma < -\sigma_2$. The case $r > 1$ follows similarly using $w^{\gamma,-\beta}$.

Lemma 2.8 *Let $w \in \tilde{W}$ and $\delta \in \mathbb{R}$. Then there exists $C > 0$ such that, for all $x \in X$ and $r > 0$,*

$$\int_{X \setminus B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n+\delta}} d\mu(y) \le C \int_r^{\infty} \frac{w(t)}{t^{\delta}} \frac{dt}{t}. \tag{15}$$

PROOF. Assume again $w \in \Delta(\sigma_1, \sigma_2)$ and now consider for $j = 0, 1, ...$

$$A_j = \{y \in X : 2^j r \le d(x,y) < 2^{j+1} r\}.$$

As above

$$C^{-1} w(2^j r) \le w(d(x,y)) \le C w(2^j r), y \in A_j. \tag{16}$$

Using again (1) we have

$$\begin{aligned}
\int_{X \setminus B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n+\delta}} d\mu(y) &= \sum_{j=0}^{\infty} \int_{A_j} \frac{w(d(x,y))}{d(x,y)^{n+\delta}} d\mu(y) \\
&\approx C \sum_{j=0}^{\infty} (2^j r)^{-\delta-n} w(2^j r) \int_{A_j} d\mu(y) \\
&\le C \sum_{j=0}^{\infty} (2^j r)^{-\delta-n} w(2^j r) \mu(B(x, 2^{j+1} r)) \\
&\le C \sum_{j=0}^{\infty} (2^j r)^{-\delta} w(2^j r) \\
&\approx C \sum_{j=0}^{\infty} (2^j r)^{-\delta} \int_{2^j r}^{2^{j+1} r} w(t) \frac{dt}{t} \\
&\le C \sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1} r} \frac{w(t)}{t^{\delta}} \frac{dt}{t} = C \int_r^{\infty} \frac{w(t)}{t^{\delta}} \frac{dt}{t}.
\end{aligned}$$

□

Corollary 2.9 *Let $w \in \tilde{W}$ and $M_w < \delta$. Then there exists $C > 0$ such that, for all $x \in X$ and $r > 0$,*

$$\int_{X \setminus B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n+\delta}} d\mu(y) \le C \frac{w(r)}{r^{\delta}}. \tag{17}$$

PROOF. From Proposition 2.5 one obtains $w \in \Delta(\sigma_1, \sigma_2)$ for some $\delta > -\sigma_2$ Invoking Lemma 2.8 and (7) we get the estimate

$$\int_r^\infty \frac{w(t)}{t^\delta} \frac{dt}{t} = \frac{1}{r^\delta} \int_1^\infty \frac{w(rs)}{s^\delta} \frac{ds}{s} \le C \frac{w(r)}{r^\delta} \int_1^\infty s^{-\sigma_2 - \delta} \frac{ds}{s} \le C \frac{w(r)}{r^\delta}.$$

□

Remark 2.3 *If $\gamma > 0$ and $\beta \in \mathbb{R}$ then (see [4, Lemma 2.2] for $\beta = 0$)*

$$\int_{X \setminus B(x,r)} \frac{\big(1 + |\log(d(x,y)|)\big)^\beta}{d(x,y)^{n+\gamma}} d\mu(y) \le C \frac{1}{r^\gamma} (1 + |\log r|)^\beta, \quad 0 < r < \infty, \tag{18}$$

To obtain (18) for $0 < r \le 1$ we use Corollary 2.9 with $\delta = 0$ applied to $w^{-\gamma,\beta}$, which belongs to $\Delta(\sigma_1, \sigma_2)$ for $\sigma_2 < \gamma < \sigma_1$. The case $r > 1$ follows similarly using the weight $w^{-\gamma,-\beta}$.

3 The weighted fractional kernels

Definition 3.1 *Let $w \in \tilde{W}$. A $\mathcal{B}(X) \times \mathcal{B}(X)$-measurable function $k_w : X \times X \to \mathbb{C}$ is said to be a fractional kernel of weight w if*

$$|k_w(x,y)| \le C \frac{w(d(x,y))}{d(x,y)^n}, \qquad x, y \in X, x \ne y. \tag{19}$$

Denote by K_w the operator given by

$$K_w f(x) = \int_X k_w(x,y) f(y) d\mu(y), x \in X.$$

Note that if $\int_0^1 \frac{w(t)}{t} < \infty$, in particular if $w \in \Delta(\sigma_1, \sigma_2)$ with $\sigma_1 < 0$, then K_w is well defined on bounded functions f with bounded support (due to Lemma 2.6), or if $w \in \tilde{W}$ and $w(t) \le Ct^n$ for $0 < t < \infty$ then K_w is well defined on integrable functions f.

Let us extend the definition of such operator to more general functions depending on the properties of w.

Theorem 3.2 *Let $w \in \tilde{W}$, $1 < p < \infty$ and let k_w be a fractional kernel of weight w. Assume that*

$$M_\infty(w) < n/p \text{ and } m(w) > 0. \tag{20}$$

Then K_w defines a bounded operator from $L^p(\mu) \to L^p(\mu) + L^\infty(\mu)$.

PROOF. Let $f \in L^p(\mu)$. We shall see first that

$$\int_{B(x,1)} |K_w(x,y)||f(y)|d\mu(y) + \int_{X\setminus B(x,1)} |K_w(x,y)||f(y)|d\mu(y) < \infty, \mu - a.e.$$

On the one hand, using that $w^{p'} \in \tilde{W}$, Lemma 2.8 gives

$$\begin{aligned}
II(f,x) &= \int_{X\setminus B(x,1)} |K_w(x,y)||f(y)|d\mu(y) \\
&\leq C\int_{X\setminus B(x,1)} \frac{w(d(x,y))}{d(x,y)^n}|f(y)|d\mu(y) \\
&\leq C(\int_{X\setminus B(x,1)} |f(y)|^p d\mu(y))^{1/p}(\int_{X\setminus B(x,1)} \frac{w^{p'}(d(x,y))}{d(x,y)^{np'}}d\mu(y))^{1/p'} \\
&\leq C(\int_1^\infty \frac{w^{p'}(t)}{t^{n(p'-1)}}\frac{dt}{t})^{1/p'}(\int_{X\setminus B(x,1)} |f(y)|^p d\mu(y))^{1/p} \\
&= C(\int_0^1 \frac{w_\infty^{p'}(t)}{t^{n(1-p')}}\frac{dt}{t})^{1/p'}(\int_{X\setminus B(x,1)} |f(y)|^p d\mu(y))^{1/p}
\end{aligned}$$

Since $m(w_\infty^{p'}) = p'm(w_\infty) = -p'M_\infty(w)$ and $n(1-p') < p'm(w_\infty)$, Theorem 2.3 gives that $w_\infty^{p'} \in \mathcal{Z}^{n(1-p')}([0,1])$. In particular $\int_0^1 \frac{w_\infty^{p'}(t)}{t^{n(1-p')}}\frac{dt}{t} < \infty$, and therefore $\int_{X\setminus B(x,1)} k_w(x,y)f(y)d\mu(y) \in L^\infty(\mu)$ and with norm bounded by $C\|f\|_{L^p(\mu)}$.

On the other hand, using Hölder's inequality and Lemma 2.6, we have

$$\begin{aligned}
I(f,x) &= \int_{B(x,1)} |K_w(x,y)||f(y)|d\mu(y) \\
&\leq C\int_{B(x,1)} \frac{w(d(x,y))}{d(x,y)^n}|f(y)|d\mu(y) \\
&\leq C(\int_{B(x,1)} \frac{w(d(x,y))}{d(x,y)^n}|f(y)|^p d\mu(y))^{1/p}(\int_{B(x,1)} \frac{w(d(x,y))}{d(x,y)^n}d\mu(y))^{1/p'} \\
&\leq C(\int_0^1 \frac{w(t)}{t}dt)^{1/p'}(\int_{B(x,1)} \frac{w(d(x,y))}{d(x,y)^n}|f(y)|^p d\mu(y))^{1/p}
\end{aligned}$$

Now the assumption $m(w_0) = m(w) > 0$ and Theorem 2.3 give $\int_0^1 \frac{w(t)}{t} dt = A < \infty$.

Now integrating $I^p(f,x)$ we have

$$\begin{aligned} \int_X I(f,x)^p d\mu(x) &\leq CA^{p/p'} \int_X \Big(\int_{B(y,1)} \frac{w(d(x,y))}{d(x,y)^n} d\mu(x) \Big) |f(y)|^p d\mu(y) \\ &\leq CA^{p/p'} (\int_0^1 \frac{w(t)}{t} dt) \int_X |f(y)|^p d\mu(y) \leq C\|f\|^p_{L^p(\mu)}. \end{aligned}$$

Therefore $I(f,x) < \infty$ μ-a.e. and, in particular, $\int_{B(x,1)} k_w(x,y)f(y)d\mu(y)$ is well defined μ-a.e. We conclude now that $K_w f(x) \in L^p(\mu) + L^\infty(\mu)$ and $\|K_w f\|_{L^p(\mu)+L^\infty(\mu)} \leq C\|f\|_{L^p(\mu)}$. □

In [4, Theorem 3.2] it was shown that for $w(t) = t^\alpha$ and $1 \leq p < n/\alpha$ the operator K_α maps $L^p(\mu)$ into $L^{q,\infty}(\mu)$ for $1/q = 1/p - \alpha/n$ extending to the non-doubling setting the Hardy-Littlewood-Sobolev inequality which holds for $\mathbb{R}^n$ and the Lebesgue measure (see [20]).

Theorem 3.3 *Let $w \in \tilde{W}$ with $0 < m_w \leq M_w < n$ and let k_w be a fractional kernel of weight w. If $1 \leq p < n/M_w$, $0 < \varepsilon < m_w$ and $0 < \delta < n - M_w$ then there exists $A > 0$ such that, for $1/q_1 = 1/p - (m_w - \varepsilon)/n$ and $1/q_2 = 1/p - (M_w + \delta)/n$, we have for every f with $\|f\|_{L^p(\mu)} = 1$*

$$\mu\{x : |K_w(f)(x)| > \lambda\} \leq \frac{C}{\lambda^{q_2}}, \quad 0 < \lambda \leq A, \tag{21}$$

$$\mu\{x : |K_w(f)(x)| > \lambda\} \leq \frac{C}{\lambda^{q_1}}, \quad \lambda \geq A \tag{22}$$

PROOF. From Proposition 2.5 we have $w \in \Delta(\sigma_1, \sigma_2)$ for all $0 < -\sigma_1 < m_w \leq M_w < -\sigma_2 < n$. Put $\sigma_1 = \varepsilon - m_w$ and $\sigma_2 = -M_w - \delta$. Now, let $1 < p < n/M_w$, $f \in L^p(\mu)$ and $r > 0$ and define

$$I_r(f,x) = \int_{B(x,r)} |K_w(x,y)||f(y)|d\mu(y), \quad x \in X,$$

$$II_r(f,x) = \int_{X \setminus B(x,r)} |K_w(x,y)||f(y)|d\mu(y), \quad x \in X.$$

Arguing as in Theorem 3.2 and using that $m_w > 0$ in Corollary 2.7, we obtain

$$I_r(f,x) \le Cw(r)^{1/p'}\Big(\int_{B(x,r)} \frac{w(d(x,y))}{d(x,y)^n}|f(y)|^p d\mu(y)\Big)^{1/p} \tag{23}$$

Now, using Fubini's theorem and Corollary 2.7 again, we have

$$\begin{aligned}\int_X I_r(f,x)^p d\mu(x) &\le Cw(r)^{p/p'}\int_X \Big(\int_{B(y,r)} \frac{w(d(x,y))}{d(x,y)^n} d\mu(x)\Big)|f(y)|^p d\mu(y)\\ &\le Cw(r)^p \int_X |f(y)|^p d\mu(y).\end{aligned}$$

On the other hand

$$II_r(f,x) \le C\Big(\int_{X\setminus B(x,r)} |f(y)|^p d\mu(y)\Big)^{1/p}\Big(\int_{X\setminus B(x,r)} \frac{w^{p'}(d(x,y))}{d(x,y)^{np'}} d\mu(y)\Big)^{1/p'}$$

and now using that $M_{w^{p'}} = p'M_w < (p'-1)n$ and Corollary 2.9, we have

$$II_r(f,x) \le Cr^{-n/p}w(r)\Big(\int_{X\setminus B(x,r)} |f(y)|^p d\mu(y)\Big)^{1/p}.$$

Now, for each $\|f\|_p = 1$, the estimates (6) and (7) allow us to write

$$II_r(f,x) \le C_0 r^{-n/p}\max\{r^{-\sigma_1}, r^{-\sigma_2}\} = \phi(r).$$

Denoting

$$\phi(r) = \begin{cases} C_0 r^{-n/p-\sigma_1}, & 0 < r \le 1;\\ C_0 r^{-n/p-\sigma_2}, & 1 \le r < \infty,\end{cases}$$

we have that ϕ is continuous, decreasing in $(0,\infty)$, $\lim_{r\to 0}\phi(r) = \infty$ and $\lim_{r\to\infty}\phi(r) = 0$. Hence for any $\lambda > 0$ there is a unique $0 < r < \infty$ such that $\phi(r) = \lambda/2$ and $II_r(f,x) \le \lambda/2$ for all $x \in X$. Hence we have

$$\begin{aligned}\mu\{x : |K_w(f)(x)| > \lambda\} &\le \mu\{x : I_r(f,x) > \lambda/2\}\\ &\le C\lambda^{-p}\|I_r(f,.)\|_p^p\\ &\le C\lambda^{-p}w(r)^p\\ &\le C\lambda^{-p}r^n\phi(r)^p\\ &= C[\phi^{-1}(\lambda/2)]^n.\end{aligned}$$

To finish the proof observe that if $\lambda \ge 2C_0$ then $\phi^{-1}(\lambda/2) = C_1\lambda^{-q_1/n}$ where $n/q_1 = n/p + \sigma_1$ and that if $0 < \lambda \le 2C_0$ then $\phi^{-1}(\lambda/2) = C_2\lambda^{-q_2/n}$ where $n/q_2 = n/p + \sigma_2$.

The case $p = 1$ is similar with the obvious modifications. □

Corollary 3.4 *(see [4, Theorem 3.2] Let* $w \in \tilde{W}$ *with* $m(w) = M(w) = m_\infty(w) = M_\infty(w) = \alpha$ *and* k_w *is a fractional kernel of weight* w*. Then for each* $1 < p < \frac{n}{\alpha}$ *and* $1/q = 1/p - \alpha/n$ *we have that* K_w *extends to a bounded operator from* $L^p(\mu)$ *into* $L^q(\mu)$*.*

4 Boundedness in Lipschitz spaces

Definition 4.1 *Let* $\phi : (0,\infty) \to (0,\infty)$ *be a continuous function. A function* $f : X \to \mathbb{C}$ *is said to satisfy a Lipschitz condition of order* ϕ *if*

$$|f(x) - f(y)| \leq C\phi(d(x,y)), x, y \in X, x \neq y. \tag{24}$$

The smallest constant satisfying (24) will be denoted $\|f\|_{Lip(\phi)}$*. It is easy to see that* $\|\cdot\|_{Lip(\phi)}$ *is a norm on the linear space of all Lipschitz functions of weight* w*, modulo constants, and* $Lip(\phi)$ *is complete under this norm.*

Remark 4.1 *If* $\lim_{t\to 0^+} \phi(t) = 0$ *then functions in* Lip_ϕ *are continuous.*

Remark 4.2 *Assume that there exist constants* $C > 1$ *and* $K > 1$ *so that* $K^{-1}\phi(t) \leq \phi(s) \leq K\phi(t)$ *whenever* $C^{-1}t \leq s \leq Ct$*. In this case* $Lip(\phi)$ *defines the same space for all equivalent distances in* X *and with equivalent norms.*

Definition 4.2 *Let* k_w *be a fractional kernel of weight* w*. We say that* k_w *has regularity* $\varepsilon > 0$ *if it satisfies*

$$|k_w(x,z) - k_w(y,z)| \leq C\Big(\frac{d(x,y)}{d(x,z)}\Big)^\varepsilon \frac{w(d(x,y))}{d(x,z)^n}, d(x,z) \geq 2d(x,y) > 0. \tag{25}$$

For a given $x_0 \in X$ *define*

$$\tilde{K}_w f(x) = \int_X \Big(k_w(x,y) - k_w(x_0,y)\Big) f(y)d\mu(y). \tag{26}$$

Note that, from Lemma 2.8, if f is bounded with $supp(f) \cap B(x_0, 2R) = \emptyset$ then $K_w f(x)$ is well defined for any $x \in B(x_0, R)$.

Example 4.1 *Let* $k_w(x,y) = \frac{w(d(x,y))}{d(x,y)^n}$ *where* $w \in \tilde{W}$ *is derivable and*

$$\sup_{t>0} \Big|\frac{tw'(t)}{w(t)} - n\Big| < \infty.$$

Then k_w *has regularity* 1.

PROOF. Consider $w_1(t) = \frac{w(t)}{t^n}$. By the mean value theorem

$$|w_1(t) - w_1(s)| \leq |w_1'((1-\theta)s + \theta t)||t - s|.$$

Hence, setting $t(\theta, x, y, z) = t_0 = (1-\theta)d(x,z) + \theta d(y,z)$ then

$$\begin{aligned}
|k_w(x,z) - k_w(y,z)| &\leq |w_1'(t_0)||d(x,z) - d(y,z)| \\
&\leq \frac{|t_0 w'(t_0) - n w(t_0)|}{t_0^{n+1}} d(x,y) \\
&\leq C \frac{w(t_0)}{t_0^{n+1}} d(x,y).
\end{aligned}$$

Let $x, y, z \in X$ such that $d(x,z) \geq 2d(x,y)$, i.e. $d(x,z) - d(x,y) \geq d(x,y)$. It is elementary to see that

$$\frac{3}{2} d(x,z) \geq d(y,z) \geq \frac{1}{2} d(x,z) \geq d(x,y).$$

This shows that

$$\frac{1}{2} d(x,z) \leq t(\theta, x, y, z) \leq \frac{3}{2} d(x,z),$$

and allows to conclude that

$$|k_w(x,z) - k_w(y,z)| \leq C \frac{w(d(x,z))}{d(x,z)^{n+1}} d(x,y).$$

$\square$

Theorem 4.3 *Let $w \in \tilde{W}$ with $m_w > 0$. Assume that k_w be a fractional kernel with regularity $0 < \varepsilon < M_w$ and*

$$\max\{n/m_w, 1\} < p < n/(M_w - \varepsilon).$$

Then $\tilde{K}_w$ is bounded from $L^p(\mu)$ to $Lip(\phi)$ for $\phi(t) = t^{-n/p} w(t)$.

PROOF. We have $n/p < m_w \leq M_w < n/p + \varepsilon$. Let $f \in L^p(\mu)$ for $p \neq \infty$, $x, y \in X$ with $x \neq y$ and $r = d(x,y)$. Then

$$\begin{aligned}
|\tilde{K}_w f(x) - \tilde{K}_w f(y)| &\leq \int_X |k_w(x,z) - k_w(y,z)||f(z)| d\mu(z) \\
&\leq \int_{B(x,2r)} |k_w(x,z)||f(z)| d\mu(z) + \int_{B(x,2r)} |k_w(y,z)||f(z)| d\mu(z) \\
&+ \int_{X \setminus B(x,2r)} |k_w(x,z) - k_w(y,z)||f(z)| d\mu(z).
\end{aligned}$$

First, using Hölder's inequality and Corollary 2.7 (because $m_{w^{p'}} = p'm_w > n(p'-1)$), we estimate

$$\begin{aligned}
&\int_{B(x,2r)} |k_w(x,z)||f(z)|d\mu(z) \\
\leq\ & C\int_{B(x,2r)} \frac{w(d(x,z))}{d(x,z)^n}|f(z)|d\mu(z) \\
\leq\ & C\Big(\int_{B(x,2r)} \frac{w^{p'}(d(x,z))}{d(x,z)^{np'}}d\mu(z)\Big)^{1/p'}\Big(\int_{B(x,2r)} |f(z)|^p d\mu(z)\Big)^{1/p} \\
\leq\ & C\frac{w(2r)}{r^{n/p}}\|f\|_{L^p(\mu)}.
\end{aligned}$$

The second term is estimated similarly using $B(x,2r) \subset B(y,3r)$,

$$\int_{B(x,2r)} |k_w(y,z)||f(z)|d\mu(z) \leq C\frac{w(3r)}{r^{n/p}}\|f\|_{L^p(\mu)}.$$

Finally we use (25) and Corollary 2.9 (since $M_{w^{p'}} = p'M_w < n(p'-1)+\varepsilon p'$) to obtain

$$\begin{aligned}
&\int_{X\setminus B(x,2r)} |k_w(x,z)-k_w(y,z)||f(z)|d\mu(z) \\
\leq\ & Cd(x,y)^\varepsilon\int_{X\setminus B(x,2r)} \frac{w(d(x,z))}{d(x,z)^{n+\varepsilon}}|f(z)|d\mu(z) \\
\leq\ & Cd(x,y)^\varepsilon\Big(\int_{X\setminus B(x,2r)} \frac{w^{p'}(d(x,z))}{d(x,z)^{(n+\varepsilon)p'}}d\mu(z)\Big)^{1/p'}\Big(\int_{X\setminus B(x,2r)} |f(z)|^p d\mu(z)\Big)^{1/p} \\
\leq\ & C\frac{w(2r)}{r^{n/p}}\|f\|_{L^p(\mu)}.
\end{aligned}$$

Therefore, using that $w(r) \approx w(2r) \approx w(3r)$ and $r = d(x,y)$ one gets

$$|\tilde{K}_w f(x) - \tilde{K}_w(y)| \leq C\frac{w(d(x,y))}{d(x,y)^{n/p}}\|f\|_p.$$

Analogue, but easier, proof works in $p = \infty$. □

We write k_α for k_w in the case $w = t^\alpha$.

Corollary 4.4 *(see [4, Theorem 5.2]) Let $0 < \alpha < n$ and k_α be a fractional kernel with regularity $\varepsilon > 0$. If $n/\alpha < p \leq \infty$ and $\alpha - n/p < \varepsilon$, then $\tilde{K}_\alpha$ maps boundedly $L^p(\mu)$ into Lip$(\alpha - n/p)$.*

Let us now analyze the boundedness of $\tilde{K}_w$ on Lipschitz spaces.

Theorem 4.5 *Let k_w be a fractional kernel with weight $w \in \tilde{W}$ with regularity ε such that $m_w > 0$. Assume that $u \in \tilde{W}$ with $m_u > 0$ and $M_{uw} < \varepsilon$. Then $\tilde{K}_w(1) = 0$ if and only if $\tilde{K}_w$ maps continuously $Lip(u)$ into $Lip(uw)$.*

PROOF. Assume $\tilde{K}_w(1) = 0$. Equivalently

$$\int_X (k_w(x,z) - k_w(y,z))d\mu(z) = 0, x, y \in X.$$

If $f \in Lip(u)$, $x \neq y$ and $r = d(x,y)$ then we can write

$$\begin{aligned}
|\tilde{K}_w f(x) - \tilde{K}_w(y)| &= |\int_X (k_w(x,z) - k_w(y,z))(f(z) - f(x))d\mu(z)| \\
&\leq \int_{B(x,2r)} |k_w(x,z)||f(z) - f(x)|d\mu(z) \\
&+ \int_{B(x,2r)} |k_w(y,z)||f(z) - f(x)|d\mu(z) \\
&+ \int_{X \setminus B(x,2r)} |k_w(x,z) - k_w(y,z)||f(z) - f(x)|d\mu(z).
\end{aligned}$$

Now, since $m_{uw} > 0$ (see Theorem 2.5), one gets

$$\begin{aligned}
\int_{B(x,2r)} |k_w(x,z)||f(z) - f(x)|d\mu(z) &\leq C \int_{B(x,2r)} \frac{w(d(x,z))}{d(x,z)^n} u(d(x,z))d\mu(z) \\
&\leq Cu(2r)w(2r)
\end{aligned}$$

by virtue of Corollary 2.9.

Using, as above, $B(x,2r) \subset B(y,3r)$ one also gets

$$\begin{aligned}
&\int_{B(x,2r)} |k_w(y,z)||f(z) - f(x)|d\mu(z) \\
\leq &\int_{B(y,3r)} |k_w(y,z)|(|f(z) - f(y)| + |f(y) - f(x)|)d\mu(z) \\
\leq &C \int_{B(y,3r)} \frac{w(d(y,z))}{d(y,z)^n} u(d(y,z))d\mu(z) + Cu(d(x,y)) \int_{B(y,3r)} \frac{w(d(y,z))}{d(y,z)^n} d\mu(z).
\end{aligned}$$

Since $w(3t) \approx w(2t) \approx w(t)$ and $u(3t) \approx u(2t) \approx u(t)$, Corollary 2.7 implies that

$$\int_{B(y,3r)} \frac{w(d(y,z))u(d(y,z))}{d(y,z)^n} d\mu(z) + u(d(x,y)) \int_{B(y,3r)} \frac{w(d(y,z))}{d(y,z)^n} d\mu(z) \le Cu(r)w(r).$$

Finally, we have

$$\begin{aligned} & \int_{X\setminus B(x,2r)} |k_w(x,z) - k_w(y,z)||f(z) - f(x)| d\mu(z) \\ \le \quad & Cd(x,y)^\varepsilon \int_{X\setminus B(x,2r)} \frac{w(d(x,z))}{d(x,z)^{n+\varepsilon}} u(d(x,z)) d\mu(z). \end{aligned}$$

Also using Corollary 2.9 we have $\int_{X\setminus B(x,2r)} \frac{w(d(x,z))u(d(x,z))}{d(x,z)^{n+\varepsilon}} d\mu(z) \le C\frac{w(2r)u(2r)}{r^\varepsilon}$. Hence, the previous estimates imply

$$|\tilde{K}_w f(x) - \tilde{K}_w f(x)| \le Cu(r)w(r).$$

Conversely, if we assume that $\tilde{K}_w$ is bounded from $Lip(u)$ to $Lip(uw)$ then $\tilde{K}(1)$ should have norm zero in $Lip(uw)$, that is $\tilde{K}(1)$ is constant, but since $\tilde{K}_w(1)(x_0) = 0$ the constant should be zero. $\square$

Applying the previous result for $w(t) = t^\alpha$ and $u(t) = t^\beta$ we recover the following theorem.

Corollary 4.6 *(see [4, Theorem 5.3]) Let $\alpha, \beta > 0$ and k_α be a fractional kernel with regularity $\varepsilon > 0$ with $\alpha + \beta < \varepsilon$. Then $\tilde{K}_\alpha$ maps boundedly $Lip(\beta)$ into $Lip(\alpha + \beta)$ if and only if $\tilde{K}_\alpha(1) = 0$.*

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Departamento de Análisis Matemático
Universidad de Valencia
46100 Burjassot (Valencia)
Spain
oblasco@uv.es
vicencasap@terra.es

Departamento de Matemática Aplicada
Universidad Politécnica de Valencia
46022 Valencia
Spain
jmotos@mat.upv.es