Multipliers on generalized mixed-norm sequence spaces

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Abstract

Given $1 \leq p, q \leq \infty$ and sequences of integers $(n_k)_k$ and $(n'_k)_k$ such that $n_k \leq n'_k \leq n_{k+1}$, the generalized mixed-norm space $\ell^{\mathcal{I}}(p,q)$ is defined as those sequences $(a_j)_j$ such that $((\sum_{j\in I_k} |a_j|^p)^{1/p})_k \in \ell^q$ where $I_k = \{j \in \mathbb{N}_0 \ s.t.\ n_k \leq j < n'_k\},\ k \in \mathbb{N}_0$.

The necessary and sufficient conditions for a sequence $\lambda = (\lambda_j)_j$ to belong to the space of multipliers $(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$, for different sequences $\mathcal{I}$ and $\mathcal{J}$ of intervals in $\mathbb{N}_0$, are determined.

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1 Introduction

Let $\mathcal{S}$ the space of complex-valued sequences with the locally convex vector topology given by means of the semi-norms $p_j(\lambda) = |\lambda_j|$ where $\lambda = (\lambda_j)_{j\in\mathbb{N}_0}$. Given two Banach spaces A, B continuously contained in $\mathcal{S}$ we write (A, B) for the space of multipliers from A into B. More precisely,

$$(A, B) = \{\lambda = (\lambda_j)_j : (\lambda_j a_j)_j \in B \ \forall (a_j)_j \in A\}.$$

We shall use the notation $supp(a) = \{n \in \mathbb{N}_0 : a_n \neq 0\}$ and $\lambda * a$ for the sequence $(\lambda_j a_j)_{j\in\mathbb{N}_0}$ where $\lambda = (\lambda_j)_{j\in\mathbb{N}_0}$ and $a = (a_j)_{j\in\mathbb{N}_0}$.

Of course for the classical ℓ^p spaces one easily sees that $(\ell^{p_1}, \ell^{p_2}) = \ell^p$ where $1/p = (1/p_2 - 1/p_1)^+$. We use the notation $p_2 \ominus p_1 = p$ to mean $\frac{1}{p_2\ominus p_1} = \frac{1}{p_2} - \frac{1}{p_1}$ whenever $p_1 > p_2$ and $p = \infty$ whenever $p_1 \leq p_2$.

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The above result can be extended (see [K]) to the class of mixed norm sequence spaces, denoted $\ell(p,q)$, which are defined by the condition

$$\Big(\sum_{n=0}^{\infty}\big(\sum_{2^n-1\le k<2^{n+1}-1}|a_k|^p\big)^{q/p}\Big)^{1/q}<\infty.$$

Theorem 1.1. *Let $1\le r,s,u,v\le\infty$. Then*

$$(\ell(r,s),\ell(u,v))=\ell(u\ominus r,v\ominus s).$$

In particular the Köthe dual of $\ell(p,q)$, defined by $(\ell(p,q),\ell^1)$ becomes $\ell(p',q')$ for $1\le p,q<\infty$ and $\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$.

Also multipliers between sequence spaces given by Taylor coefficients of holomorphic functions in the disk have been deeply studied in the literature. Since the time of Hardy and Littlewood, mixed-norm and related spaces have been used to study function spaces on the unit disk, and later to study multipliers between such spaces. Special emphasis has been put on the case where the spaces A and B correspond to the sequence space of Taylor coefficient of analytic functions such as Hardy spaces, Bergman spaces, mixed norm spaces of analytic functions, etc. The theory of Hardy spaces and mixed norm spaces of analytic functions was originated in the work of Hardy and Littlewood (see [HL1, HL2]) who implicitly considered the spaces $H(p,q,\alpha)$ of functions $f\in\mathcal{H}(\mathbb{D})$ such that

$$\Big(\int_0^1(1-r)^{q\alpha-1}M_p^q(f,r)dr\Big)^{1/q}<\infty.$$

Their work on these spaces was continued by Flett and Sledd (see [F1, F2, F3, S1, S2]) and later on by Pavlovic (see [P1, P2]). Multipliers on Hardy spaces were in fashion for a long time and much work was done on them and related spaces. However nowadays complete descriptions of multipliers between Hardy spaces (H^p,H^q) for certain values of p and q remain still open. The reader is referred to the surveys (see [CL, O]) for lots of results and references. Also many results on multipliers between mixed norm spaces of analytic functions have been established in the last decades (see [B1, B2, B3] and references thereby). For such a purpose the use of solid spaces (sequence spaces whose norm depends only on the size of the coefficients), and in particular $\ell(p,q)$ spaces, is a rather important tool. It is worth o mentioning that the smallest solid space contained or which contains one of classical Hardy, Bergman and $H(p,q,\alpha)$ is actually $H(2,q,\alpha)$ for some values p,q and α (see [B2, B3]) and this last space can be identified with certain weighted $\ell(2,q)$, due to Plancherel's theorem.

Another appearance of mixed norm spaces comes with the use of lacunary sequences, that is $a = (a_n)$ such that $supp(a) \subset \{n_k : k \in \mathbb{N}_0\}$ for a sequence of integers satisfying $\inf n_{k+1}/n_k = \lambda > 0$. Recently (see [KD-SAA]) the description of the Taylor coefficient of analytic functions $F(z) = \sum_{k=0}^{\infty} b_k z^{n_k}$, where n_k is a lacunary sequence, belonging to the weighted Bergman-Besov space $B^1(\rho)$ has been achieved under certain conditions on the weight. It corresponds again with certain weighted $\ell(2,1)$.

In this paper we consider families of intervals $\mathcal{I} = \{I_k : k \in \mathbb{N}_0\}$ where $I_k = \{j \in \mathbb{N}_0 \ s.t. \ n_k \leq j < n'_k\}$ for some increasing sequences $(n_k)_k$ and (n'_k) such that $n_k \leq n'_k \leq n_{k+1}$ and we use the notation $\Lambda_{\mathcal{I}} = \cup I_k$. We shall introduce the spaces $\ell^{\mathcal{I}}(p,q)$ given by sequences $a = (a_j)_{j \in \Lambda_{\mathcal{I}}}$ verifying

$$((\sum_{j \in I_k} |a_j|^p)^{1/p})_k \in \ell^q$$

and the obvious modifications for $p = \infty$ or $q = \infty$.

In particular $\ell(p,q) = \ell^{\mathcal{I}}(p,q)$ for $I_k = [2^k - 1, 2^k - 1) \cap \mathbb{N}_0$. Also a lacunary sequence $a = (a_n)_n$ corresponds to $supp(a) \subseteq \Lambda_{\mathcal{I}}$ where $\mathcal{I} = \{I_k : k \in \mathbb{N}_0\}$ with $I_k = \{n_k\}$ (that is $n'_k = n_k + 1$) for some $\inf_k n_{k+1}/n_k = \lambda > 1$.

We shall give the necessary and sufficient conditions for a sequence $\lambda = (\lambda_j)_j$ to belong to the multiplier space $(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$ whenever $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. We also get some applications to multipliers between certain weighted mixed norm spaces of analytic functions. The paper is organized as follows. Section 2 contains the definitions and first properties of the spaces $\ell^{\mathcal{I}}(p,q)$, studying inclusions between them and conditions for coincidence results $\ell^{\mathcal{I}}(p,q) = \ell^{\mathcal{J}}(p,q)$. Section 3 contains the main result, which is split into three subsections: The case where intervals in $\mathcal{J}$ are union of intervals in $\mathcal{I}$, to be denoted $\mathcal{I} \leq \mathcal{J}$, the case where for each $I \in \mathcal{I}$ there exits $J \in \mathcal{J}$ such that either $I \subseteq J$ or $J \subseteq I$ and finally the case where there exists $(I,J) \in \mathcal{I} \times \mathcal{J}$ such that $I \cap J \neq \emptyset$ and $I \cap J \notin \mathcal{I} \cup \mathcal{J}$. In Section 4 we include some application to multipliers on spaces of analytic functions and extend some recent result on weighted Bergman-Besov classes.

From now on, we will write $A \approx B$ whenever there exist $C > 0$ such that $C^{-1}A \leq B \leq CA$ and, as usual, $\#I$ stands for the cardinal of I, $1/p + 1/p' = 1$ for $1 \leq p \leq \infty$ and also C denotes a constant that may vary from line to line.

2 Generalized mixed-norm spaces

Definition 2.1. Let $1 \leq p,q \leq \infty$ and let $\mathcal{I}$ be a collection of disjoint intervals in $\mathbb{N}_0$, say $I_k = \mathbb{N}_0 \cap [n_k, n'_k)$ where $n_k \leq n'_k \leq n_{k+1}$. We set

$\Lambda_{\mathcal{I}} = \cup_{k\in\mathbb{N}_0} I_k$. We write $\ell^{\mathcal{I}}(p,q)$ for the space of sequences $(a_n)_{n\in\Lambda_{\mathcal{I}}}$ verifying

$$((\sum_{j\in I_k} |a_j|^p)^{1/p})_k \in \ell^q.$$

This space becomes a Banach space under the norm

$$\|a\|^{\mathcal{I}}_{p,q} = \left(\sum_{k=0}^{\infty} \left\{\sum_{j\in I_k} |a_j|^p\right\}^{q/p}\right)^{1/q}$$

with the obvious modifications for $p = \infty$ or $q = \infty$.

Remark 2.2. Of course $\ell^{\mathcal{I}}(p,p) = \{(a_n)_{n\in\Lambda_{\mathcal{I}}} : (\sum_n |a_n|^p)^{1/p} < \infty\}$. In particular $\ell^{\mathcal{I}}(p,p) = \ell^p$ whenever $\Lambda_{\mathcal{I}} = \mathbb{N}_0$.

An elementary approach, using Hölder's inequality, leads to the duality

$$\ell^{\mathcal{I}}(p,q)^* = \ell^{\mathcal{I}}(p',q')$$

for $1 \le p, q < \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$.

Remark 2.3. It is clear that $(a_j)_j \in \ell^{\mathcal{I}}(p,q) \Leftrightarrow (a_j^p)_j \in \ell^{\mathcal{I}}(1, q/p)$ in the case $p < q$ and also $(a_j)_j \in \ell^{\mathcal{I}}(p,q) \Leftrightarrow (a_j^q)_j \in \ell^{\mathcal{I}}(\frac{p}{q}, 1)$ in the case $p > q$.

Moreover, for $a^p = (a_j^p)_j$,

$$\|a\|^{\mathcal{I}}_{p,q} = \left(\|a^p\|^{\mathcal{I}}_{1,q/p}\right)^{1/p} = \left(\|a^q\|^{\mathcal{I}}_{p/q,1}\right)^{1/q} \tag{1}$$

Remark 2.4. Let $a \in \ell^{\mathcal{I}}(p,q)$.

(i) If $\mathcal{I}'$ is a sub-collection of intervals in $\mathcal{I}$ then $\|a\|^{\mathcal{I}'}_{p,q} \le \|a\|^{\mathcal{I}}_{p,q}$.

(ii) If $\mathcal{I} = \mathcal{I}' \cup \mathcal{I}''$ for two disjoint collections $\mathcal{I}'$ and $\mathcal{I}''$ then $\|a\|^{\mathcal{I}}_{p,q} = \left((\|a\|^{\mathcal{I}'}_{p,q})^q + (\|a\|^{\mathcal{I}''}_{p,q})^q\right)^{1/q}$.

We would like to analyze the embedding between $\ell^{\mathcal{I}}(p_1,q_1)$ and $\ell^{\mathcal{I}}(p_2,q_2)$.

Proposition 2.5. *Let $\mathcal{I}$ be a collection of disjoint intervals in $\mathbb{N}_0$ and let $1 \le p_1, p_2, q \le \infty$ with $p_1 \ne p_2$. Then $\ell^{\mathcal{I}}(p_1,q) = \ell^{\mathcal{I}}(p_2,q)$ (with equivalent norms) if and only if*

$$\sup_{k\in\mathbb{N}_0} \#I_k < \infty \tag{2}$$

In particular if $\sup_{k\in\mathbb{N}_0} \#I_k < \infty$ *then*

$$\ell^{\mathcal{I}}(p,q) = \{(a_n)_{n\in\Lambda_{\mathcal{I}}} : (\sum_n |a_n|^q)^{1/q} < \infty\}.$$

Proof. $\Longrightarrow$) Assume, for instance, $p_1 < p_2$ and that $\|a\|^{\mathcal{I}}_{p_1,q} \approx \|a\|^{\mathcal{I}}_{p_2,q}$ for all a supported in $\Lambda_{\mathcal{I}}$. Hence taking $a = \chi_{I_k}$ one concludes that $(n'_k - n_k)^{1/p_1 - 1/p_2} \leq C$ for any k. Hence $\sup_k \# I_k < \infty$.

$\Longleftarrow$) Note that $\# I_k = (n'_k - n_k)$ and assume $M = \sup_k (n_k - n'_k)$. Then

$$\|a\|^{\mathcal{I}}_{p_1,q} = \left(\sum_{k=0}^{\infty} \left\{ \sum_{j \in I_k} |a_j|^{p_1} \right\}^{q/p_1} \right)^{1/q} \approx \left(\sum_{k=0}^{\infty} \left\{ \sum_{j \in I_k} |a_j|^{p_2} \right\}^{q/p_2} \right)^{1/q} = \|a\|^{\mathcal{I}}_{p_2,q}$$

since $\|.\|_{p_1} \approx \|.\|_{p_2}$ in $\mathbb{C}^M$.

$\square$

Proposition 2.6. *Let $1 \leq p_1, q_1, p_2, q_2 \leq \infty$ and let $\mathcal{I}$ be a collection of disjoint intervals in $\mathbb{N}_0$ with $sup_k \# I_k = \infty$.*
Then $\ell^{\mathcal{I}}(p_1, q_1) \subseteq \ell^{\mathcal{I}}(p_2, q_2)$ if and only if $p_1 \leq p_2$ and $q_1 \leq q_2$.

Proof. $\Longrightarrow$) Assume that there exists $C > 0$ such that $\|a\|^{\mathcal{I}}_{p_2,q_2} \leq C\|a\|^{\mathcal{I}}_{p_1,q_1}$ for all a supported in $\Lambda_{\mathcal{I}}$. Hence taking $k \in \mathbb{N}_0$ and $a = \chi_{I_k}$ one concludes that $(\# I_k)^{1/p_2 - 1/p_1} \leq C$. Hence $p_1 \leq p_2$. Let $N \in \mathbb{N}_0$ and consider $a = \sum_{k=1}^{N} \chi_{n_k}$. Applying the above inequality we obtain $N^{1/q_2 - 1/q_1} \leq C$. Therefore $q_1 \leq q_2$.

$\Longleftarrow$) Let us denote

$$\ell^q(\ell^p) = \{(x_k)_{k \in \mathbb{N}_0} : x_k \in \ell^p, (\sum_{k=0}^{\infty} \|x_k\|^q_{\ell^p})^{1/q} < \infty.\}$$

Hence the mapping

$$(a_n)_{n \in \mathbb{N}_0} \to ((a_j)_{j \in I_k})_{k \in \mathbb{N}_0}$$

is an isometric embedding from $\ell^{\mathcal{I}}(p,q)$ into $\ell^q(\ell^p)$. Taking into account that $\ell^{r_1}(E) \subseteq \ell^{r_2}(E)$ for any Banach space E and $r_1 \leq r_2$ we conclude that

$$\ell^{\mathcal{I}}(p, q_1) \subseteq \ell^{\mathcal{I}}(p, q_2) \text{ and } \ell^{\mathcal{I}}(p_1, q) \subseteq \ell^{\mathcal{I}}(p_2, q).$$

Therefore

$$\ell^{\mathcal{I}}(p_1, q_1) \subseteq \ell^{\mathcal{I}}(p_2, q_1) \subseteq \ell^{\mathcal{I}}(p_2, q_2).$$

$\square$

We would like to analyze the embedding between $\ell^{\mathcal{I}}(p,q)$ and $\ell^{\mathcal{J}}(p,q)$ for $\mathcal{I} \neq \mathcal{J}$ whenever $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$.

Proposition 2.7. *Let $\mathcal{I} = \{I_l : l \in \mathbb{N}_0\}$ and $\mathcal{J} = \{J_k : k \in \mathbb{N}_0\}$. If $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$, $p \leq q$ (respect. $q \leq p$) and $\sup_k \# J_k < \infty$ (respect. $\sup_l \# I_l < \infty$) then*

$$\ell^{\mathcal{I}}(p,q) \subseteq \ell^{\mathcal{J}}(p,q) (\text{ respect. } \ell^{\mathcal{J}}(p,q) \subseteq \ell^{\mathcal{I}}(p,q)).$$

Proof. From Proposition 2.5 gives $\ell^{\mathcal{J}}(p,q) = \ell^{\mathcal{J}}(q,q)$ and clearly $\ell^{\mathcal{J}}(q,q) = \ell^{\mathcal{I}}(q,q)$. Then the result follows using $\ell^{\mathcal{I}}(p,q) \subseteq \ell^{\mathcal{I}}(q,q)$ whenever $p \leq q$. □

Let us mention another particular case where they coincide.

Proposition 2.8. *Let $\mathcal{I}$ such that $I_k = [n_k, n'_k) \cap \mathbb{N}_0$ with $n'_{2k} = n_{2k+1}$ and define*

$$\mathcal{J} = \{J_k = I_{2k} \cup I_{2k+1} : k \in \mathbb{N}_0\}.$$

Then $\ell^{\mathcal{I}}(p,q) = \ell^{\mathcal{J}}(p,q)$.

Proof. Note that $J_k = I_{2k} \cup I_{2k+1}$ is again an interval in $\mathbb{N}_0$. Using that $(a+b)^\alpha \leq C_\alpha(a^\alpha + b^\alpha)$ for $a, b, \alpha > 0$ then

$$\begin{aligned}
\|a\|^{\mathcal{J}}_{p,q} &= \left(\sum_{k=0}^{\infty} \left\{\sum_{j\in J_k} |a_j|^p\right\}^{q/p}\right)^{1/q} \\
&= \left(\sum_{k=0}^{\infty} \left\{\sum_{j\in I_{2k}} |a_j|^p + \sum_{j\in I_{2k+1}} |a_j|^p\right\}^{q/p}\right)^{1/q} \\
&\leq C\left(\sum_{k=0}^{\infty} \left\{\sum_{j\in I_{2k}} |a_j|^p\right\}^{q/p} + \sum_{k=0}^{\infty} \left\{\sum_{j\in I_{2k+1}} |a_j|^p\right\}^{q/p}\right)^{1/q} \\
&\leq C\|a\|^{\mathcal{I}}_{p,q}
\end{aligned}$$

On the other hand, using now $(a^\beta + b^\beta) \leq C_\beta(a+b)^\beta$ for $a, b, \beta > 0$,

$$\begin{aligned}
\|a\|^{\mathcal{I}}_{p,q} &= \left(\sum_{k=0}^{\infty} \left\{\sum_{j\in I_{2k}} |a_j|^p\right\}^{q/p} + \left\{\sum_{j\in I_{2k+1}} |a_j|^p\right\}^{q/p}\right)^{1/q} \\
&\leq C'\left(\sum_{k=0}^{\infty} \left\{\sum_{j\in I_{2k}\cup I_{2k+1}} |a_j|^p\right\}^{q/p}\right)^{1/q} \\
&\leq C'\|a\|^{\mathcal{J}}_{p,q}
\end{aligned}$$

□

The previous idea easily generalizes using the following definition.

Definition 2.9. Let $\mathcal{I} := \{I_l : \ l \in \mathbb{N}_0\}$ and $\mathcal{J} := \{J_k : \ k \in \mathbb{N}_0\}$. We say that $\mathcal{I} \leq \mathcal{J}$ if the following conditions hold:

(i) $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$,

(ii) $F_k = F_k(\mathcal{I}, \mathcal{J}) := \{l \in \mathbb{N}_0 : I_l \subseteq J_k\} \neq \emptyset$ for all $k \in \mathbb{N}_0$,

(iii) $J_k = \cup_{l \in F_k} I_l$ for all $k \in \mathbb{N}_0$.

Proposition 2.10. *Let $1 \leq p, q \leq \infty$ and $\mathcal{I} \leq \mathcal{J}$. Then*

(i) $\ell^{\mathcal{J}}(p,q) \subseteq \ell^{\mathcal{I}}(p,q)$ for $p \leq q$.

(ii) $\ell^{\mathcal{I}}(p,q) \subseteq \ell^{\mathcal{J}}(p,q)$ for $q \leq p$.

Moreover the embeddings above are of norm 1.

Proof. (i) Case $q = \infty$: Let $a \in \ell^{\mathcal{J}}(p, \infty)$ and $l \in \mathbb{N}_0$. We know that there is k such that $I_l \subseteq J_k$. Hence

$$(\sum_{n \in I_l} |a_n|^p)^{1/p} \leq (\sum_{n \in J_k} |a_n|^p)^{1/p} \leq \|a\|^{\mathcal{J}}_{p,\infty}.$$

This gives $\|a\|^{\mathcal{I}}_{p,\infty} \leq \|a\|^{\mathcal{J}}_{p,\infty}$.

The case $p = 1$: Let $a \in \ell^{\mathcal{J}}(1,q)$ and $q \geq 1$. Therefore

$$\left(\|a\|^{\mathcal{J}}_{1,q}\right)^q = \sum_k \left(\sum_{l \in F_k} \sum_{n \in I_l} |a_n|\right)^q \geq \sum_k \sum_{l \in F_k} (\sum_{n \in I_l} |a_n|)^q = \left(\|a\|^{\mathcal{I}}_{1,q}\right)^q.$$

The case $1 < p \leq q < \infty$ follows using (1) and the previous one.

(ii) The case $p = \infty$: Let $a \in \ell^{\mathcal{I}}(\infty, q)$. Then

$$\|a\|^{\mathcal{J}}_{\infty,q} = \Big(\sum_k \sup_{l \in F_k} (\sup_{n \in I_l} |a_n|)^q\Big)^{1/q} \leq \Big(\sum_k \sum_{l \in F_k} (\sup_{n \in I_l} |a_n|)^q\Big)^{1/q} = \|a\|^{\mathcal{I}}_{\infty,q}.$$

To cover the remaning cases, from (1), we simply need to show that $\ell^{\mathcal{I}}(p,1) \subseteq \ell^{\mathcal{J}}(p,1)$ for $p \geq 1$. Now observe that

$$\begin{aligned} \|a\|^{\mathcal{J}}_{p,1} &= \sum_k (\sum_{l \in F_k} \sum_{n \in I_l} |a_n|^p)^{1/p} = \sum_k (\sum_{l \in F_k} \|a \chi_{I_l}\|_p^p)^{1/p} \\ &\leq \sum_k \sum_{l \in F_k} \|a \chi_{I_l}\|_p = \sum_l (\sum_{n \in I_l} |a_n|^p)^{1/p} = \|a\|^{\mathcal{I}}_{p,1}. \end{aligned}$$

□

Theorem 2.11. *Let $\mathcal{I} \leq \mathcal{J}$ and $1 \leq p, q \leq \infty$ with $p \neq q$.*

$\ell^{\mathcal{I}}(p,q) = \ell^{\mathcal{J}}(p,q)$ (with equivalent norms) if and only if $\sup_k \# F_k < \infty$.

Proof. $\Longrightarrow$) Assume that $\|a\|^{\mathcal{J}}_{p,q} \approx \|a\|^{\mathcal{I}}_{p,q}$ for all a finitely supported. Let $k \in \mathbb{N}_0$ and define

$$a^{(k)} = \sum_{l \in F_k} (\#I_l)^{-1/p} \chi_{I_l}.$$

Then $\|a\|^{\mathcal{J}}_{p,q} = (\#F_k)^{1/p}$ and $\|a\|^{\mathcal{I}}_{p,q} = (\#F_k)^{1/q}$.

One concludes that $C_2 \le (\#F_k)^{1/p-1/q} \le C_1$ which implies, in the case $p \neq q$, $\sup_{k\in\mathbb{N}_0}(\#F_k) < \infty$.

$\Longleftarrow$) Case $p < q$: From Proposition 2.10 we only need to show $\ell^{\mathcal{I}}(p,q) \subseteq \ell^{\mathcal{J}}(p,q)$. Using now Hölder's inequality for $q/p > 1$

$$\left\{\sum_{n\in J_k} |a_n|^p\right\}^{1/p} \le \left\{\sum_{l\in F_k}\sum_{n\in I_l} |a_n|^p\right\}^{1/p} \le \left\{\sum_{l\in F_k} (\sum_{n\in I_l} |a_n|^p)^{q/p}\right\}^{1/q} (\#F_k)^{\frac{1}{p\ominus q}}.$$

Therefore, if $M = \sup_k F_k$, we have

$$\begin{aligned}\|a\|^{\mathcal{J}}_{p,q} &= \Big(\sum_{k=0}^{\infty}(\sum_{n\in J_k} |a_n|^p)^{q/p}\Big)^{1/q} \le M^{\frac{1}{p\ominus q}}\Big(\sum_{k\in\mathbb{N}_0}\sum_{l\in F_k}(\sum_{n\in I_l} |a_n|^p)^{q/p}\Big)^{1/q} \\ &= M^{\frac{1}{p\ominus q}}\Big(\sum_{l\in\mathbb{N}_0}(\sum_{n\in I_l} |a_n|^p)^{q/p}\Big)^{1/q} = M^{\frac{1}{p\ominus q}}\|a\|^{\mathcal{I}}_{p,q}.\end{aligned}$$

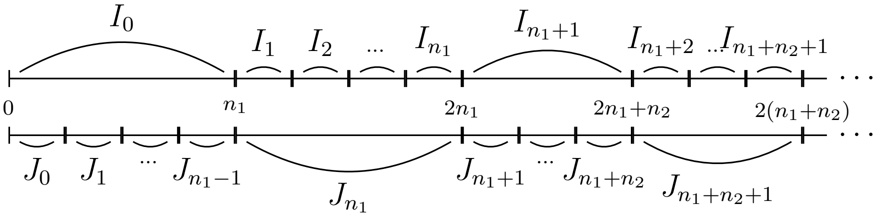
Case $p > q$: Using again Proposition 2.10 we shall show $\ell^{\mathcal{J}}(p,q) \subseteq \ell^{\mathcal{I}}(p,q)$. Using $1/q = 1/q \ominus p + 1/p$

$$\begin{aligned}\|a\|^{\mathcal{I}}_{p,q} &= (\sum_{l} \|a\chi_{I_l}\|_p^q)^{1/q} = (\sum_{k}\sum_{l\in F_k} \|a\chi_{I_l}\|_p^q)^{1/q} \\ &\le \left(\sum_{k}(\sum_{l\in F_k} \|a\chi_{I_l}\|_p^p)^{q/p}(\#F_k)^{q/q\ominus p}\right)^{1/q} \\ &\le M^{\frac{1}{q\ominus p}}\left(\sum_{k}(\sum_{n\in J_k} |a_n|^p)^{q/p}\right)^{1/q} \le M^{\frac{1}{q\ominus p}}\|a\|^{\mathcal{J}}_{p,q}.\end{aligned}$$

□

Let us now exhibit an example where neither $\ell(p,q)^{\mathcal{I}} \subseteq \ell^{\mathcal{J}}(p,q)$ nor $\ell^{\mathcal{J}}(p,q) \subseteq \ell^{\mathcal{I}}(p,q)$.

Example 2.12. Let $1 \le p < q < \infty$ and take $\mathcal{I}, \mathcal{J}$ as showed below:



with:

$card(I_0) = n_1$	$card(J_0) = ... = card(J_{n_1}) = 1$
$card(I_1) = ... = card(I_{n_1}) = 1$	$card(J_{n_1}) = n_1$
$card(I_{n_1+1}) = n_2$	$card(J_{n_1+1}) = ... = card(J_{n_1+n_2}) = 1$
$card(I_{n_1+2}) = ... = card(I_{n_1+n_2+1}) = 1$	$card(J_{n_1+n_2+1}) = n_2, ...$
$card(I_{n_1+n_2+2}) = n_3, ...$	...

Let's see that neither $\ell^{\mathcal{J}}(p,q) \subset \ell^{\mathcal{I}}(p,q)$ nor $\ell^{\mathcal{I}}(p,q) \subset \ell^{\mathcal{J}}(p,q)$.
Taking

$$a = (\overbrace{\beta_1, ..., \beta_1}^{n_1}, \overbrace{0, ..., 0}^{n_1}, \overbrace{\beta_2, ..., \beta_2}^{n_2}, 0, ...)$$

and

$$b = (\underbrace{0, ..., 0}_{n_1}, \underbrace{\beta_1, ..., \beta_1}_{n_1}, \underbrace{0, ..., 0}_{n_2}, \beta_2, ...)$$

we have:

$$\|a\|^{\mathcal{I}}_{p,q} = \|b\|^{\mathcal{J}}_{p,q} = (\sum_j \beta_j^q n_j^{q/p})^{1/q}$$

$$\|a\|^{\mathcal{J}}_{p,q} = \|b\|^{\mathcal{I}}_{p,q} = (\sum_j \beta_j^q n_j)^{1/q}$$

Then it is enough to consider $q > p$ and $\beta_j = n_j^{-1/p} j^{-1/q}$. Now

$$(\sum_j \beta_j^q n_j^{q/p})^{1/q} = (\sum_j j^{-1})^{1/q} = \infty$$

and, since $n_j \geq j$,

$$(\sum_j \beta_j^q n_j)^{1/q} = (\sum_j j^{-1} n_j^{1-q/p})^{1/q} \leq (\sum_j j^{-q/p})^{1/q} < \infty.$$

Hence we have $a \in \ell^{\mathcal{J}}(p,q) \backslash \ell^{\mathcal{I}}(p,q)$ and $b \in \ell^{\mathcal{I}}(p,q) \backslash \ell^{\mathcal{J}}(p,q)$.

We would like to explain a procedure to analyze the general case $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$.

Definition 2.13. Let $\mathcal{I}$ and $\mathcal{J}$ families of disjoint intervals in $\mathbb{N}_0$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. For each $k \in \mathbb{N}_0$ we use the notation, as above, $F_k = \{l \in \mathbb{N}_0 : I_l \subseteq J_k\}$ which now might be empty. We also define

$$\tilde{F}_k = \{l \in \mathbb{N}_0 : J_k \cap I_l \neq \emptyset\}.$$

We write ϕ and Φ for the mappings given by

$$\phi(k) = \min \tilde{F}_k \text{ and } \Phi(k) = \max \tilde{F}_k.$$

Similarly, interchanging $\mathcal{I}$ and $\mathcal{J}$, we define G_l, $\tilde{G}_l$, $\psi(l)$ and $\Psi(l)$.

Definition 2.14. We define the "left" and "right" part of the interval J_k by

$$\check{J}_k = J_k \cap I_{\phi(k)} \text{ and } \hat{J}_k = J_k \cap I_{\Phi(k)}$$

and, denoting $J'_k = \cup_{l \in F_k} I_l$ and $\tilde{J}_k = \cup_{l \in \tilde{F}_k} I_l$, we have

$$J'_k \subseteq J_k \subseteq \tilde{J}_k \tag{3}$$

and

$$J_k = J'_k \cup \hat{J}_k \cup \check{J}_k, \tag{4}$$

where $J'_k = \emptyset$ whenever $F_k = \emptyset$.

Similarly, interchanging $\mathcal{I}$ and $\mathcal{J}$ we consider $\check{I}_l, \hat{I}_l, I'_l$ and $\tilde{I}_l$.

With this notation out of the way we can classify intervals in $\mathcal{J}$ into four different types (according to $\mathcal{I}$). Note that each interval $J \in \mathcal{J}$ there are four possibilities: J coincides with I for some $I \in \mathcal{I}$, J can be written as a union of at least two intervals in $\mathcal{I}$, J is strictly contained into some interval $I \in \mathcal{I}$ or there exists $I \in \mathcal{I}$ which overlaps with J and its complement J^c.

Therefore we decompose $\mathbb{N}_0$ into four disjoint sets defined as follows:

Definition 2.15. Let $\mathcal{I}$ and $\mathcal{J}$ families of disjoint intervals in $\mathbb{N}_0$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. We introduce

$$N^{\mathcal{J}}_{equal} = \{k \in \mathbb{N}_0 : \#(\tilde{F}_k \setminus F_k) = 0, \#\tilde{F}_k = 1\}, \tag{5}$$

$$N^{\mathcal{J}}_{big} = \{k \in \mathbb{N}_0 : \#(\tilde{F}_k \setminus F_k) = 0, \#\tilde{F}_k \geq 2\}, \tag{6}$$

$$N^{\mathcal{J}}_{small} = \{k \in \mathbb{N}_0 : \#(\tilde{F}_k \setminus F_k) > 0, \#\tilde{F}_k = 1\}, \tag{7}$$

$$N^{\mathcal{J}}_{inter} = \{k \in \mathbb{N}_0 : \#(\tilde{F}_k \setminus F_k) > 0, \#\tilde{F}_k \geq 2\}. \tag{8}$$

We define the sets $N^{\mathcal{I}}_{equal}, N^{\mathcal{I}}_{big}, N^{\mathcal{I}}_{small}$ and $N^{\mathcal{I}}_{inter}$ similarly.

Remark 2.16. Using (4) we can also give a description of the sets above in terms of ϕ and Φ:

$N^{\mathcal{J}}_{equal} = \{k : \phi(k) = \Phi(k), J_k = I_{\phi(k)}\}$.

$N^{\mathcal{J}}_{big} = \{k : \phi(k) < \Phi(k), J_k = \tilde{J}_k\}$.

$N^{\mathcal{J}}_{small} = \{k : \phi(k) = \Phi(k), J_k \subsetneq I_{\phi(k)}\}$.

$N^{\mathcal{J}}_{inter} = \{k : \phi(k) < \Phi(k), J_k \subsetneq \tilde{J}_k\}$.

Using the above decomposition we can generalize Proposition 2.7, Proposition 2.10 and Theorem 2.11. Note that $\sup_k \# J_k < \infty$ implies $\sup_k \# \tilde{F}_k < \infty$ and also that $\mathcal{I} \leq \mathcal{J}$ corresponds to the case where $N^{\mathcal{J}}_{inter} \cup N^{\mathcal{J}}_{small} = \emptyset$ or equivalently $\# \tilde{G}_l = 1$ for any $l \in \mathbb{N}_0$.

Theorem 2.17. *Let $1 \leq p < q \leq \infty$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. Then*

$$\ell^{\mathcal{I}}(p,q) \subseteq \ell^{\mathcal{J}}(p,q) \Longleftrightarrow \sup\{\# \tilde{F}_k ; k \in \mathbb{N}_0\} < \infty.$$

Proof. $\Longrightarrow$) Arguing as in Theorem 2.11, for $k \in \mathbb{N}_0$ we consider

$$a^{(k)} = \sum_{l \in \tilde{F}_k} (\#(I_l \cap J_k))^{-1/p} \chi_{I_l \cap J_k}.$$

Hence

$$\|a^{(k)}\|^{\mathcal{J}}_{p,q} = (\sum_{n \in J_k} |a_n|^p)^{1/p} = (\sum_{l \in \tilde{F}_k} \sum_{n \in I_l \cap J_k} |a_n|^p)^{1/p} = (\# \tilde{F}_k)^{1/p}$$

and

$$\|a^{(k)}\|^{\mathcal{I}}_{p,q} = (\sum_{l \in \tilde{F}_k} (\sum_{n \in I_l \cap J_k} |a_n|^p)^{q/p})^{1/q} = (\# \tilde{F}_k)^{1/q}.$$

Therefore using that $\|a^{(k)}\|^{\mathcal{J}}_{p,q} \leq C \|a^{(k)}\|^{\mathcal{I}}_{p,q}$ and $p < q$ we conclude that $\sup\{\# \tilde{F}_k ; k \in \mathbb{N}_0\} < \infty$.

$\Longleftarrow$) Denote $\sup_k (\# F_k) = M \geq 0$ and let $k \in \mathbb{N}_0$.

Case $q = \infty$: If $k \in N^{\mathcal{J}}_{small} \cup N^{\mathcal{J}}_{equal}$ then

$$(\sum_{n \in J_k} |a_n|^p)^{1/p} \leq (\sum_{n \in I_{\phi(k)}} |a_n|^p)^{1/p} \leq \|a\|^{\mathcal{I}}_{p,\infty}.$$

If $k \in N^{\mathcal{J}}_{big} \cup N^{\mathcal{J}}_{inter}$ we have

$$\begin{aligned}(\sum_{n \in J_k} |a_n|^p)^{1/p} &= (\sum_{l \in F_k} \sum_{n \in I_l} |a_n|^p + \sum_{n \in \check{J}_k \cup \hat{J}_k} |a_n|^p)^{1/p} \\ &\leq (\sum_{l \in F_k} \sum_{n \in I_l} |a_n|^p)^{1/p} + (\sum_{n \in I_{\phi(k)}} |a_n|^p)^{1/p} + (\sum_{n \in I_{\Phi(k)}} |a_n|^p)^{1/p} \\ &\leq C \left\{ \sup_{l \in F_k} (\sum_{n \in I_l} |a_n|^p)^{1/p} (\# F_k)^{1/p} + 2 \|a\|^{\mathcal{I}}_{p,\infty} \right\}\end{aligned}$$

This shows $\ell^{\mathcal{I}}(p,\infty) \subseteq \ell^{\mathcal{J}}(p,\infty)$.

Case $q < \infty$: Arguing as in Proposition 2.10 we simply show that $\ell^{\mathcal{I}}(1,q) \subseteq \ell^{\mathcal{J}}(1,q)$ for $q > 1$.

Observe that

$$\begin{aligned}\sum_{k\in N^{\mathcal{J}}_{small}} (\sum_{n\in J_k} |a_n|)^q &\leq \sum_{l\in N^{\mathcal{I}}_{big}\cup N^{\mathcal{I}}_{inter}} \sum_{\phi(k)=l} (\sum_{n\in J_k} |a_n|)^q \\ &\leq \sum_{l\in N^{\mathcal{I}}_{big}\cup N^{\mathcal{I}}_{inter}} (\sum_{\phi(k)=l} \sum_{n\in J_k} |a_n|)^q \\ &= \sum_{l\in N^{\mathcal{I}}_{big}\cup N^{\mathcal{I}}_{inter}} (\sum_{n\in I_l} |a_n|)^q \\ &\leq \Big(\|a\|^{\mathcal{I}}_{1,q}\Big)^q\end{aligned}$$

Also we have

$$\begin{aligned}\sum_{k\in N^{\mathcal{J}}_{equal}\cup N^{\mathcal{J}}_{big}} (\sum_{n\in J_k} |a_n|)^q &\leq \sum_{k\in N^{\mathcal{J}}_{equal}\cup N^{\mathcal{J}}_{big}} (\sum_{l\in F_k}\sum_{n\in I_l} |a_n|)^q \\ &\leq \sum_{k\in N^{\mathcal{J}}_{equal}\cup N^{\mathcal{J}}_{big}} (\#F_k)^{q-1} \sum_{l\in F_k}(\sum_{n\in I_l} |a_n|)^q \\ &\leq M^{q-1} \sum_{k\in N^{\mathcal{J}}_{equal}\cup N^{\mathcal{J}}_{big}} \sum_{l\in F_k}(\sum_{n\in I_l} |a_n|)^q \\ &\leq M^{q-1}\Big(\|a\|^{\mathcal{I}}_{1,q}\Big)^q.\end{aligned}$$

Finally

$$\begin{aligned}\sum_{k\in N^{\mathcal{J}}_{inter}} (\sum_{n\in J_k} |a_n|)^q &\leq \sum_{k\in N^{\mathcal{J}}_{inter}} (\sum_{l\in F_k}\sum_{n\in I_l} |a_n| + \sum_{n\in \check{J}_k} |a_n| + \sum_{n\in \hat{J}_k} |a_n|)^q \\ &\leq C \sum_{k\in N^{\mathcal{J}}_{inter}} (\#F_k)^{q-1} \sum_{l\in F_k}(\sum_{n\in I_l} |a_n|)^q \\ &+ C \sum_{k\in N^{\mathcal{J}}_{inter}} (\sum_{n\in \check{J}_k} |a_n|)^q + C \sum_{k\in N^{\mathcal{J}}_{inter}} (\sum_{n\in \hat{J}_k} |a_n|)^q \\ &\leq CM^{q-1} \sum_{l\in N^{\mathcal{I}}_{inter}\cup N^{\mathcal{I}}_{small}} (\sum_{n\in I_l} |a_n|)^q \\ &+ C \sum_{k\in N^{\mathcal{J}}_{inter}} (\sum_{n\in I_{\phi(k)}} |a_n|)^q + \sum_{k\in N^{\mathcal{J}}_{inter}} (\sum_{n\in I_{\Phi(k)}} |a_n|)^q \\ &\leq C\Big(\|a\|^{\mathcal{I}}_{1,q}\Big)^q\end{aligned}$$

Combining the above estimates we conclude this implication. $\square$

Corollary 2.18. *Let $1 \leq p < q \leq \infty$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. Then*

$$\ell^{\mathcal{J}}(p,q) \subseteq \ell^{\mathcal{I}}(p,q) \Longleftrightarrow \sup\{\#\tilde{G}_l; l \in \mathbb{N}_0\} < \infty.$$

Next result can be achieved using duality but we include a direct proof.

Theorem 2.19. *Let $1 \leq q < p \leq \infty$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. Then*

$$\ell^{\mathcal{J}}(p,q) \subseteq \ell^{\mathcal{I}}(p,q) \Longleftrightarrow \sup\{\#\tilde{F}_k; k \in \mathbb{N}_0\} < \infty.$$

Proof. $\Longrightarrow$) Repeat the argument presented in the direct implication of Theorem 2.17.

$\Longleftarrow$) Denote again $\sup_k(\#F_k) = M$.

Case $p = \infty$: Observe first that if $l \in N^{\mathcal{I}}_{big} \cup N^{\mathcal{I}}_{equal}$ we have

$$(\sup_{n\in I_l} |a_n|)^q = |a_{n(l)}|^q \leq (\sup_{n\in J_k} |a_n|)^q$$

for some $k = k(l) \in N^{\mathcal{J}}_{small} \cup N^{\mathcal{J}}_{equal}$. Since $k(l) \neq k(l')$ for $l \neq l' \in N^{\mathcal{I}}_{big} \cup N^{\mathcal{I}}_{equal}$ we obtain

$$\sum_{l\in N^{\mathcal{I}}_{big}\cup N^{\mathcal{I}}_{equal}} (\sup_{n\in I_l} |a_n|)^q \leq \sum_{k\in N^{\mathcal{J}}_{small}\cup N^{\mathcal{J}}_{equal}} (\sup_{n\in J_k} |a_n|)^q.$$

Also if $l \in N^{\mathcal{I}}_{inter}$ then $(\sup_{n\in I_l} |a_n|)^q = |a_{n(l)}|^q$ where $n(l) \in I'_l \cup \hat{I}_l \cup \check{I}_l$. Note that $n(l) \in J_k$ for some $k \in N^{\mathcal{J}}_{small} \cup N^{\mathcal{J}}_{inter}$ and

$$1 \leq \#(\{l \in N^{\mathcal{I}}_{inter} : n(l) \in J_k\}) \leq 2.$$

Hence

$$\sum_{l\in N^{\mathcal{I}}_{inter}} (\sup_{n\in I_l} |a_n|)^q \leq 2 \sum_{k\in N^{\mathcal{J}}_{small}\cup N^{\mathcal{J}}_{inter}} (\sup_{n\in J_k} |a_n|)^q.$$

On the other hand

$$\begin{aligned}\sum_{l\in N^{\mathcal{I}}_{small}} (\sup_{n\in I_l} |a_n|)^q &\leq \sum_{k\in N^{\mathcal{J}}_{big}\cup N^{\mathcal{J}}_{inter}} \sum_{\psi(l)=k} (\sup_{n\in I_l} |a_n|)^q \\ &\leq \sum_{k\in N^{\mathcal{J}}_{big}\cup N^{\mathcal{J}}_{inter}} (\sup_{n\in J_k} |a_n|)^q (\#F_k)^q \\ &\leq M^q (\|a\|^{\mathcal{J}}_{p,\infty})^q.\end{aligned}$$

Combining the previous cases we get $\ell^{\mathcal{J}}(\infty,q) \subseteq \ell^{\mathcal{I}}(\infty,q)$.

Case $p < \infty$. Arguing as in Proposition 2.10 we simply show that $\ell^{\mathcal{J}}(p,1) \subseteq \ell^{\mathcal{I}}(p,1)$ for $p > 1$.

$$
\begin{aligned}
\|a\|^{\mathcal{I}}_{p,1} &= \sum_{l} (\sum_{n \in I_l} |a_n|^p)^{1/p} \\
&\leq \sum_{l \in N^{\mathcal{I}}_{small}} (\sum_{n \in I_l} |a_n|^p)^{1/p} \\
&+ \sum_{l \in N^{\mathcal{I}}_{equal} \cup N^{\mathcal{I}}_{big}} (\sum_{k \in G_l} \sum_{n \in J_k} |a_n|^p)^{1/p} \\
&+ \sum_{l \in N^{\mathcal{I}}_{inter}} (\sum_{k \in G_l} \sum_{n \in J_k} |a_n|^p + \sum_{n \in \check{I}_l} |a_n|^p + \sum_{n \in \hat{I}_l} |a_n|^p)^{1/p} \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

Now observe that

$$
I_1 \leq \sum_{k \in N^{\mathcal{J}}_{big} \cup N^{\mathcal{J}}_{inter}} \sum_{l \in F_k} (\sum_{n \in I_l} |a_n|^p)^{1/p} \leq \sum_{k \in N^{\mathcal{J}}_{big} \cup N^{\mathcal{J}}_{inter}} (\sum_{n \in J_k} |a_n|^p)^{1/p} \#(F_k) \leq M \|a\|^{\mathcal{J}}_{p,1}.
$$

Also note, since $p > 1$,

$$
I_2 \leq \sum_{l \in N^{\mathcal{I}}_{equal} \cup N^{\mathcal{I}}_{big}} \sum_{k \in G_l} (\sum_{n \in J_k} |a_n|^p)^{1/p} \leq \|a\|^{\mathcal{J}}_{p,1}.
$$

Finally

$$
\begin{aligned}
I_3 &\leq \sum_{l \in N^{\mathcal{I}}_{inter}} (\sum_{k \in G_l} \sum_{n \in J_k} |a_n|^p)^{1/p} + (\sum_{n \in \check{I}_l} |a_n|^p)^{1/p} + (\sum_{n \in \hat{I}_l} |a_n|^p)^{1/p} \\
&\leq \left(\sum_{k \in N^{\mathcal{J}}_{inter} \cup N^{\mathcal{J}}_{small}} (\sum_{n \in J_k} |a_n|^p)^{1/p} + \sum_{l \in N^{\mathcal{I}}_{inter}} (\sum_{n \in J_{\psi(l)}} |a_n|^p)^{1/p} + \sum_{l \in N^{\mathcal{I}}_{inter}} (\sum_{n \in J_{\Psi(l)}} |a_n|)^q \right) \\
&\leq C \sum_{k} (\sum_{n \in J_k} |a_n|^p)^{1/p} = C \|a\|^{\mathcal{J}}_{p,1}.
\end{aligned}
$$

The converse implication is now complete. □

Corollary 2.20. *Let $1 \leq q < p \leq \infty$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. Then*

$$
\ell^{\mathcal{I}}(p,q) \subseteq \ell^{\mathcal{J}}(p,q) \iff \sup\{\#\tilde{G}_l; l \in \mathbb{N}_0\} < \infty.
$$

Corollary 2.21. *Let $1 \leq p, q \leq \infty$ with $p \neq q$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. Then*

$$\ell^{\mathcal{J}}(p,q) = \ell^{\mathcal{I}}(p,q) \Longleftrightarrow \sup\{(\#\tilde{F}_k)(\#\tilde{G}_l); k, l \in \mathbb{N}_0\} < \infty.$$

Proof. It suffices to show the case $p < q$. Note that $\ell^{\mathcal{I}}(p,q) \subseteq \ell^{\mathcal{J}}(p,q)$ and $\ell^{\mathcal{J}}(p,q) \subseteq \ell^{\mathcal{I}}(p,q)$ are equivalent, due to Theorem 2.17 and Corollary 2.18, to the facts $sup_k(\#\tilde{F}_k) < \infty$ and $sup_l(\#\tilde{G}_l) < \infty$, or equivalently

$$\sup\{(\#\tilde{F}_k)(\#\tilde{G}_l); k, l \in \mathbb{N}_0\} = \sup_k(\#\tilde{F}_k) \sup_l(\#\tilde{G}_l) < \infty.$$

□

3 Multipliers on generalized mixed-norm spaces

In this section we consider $1 \leq r, s, u, v \leq \infty$ and $\mathcal{I}, \mathcal{J}$ such that $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. We define

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) = \{\lambda = (\lambda_n)_{n \in \Lambda_{\mathcal{I}} \cap \Lambda_{\mathcal{J}}} : \|(\lambda_n a_n)_{n \in \Lambda_{\mathcal{J}}}\|^{\mathcal{J}}_{u,v} \leq C \|(a_n)_{n \in \Lambda_{\mathcal{I}}}\|^{\mathcal{I}}_{r,s}\}.$$

The case $\mathcal{I} = \mathcal{J}$ can be shown repeating the proof for $\mathcal{I} = \{I_k : k \in \mathbb{N}_0\}$ where $I_k = [2^k - 1, 2^{k+1} - 1) \cap \mathbb{N}_0$ (see [K, Theorem 1]).

Theorem 3.1. $(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{I}}(u,v)) = \ell^{\mathcal{I}}(u \ominus r, v \ominus s)$.

We define the Köthe dual $\ell^{\mathcal{I}}(p,q)^K = (\ell^{\mathcal{I}}(p,q), \ell^{\mathcal{I}}(1,1))$.

Corollary 3.2. $\ell^{\mathcal{I}}(r,s)^K = \ell^{\mathcal{I}}(r', s')$.

There are some other cases where the set of multipliers can be easily determined. Using Proposition 2.5 and Corollary 2.21 one easily obtains the following results.

Proposition 3.3.
(i) If $\sup_{k \in \mathbb{N}_0} \#J_k < \infty$ then $(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) = \ell^{\mathcal{I}}(v \ominus r, v \ominus s)$.
(ii) If $\sup_{l \in \mathbb{N}_0} \#I_l < \infty$ then $(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) = \ell^{\mathcal{J}}(u \ominus s, v \ominus s)$.
(iii) If $\sup\{(\#\tilde{F}_k)(\#\tilde{G}_l); k, l \in \mathbb{N}_0\} < \infty$ then

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) = \ell^{\mathcal{J}}(u \ominus r, v \ominus s) = \ell^{\mathcal{I}}(u \ominus r, v \ominus s).$$

Also as a direct consequence of Theorem 2.17 we obtain:

Proposition 3.4. *If $r \leq u$, $s \leq v$ and $u < v$ and $\sup\{\#\tilde{F}_k; k \in \mathbb{N}_0\} < \infty$ then*

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) = \{(\lambda_n)_{n \in \Lambda_{\mathcal{I}}} : \sup_n |\lambda_n| < \infty\}.$$

Proof. It is obvious that if $(\lambda_n)_{n\in\Lambda_{\mathcal{I}}}$ is a multiplier needs to be a bounded sequence. Note that the inclusion

$$\{(\lambda_n)_{n\in\Lambda_{\mathcal{I}}} : \sup_n |\lambda_n| < \infty\} \subseteq (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$$

is equivalent to $\ell^{\mathcal{I}}(r,s) \subseteq \ell^{\mathcal{J}}(u,v)$. Now use the embedding $\ell^{\mathcal{I}}(r,s) \subseteq \ell^{\mathcal{I}}(u,v)$ and Theorem 2.17 to conclude the result. □

Definition 3.5. If $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. We define the collection of pairwise disjoint intervals in $\mathbb{N}_0$

$$\widetilde{\mathcal{I}\cap\mathcal{J}} = \{I_l \cap J_k : k \in \mathbb{N}_0, l \in \tilde{F}_k\}.$$

It coincides with $\{I_l \cap J_k : l \in \mathbb{N}_0, k \in \tilde{G}_l\}$.

Proposition 3.6. *Let* $1 \le r, s, u, v \le \infty$.

(i) If $r \le s, v \le u$ *then* $(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) \subseteq \ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(u \ominus r, v \ominus s)$.
In particular, if $\sup_k \#\tilde{F}_k < \infty$ *then*

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) \subseteq \ell^{\mathcal{J}}(u \ominus r, v \ominus s).$$

(ii) If $s \le r, u \le v$ *then* $\ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(u \ominus r, v \ominus s) \subseteq (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$.
In particular, if $\sup_l \#\tilde{G}_l < \infty$ *then*

$$\ell^{\mathcal{I}}(u \ominus r, v \ominus s) \subseteq (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)).$$

Proof. (i) Note that $\widetilde{\mathcal{I}\cap\mathcal{J}} \le \mathcal{I}$ and $\widetilde{\mathcal{I}\cap\mathcal{J}} \le \mathcal{J}$. Hence, from Proposition 2.10,

$$\ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(p,q) \subseteq \ell^{\mathcal{I}}(p,q), p \ge q \tag{9}$$

and

$$\ell^{\mathcal{J}}(p,q) \subseteq \ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(p,q), p \le q. \tag{10}$$

Now using (9), (10) and Theorem 3.1 we obtain

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) \subseteq (\ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(r,s), \ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(u,v)) = \ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(u \ominus r, v \ominus s).$$

Also we have

$$F_k(\widetilde{\mathcal{I}\cap\mathcal{J}}, \mathcal{J}) = \{(k,l) : l \in \tilde{F}_k\}$$

and

$$F_l(\widetilde{\mathcal{I}\cap\mathcal{J}}, \mathcal{I}) = \{(k,l) : k \in \tilde{G}_l\}.$$

Using now Theorem 2.11

$$\ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(p,q) = \ell^{\mathcal{J}}(p,q) \Longleftrightarrow \sup_k \#\tilde{F}_k < \infty. \tag{11}$$

$$\ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(p,q) = \ell^{\mathcal{I}}(p,q) \Longleftrightarrow \sup_l \#\tilde{G}_l < \infty. \tag{12}$$

The particular case follows now applying (11).

(ii) is similar and left to the reader. □

Our purpose is to get a final description of multipliers $(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$. We shall deal first with the case $\mathcal{I} \leq \mathcal{J}$ and get a reduction to this situation in the remaining cases.

3.1 The case $\mathcal{I} \leq \mathcal{J}$

In this section we consider $\mathcal{I}$ and $\mathcal{J}$ such that $\mathbb{N}_0 = N^{\mathcal{J}}_{big} \cup N^{\mathcal{J}}_{equal}$. This means that $\tilde{F}_k = F_k \neq \emptyset$ and $J_k = \cup_{l\in F_k} I_l$ for all k. Notice that if $l \in F_k$ means $I_l \subseteq J_k$ and we have

$$F_k = \{l \in \mathbb{N}_0 : \phi(k) \leq l \leq \Phi(k)\}.$$

We use the notation $\mathcal{J}/\mathcal{I} = \{F_k : k \in \mathbb{N}_0\}$.

We shall need the following well known fact.

Lemma 3.7. *Let $0 < u, r \leq \infty$, $A \subseteq \mathbb{N}_0$ and $(\lambda_i)_{i\in A}$. There exists $(a_i)_{i\in A}$ such that*

$$(\sum_{i\in A} |a_i|^r)^{1/r} = 1 \text{ and } (\sum_{i\in A} |\lambda_i|^{u\ominus r})^{1/u\ominus r} = (\sum_{i\in A} |a_i\lambda_i|^u)^{1/u}$$

(with the obvious modifications whenever u, r or $u \ominus r$ equals ∞.)

Proof. For $r = \infty$ (then $u \ominus r = u$) it suffices to take $a_i = 1, i \in A$.

If $r < \infty$ and $u \geq r$ (hence $u \ominus r = \infty$) it suffices to take

$$a_i = \begin{cases} 1 & i = i(A) \\ 0 & otherwise \end{cases}$$

for $i(A)$ such that $\sup_{i\in A} |\lambda_i| = |\lambda_{i(A)}|$.

If $u < r < \infty$ take

$$a_i = (\sum_{i\in A} |\lambda_i|^{u\ominus r})^{-1/r} \lambda_i^{u\ominus r/r}, i \in A.$$

Using that $1 + \frac{u\ominus r}{r} = \frac{u\ominus r}{u}$ one shows the result. □

Theorem 3.8. *If $\mathcal{I} \leq \mathcal{J}$ then*

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) = \{(\lambda_n)_n : \Big((\sum_{i\in I_l} |\lambda_i|^{u\ominus r})^{1/u\ominus r} \Big)_l \in \ell^{\mathcal{J}/\mathcal{I}}(u\ominus s, v\ominus s)\}.$$

Proof. $\subseteq$) Assume that $(\lambda_n)_n \in (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$.

We use Lemma 3.7 with $A = I_l$ to select for each for $l \in \mathbb{N}_0$, a sequence $(a_i^{(l)})_{i\in I_l}$ such that $(\sum_{i\in I_l} |a_i^{(l)}|^r)^{1/r} = 1$ and $\beta_l = (\sum_{i\in I_l} |\lambda_i|^{u\ominus r})^{1/u\ominus r} = (\sum_{i\in I_l} |\lambda_i a_i^{(l)}|^u)^{1/u}$.

Now, again use Lemma 3.7 with $A = F_k$ for each $k \in \mathbb{N}_0$, to choose $(\alpha_l)_{l\in F_k}$ verifying $(\sum_{l\in F_k} |\alpha_l|^s)^{1/s} = 1$ and $(\sum_{l\in F_k} \beta_l^{u\ominus s})^{1/u\ominus s} = (\sum_{l\in F_k} |\beta_l \alpha_l|^u)^{1/u}$.

Finally, using Lemma 3.7 for $A = \mathbb{N}_0$, one more time, take $\gamma = (\gamma_k)_k$ verifying $(\sum_k |\gamma_k|^s)^{1/s} = 1$ and $\big(\sum_k \{\sum_{l\in F_k} \beta_l^{u\ominus s}\}^{v\ominus s/u\ominus s}\big)^{1/v\ominus s} = \big(\sum_k \gamma_k^v \{\sum_{l\in F_k} \beta_l^{u\ominus s}\}^{v/u\ominus s}\big)^{1/v}$.

This procedure allows us to obtain the sequence $a = (a_i)_i$, $a_i = \gamma_k \alpha_l a_i^{(l)}$ where $i \in I_l$, $l \in F_k$ and $k \in \mathbb{N}_0$. With this choice we get that $\|a\|_{r,s}^{\mathcal{I}} = 1$ and

$$\|\beta\|_{u\ominus s, v\ominus s}^{\mathcal{J}/\mathcal{I}} = \|\lambda * a\|_{u,v}^{\mathcal{J}} \leq \|\lambda\|.$$

$\supseteq$) Let $a = (a_i)_i \in \ell^{\mathcal{I}}(r,s)$ and $\lambda = (\lambda_i)_i$ such that $(\beta_l)_l \in \ell^{\mathcal{J}/\mathcal{I}}(u\ominus s, v\ominus s)$ where

$$\beta_l = (\sum_{i\in I_l} |\lambda_i|^{u\ominus r})^{1/u\ominus r}.$$

Fix $k \in \mathbb{N}_0$

$$\begin{aligned}
\left(\sum_{i\in J_k} |\lambda_i a_i|^u\right)^{1/u} &= \left(\sum_{l\in F_k}\sum_{i\in I_l} |\lambda_i a_i|^u\right)^{1/u} \\
&\leq \left(\sum_{l\in F_k} (\sum_{i\in I_l} |\lambda_i|^{u\ominus r})^{u/u\ominus r} (\sum_{i\in I_l} |a_i|^r)^{\frac{u}{r}}\right)^{1/u} \\
&\leq \left(\sum_{l\in F_k} (\sum_{i\in I_l} |\lambda_i|^{u\ominus r})^{u\ominus s/u\ominus r}\right)^{1/u\ominus s} \left(\sum_{l\in F_k} (\sum_{i\in I_l} |a_i|^r)^{s/r}\right)^{1/s}
\end{aligned}$$

Taking the v-norm, we get to:

$$\left(\sum_k (\sum_{i\in J_k} \lambda_i a_i|^u)^{\frac{v}{u}}\right)^{1/v} \le \left(\sum_k \{\sum_{l\in F_k} \beta_l^{u\ominus s}\}^{v/u\ominus s}\{\sum_{l\in F_k}(\sum_{i\in I_l}|a_i|^r)^{s/r}\}^{v/s}\right)^{1/v}$$

$$\le \left(\sum_k \{\sum_{l\in F_k} \beta_l^{u\ominus s}\}^{v\ominus s/u\ominus s}\right)^{1/v\ominus s} \left(\sum_k \{\sum_{l\in F_k}(\sum_{i\in I_l}|a_i|^r)^{s/r}\}^{s/s}\right)^{1/s}$$

$$= \left(\sum_k \{\sum_{l\in F_k} \beta_l^{u\ominus s}\}^{v\ominus s/u\ominus s}\right)^{1/v\ominus s} \left(\sum_l \{\sum_{i\in I_l}|a_i|^r\}^{s/r}\right)^{1/s}$$

Hence $(\lambda_n)_n \in (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$ and $\|\lambda\| \le \|\beta\|^{\mathcal{J}/\mathcal{I}}_{u\ominus s, v\ominus s}$.

□

Corollary 3.9. *Let $\mathcal{J} \le \mathcal{I}$ and $1 \le r,s,u,v \le \infty$. Then*

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) = \{(\lambda_n)_n : \left((\sum_{i\in J_k}|\lambda_i|^{u\ominus r})^{1/u\ominus r}\right)_k \in \ell^{\mathcal{I}/\mathcal{J}}(v\ominus r, v\ominus s)\}.$$

Proof. Recall that $\tilde{G}_l = G_l = \{k \in \mathbb{N}_0 : J_k \subseteq I_l\}$ and $I_l = \cup_{k\in G_l} J_k$: We now denote $\mathcal{I}/\mathcal{J} = \{G_l : l \in \mathbb{N}_0\}$. Using Köthe duals we actually have

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) = (\ell^{\mathcal{J}}(u',v'), \ell^{\mathcal{I}}(r',s')).$$

Taking into account that $p' \ominus q' = q \ominus p$ for all p,q the result follows from Theorem 3.8. □

3.2 The case $\widetilde{\mathcal{I}\cap\mathcal{J}} \subseteq \mathcal{I}\cup\mathcal{J}$

Let $\mathcal{I} = \{I_l : l \in \mathbb{N}_0\}$ and $\mathcal{J} = \{J_k : k \in \mathbb{N}_0\}$ such that $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. We assume in this section that $N^{\mathcal{I}}_{inter} = \emptyset$ and $N^{\mathcal{J}}_{inter} = \emptyset$, that is to say for a given $l \in \mathbb{N}_0$ either there exists k such that $I_l \subseteq J_k$ or there exist k' such that $J_{k'} \subseteq I_l$. In other words each interval in $\widetilde{\mathcal{I}\cap\mathcal{J}}$ belongs either to $\mathcal{I}$ or to $\mathcal{J}$.

To extend the result on multipliers to this setting we shall use the following lemma whose easy proof is left to the reader.

Lemma 3.10. *Let $\mathcal{I} = \{I_l : l \in \mathbb{N}_0\}$ and $\mathcal{J} = \{J_k : k \in \mathbb{N}_0\}$ such that $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$ and let $\mathcal{I}_i$ (respect. $\mathcal{J}_i$) for $i = 1, \cdots, m$ sub-collections of $\mathcal{I}$ (respect. $\mathcal{J}$) with $\mathcal{I} = \cup_{i=1}^m \mathcal{I}_i$ (respect. $\mathcal{J} = \cup_{i=1}^m \mathcal{J}_i$) satisfying $\Lambda_{\mathcal{I}_i} = \Lambda_{\mathcal{J}_i}$ for $i = 1, \cdots, m$. Then*

$$\lambda = (\lambda_n)_{n\in\Lambda_{\mathcal{I}}} \in (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$$

if and only if

$$\lambda^{(i)} = (\lambda_n)_{n\in\Lambda_{\mathcal{I}_i}} \in (\ell^{\mathcal{I}_i}(r,s), \ell^{\mathcal{J}_i}(u,v)), i = 1, \cdots, m.$$

Moreover $\|\lambda\| \approx \sum_{i=1}^m \|\lambda^{(i)}\|$.

Theorem 3.11. *Let* $\widetilde{\mathcal{I}\cap\mathcal{J}} \subseteq \mathcal{I}\cup\mathcal{J}$. *Then* $(\lambda_n)_n \in (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$ *if and only if it satisfies the conditions*

$$((\sum_{i\in J_k} |\lambda_i|^{u\ominus r})^{1/u\ominus r})_{k\in N^{\mathcal{J}}_{equal}} \in \ell^{v\ominus s}, \tag{13}$$

$$((\sum_{i\in I_l} |\lambda_i|^{u\ominus r})^{1/u\ominus r})_{l\in N^{\mathcal{I}}_{small}} \in \ell^{\mathcal{F}}(u\ominus s, v\ominus s), \tag{14}$$

.

$$((\sum_{i\in J_k} |\lambda_i|^{u\ominus r})^{1/u\ominus r})_{k\in N^{\mathcal{J}}_{small}} \in \ell^{\mathcal{G}}(v\ominus r, v\ominus s), \tag{15}$$

where $\mathcal{F} = \{F_k : k \in N^{\mathcal{J}}_{big}\}$ *and* $\mathcal{G} = \{G_l : l \in N^{\mathcal{I}}_{big}\}$.

Proof. Let us consider the following collection of intervals

$$\mathcal{J}_b = \{J_k : k \in N^{\mathcal{J}}_{big}\}, \quad \mathcal{J}_e = \{J_k : k \in N^{\mathcal{J}}_{equal}\}, \text{ and } \mathcal{J}_s = \{J_k : k \in N^{\mathcal{J}}_{small}\}$$

and similarly for $\mathcal{I}$.

If $J_k \in \mathcal{J}_b$ (respect. $I_l \in \mathcal{I}_b$) we have $F_k = \{l \in \mathbb{N}_0 : I_l \subsetneq J_k\} \neq \emptyset$ (respect. $G_l = \{k \in \mathbb{N}_0 : J_k \subsetneq I_l\} \neq \emptyset$) and

$$J_k = \cup_{l\in F_k} I_l, I_l \in \mathcal{I}_s \quad (\text{ respect. } I_l = \cup_{l\in G_l} J_k, J_k \in \mathcal{J}_s). \tag{16}$$

Hence $\mathcal{J} = \mathcal{J}_e \cup \mathcal{J}_b \cup \mathcal{J}_s$, $\mathcal{I} = \mathcal{I}_e \cup \mathcal{I}_b \cup \mathcal{I}_s$ and

$$\mathcal{J}_e = \{J_k : k \in N^{\mathcal{J}}_{equal}\} = \{I_l : l \in N^{\mathcal{I}}_{equal}\} = \mathcal{I}_e.$$

Observe that $\mathcal{I}_s \leq \mathcal{J}_b$ and $\mathcal{J}_s \leq \mathcal{I}_b$ and, in particular, $\mathcal{G} = \mathcal{I}_b/\mathcal{J}_s$ and $\mathcal{F} = \mathcal{J}_b/\mathcal{I}_s$.

We use Lemma 3.10 and observe that, denoting $\Lambda_0 = \Lambda_{\mathcal{J}_e}$, $\Lambda_1 = \Lambda_{\mathcal{J}_b} = \Lambda_{\mathcal{I}_s}$ and $\Lambda_2 = \Lambda_{\mathcal{J}_s} = \Lambda_{\mathcal{I}_b}$,

$$(\lambda_n)_{n\in\Lambda_0} \in (\ell^{\mathcal{J}_e}(r,s), \ell^{\mathcal{I}_e}(u,v))$$

corresponds to (13) invoking Theorem 3.1, also that

$$(\lambda_n)_{n\in\Lambda_1} \in (\ell^{\mathcal{I}_s}(r,s), \ell^{\mathcal{J}_b}(u,v))$$

corresponds to (14) invoking Theorem 3.8 and, finally,

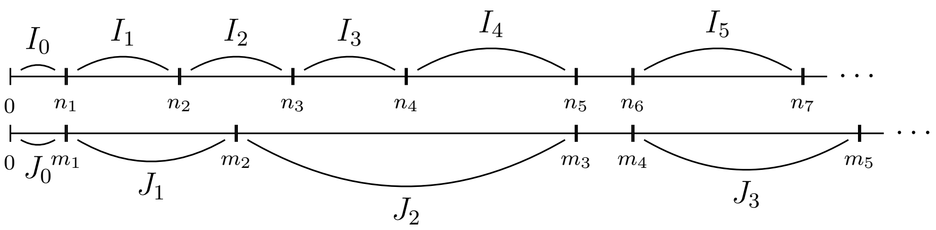
$$(\lambda_n)_{n\in\Lambda_2} \in (\ell^{\mathcal{I}_b}(r,s), \ell^{\mathcal{J}_s}(u,v))$$

corresponds to (15) invoking Corollary 3.9. □

3.3 The general case

In this section we assume that there exist $k \in \mathbb{N}_0$ and $l \in \tilde{F}_k$ such that $I_l \cap J_k \in \widetilde{\mathcal{I} \cap \mathcal{J}}$ and $I_l \cap J_k \notin \mathcal{I} \cup \mathcal{J}$.

Since the notation may be a bit confusing, we will illustrate the idea. Let $\mathcal{I}, \mathcal{J}$ be different partitions of $\mathbb{N}_0$, for example:



The situation we are handling now corresponds to $N^{\mathcal{J}}_{inter} \neq \emptyset$ (and hence $N^{\mathcal{I}}_{inter} \neq \emptyset$).

Definition 3.12.

$$\mathcal{J}' = \{J'_k = \cup_{l \in F_k} I_l : k \in \mathbb{N}_0, \#F_k > 0\},$$

$$\mathcal{H} = \widetilde{\mathcal{I} \cap \mathcal{J}} \setminus (\mathcal{I} \cup \mathcal{J}),$$

$$\mathcal{J}_s = \{J_k : k \in N^{\mathcal{J}}_{small}\}.$$

Denote $\mathcal{J}'' = \mathcal{J}' \cup \mathcal{J}_s$ and $\mathcal{J}_{new} = \mathcal{J}'' \cup \mathcal{H}$.

We use similar notations for $\mathcal{I}$.

Recalling that $\phi(k) = \min \tilde{F}_k$ and $\Phi(k) = \max \tilde{F}_k$ for $k \in \mathbb{N}_0$. We easily observe that $\phi(N^{\mathcal{J}}_{equal}) \subseteq N^{\mathcal{I}}_{equal}$, $\phi(N^{\mathcal{J}}_{big}) \subseteq N^{\mathcal{I}}_{small}$, $\phi(N^{\mathcal{J}}_{small}) \subseteq N^{\mathcal{I}}_{big} \cup N^{\mathcal{I}}_{inter}$ and $\phi(N^{\mathcal{J}}_{inter}) \subseteq N^{\mathcal{I}}_{small} \cup N^{\mathcal{I}}_{inter}$. Same results hold for Φ.

Lemma 3.13.

$$\mathcal{H} = \{\hat{J}_k : k \in N^{\mathcal{J}}_{inter}, \phi(k) \in N^{\mathcal{I}}_{inter}\} \cup \{\check{J}_k : k \in N^{\mathcal{J}}_{inter}, \Phi(k) \in N^{\mathcal{I}}_{inter}\}.$$

Proof. $\subseteq$) Let $I \in \mathcal{H}$. Since $I \in \widetilde{\mathcal{I} \cap \mathcal{J}}$ then there exist $k \in \mathbb{N}_0$ and $l \in \tilde{F}_k$ such that $I = I_l \cap J_k$. On the other hand, since $I \notin \mathcal{I} \cup \mathcal{J}$ we have that $I \subsetneq I_l$ and $I \subsetneq J_k$. Hence either $\phi(k) = l$ and $\Psi(l) = k$ or $\Phi(k) = l$ and $\psi(l) = k$. This gives either $k \in N^{\mathcal{J}}_{inter}$ and $\phi(k) \in N^{\mathcal{I}}_{inter}$ (and hence $I = \hat{J}_k$) or $k \in N^{\mathcal{J}}_{inter}$ and $\Phi(k) \in N^{\mathcal{I}}_{inter}$ (and hence $I = \check{J}_k$).

$\supseteq$) Let $k \in N^{\mathcal{J}}_{inter}$ with $\phi(k) \in N^{\mathcal{I}}_{inter}$ and consider $\hat{J}_k = J_k \cap I_{\phi(k)} \in \widetilde{\mathcal{I} \cap \mathcal{J}}$. Then $\hat{J}_k \subsetneq J_k$ (hence $\hat{J}_k \notin \mathcal{J}$) and $\hat{J}_k \subsetneq I_\phi(k)$ (hence $\hat{J}_k \notin \mathcal{I}$). Similarly for $\check{J}_k$ in the case $k \in N^{\mathcal{J}}_{inter}$ with $\Phi(k) \in N^{\mathcal{I}}_{inter}$

□

Remark 3.14. Note that $\hat{J}_k = J_k \cap I_l$ if and only if $\check{I}_l = I_l \cap J_k$. Therefore

$$\mathcal{H} = \{\hat{I}_l : l \in N^{\mathcal{I}}_{inter}, \psi(l) \in N^{\mathcal{J}}_{inter}\} \cup \{\check{I}_l : k \in N^{\mathcal{I}}_{inter}, \Psi(l) \in N^{\mathcal{J}}_{inter}\}.$$

Lemma 3.15.

$$\widetilde{\mathcal{I}'' \cap \mathcal{J}''} \subseteq \mathcal{I}_s \cup \mathcal{J}_s \cup \mathcal{I}_e \subseteq \mathcal{I}'' \cup \mathcal{J}''.$$

Proof. Let $I \in \mathcal{I}' \cup \mathcal{I}_s$ and $J \in \mathcal{J}' \cup \mathcal{J}_s$ with $I \cap J \neq \emptyset$. The case $I \in \mathcal{I}_s$ and $J \in \mathcal{J}_s$ can not hold. If $I \in \mathcal{I}_s$ and $J \in \mathcal{J}'$ then $I \cap J = I \in \mathcal{I}_s$. Similarly if $I \in \mathcal{I}'$ and $J \in \mathcal{J}_s$ then $I \cap J = J \in \mathcal{J}_s$. Finally if $I \in \mathcal{I}'$ and $J \in \mathcal{J}'$ then $I = J \in \mathcal{I}_e = \mathcal{J}_e$. $\square$

Theorem 3.16. *$\lambda \in (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$ if and only if $(\lambda_n)_n$ satisfies*

$$\Big((\sum_{i \in J_k} |\lambda_i|^{u \ominus r})^{1/u \ominus r} \Big)_{k \in N^{\mathcal{J}}_{equal}} \in \ell^{v \ominus s} \tag{17}$$

$$\Big((\sum_{i \in I_l} |\lambda_i|^{u \ominus r})^{1/u \ominus r} \Big)_{l \in N^{\mathcal{I}}_{small}} \in \ell^{\mathcal{F}}(u \ominus s, v \ominus s) \tag{18}$$

$$\Big((\sum_{i \in J_k} |\lambda_i|^{u \ominus r})^{1/u \ominus r} \Big)_{k \in N^{\mathcal{J}}_{small}} \in \ell^{\mathcal{G}}(v \ominus r, v \ominus s) \tag{19}$$

$$\Big((\sum_{i \in \check{J}_k} |\lambda_i|^{u \ominus r})^{1/u \ominus r} \Big)_{k \in \Lambda_r} + \Big((\sum_{i \in \hat{J}_k} |\lambda_i|^{u \ominus r})^{1/u \ominus r} \Big)_{k \in \Lambda_l} \in \ell^{v \ominus s} \tag{20}$$

where

$$\Lambda_r = \{k \in N^{\mathcal{J}}_{inter}, \Phi(k) \in N^{\mathcal{I}}_{inter}\} \textit{ and } \Lambda_l = \{k \in N^{\mathcal{J}}_{inter}, \phi(k) \in N^{\mathcal{I}}_{inter}\},$$

$$\mathcal{G} = \{G_l : l \in N^{\mathcal{I}}_{big} \cup N^{\mathcal{I}}_{inter}, \#G_l > 0\}$$

and

$$\mathcal{F} = \{F_k : k \in N^{\mathcal{J}}_{big} \cup N^{\mathcal{J}}_{inter}, \#F_k > 0\}.$$

Proof. Using $J_k = J'_k \cup \hat{J}_k \cup \check{J}_k$ and Lemma 3.13 one obtains $\mathcal{J}_{new} \leq \mathcal{J}$ and $\mathcal{I}_{new} \leq \mathcal{I}$. Clearly $\#F_l(\mathcal{I}_{new}, \mathcal{I}) \leq 3$ and $\#F_k(\mathcal{J}_{new}, \mathcal{J}) \leq 3$ for all k. Therefore, using Proposition 2.11, we have $\ell^{\mathcal{J}_{new}}(p,q) = \ell^{\mathcal{J}}(p,q)$ and $\ell^{\mathcal{I}_{new}}(p,q) = \ell^{\mathcal{I}}(p,q)$, which gives

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) = (\ell^{\mathcal{I}_{new}}(r,s), \ell^{\mathcal{J}_{new}}(u,v)). \tag{21}$$

Taking into account Lemma 3.13 and Remark 3.14 we observe that $\Lambda_{\mathcal{H}} = \Lambda_r \cup \Lambda_l$ and $\Lambda_{\mathcal{I}''} = \Lambda_{\mathcal{J}''}$.

Since $\mathcal{J}_{new} = \mathcal{J}'' \cup \mathcal{H}$ and $\mathcal{I}_{new} = \mathcal{I}'' \cup \mathcal{H}$ we can apply Lemma 3.10 to conclude that $\lambda \in (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$ if and only if $(\lambda_n)_{n\in\Lambda_{\mathcal{H}}} \in (\ell^{\mathcal{H}}(r,s), \ell^{\mathcal{H}}(u,v))$ and $(\lambda_n)_{n\notin\Lambda_{\mathcal{H}}} \in (\ell^{\mathcal{I}''}(r,s), \ell^{\mathcal{J}''}(u,v))$.

Now apply Theorem 3.1 to obtain $(\lambda_n)_{n\in\Lambda_{\mathcal{H}}} \in \ell^{\mathcal{H}}(u \ominus r, v \ominus s)$ which corresponds to (20).

On the other hand, comparing $\mathcal{I}''$ and $\mathcal{J}''$ we notice that $I \in \mathcal{I}''_{big}$ corresponds to $I = I'_l$ for some $l \in N^{\mathcal{I}}_{big} \cup N^{\mathcal{I}}_{inter}$ and $\#G_l \geq 1$. Hence we obtain that $\mathcal{G} = \{G_I : I \in \mathcal{I}''_{big}\}$ and similarly $\mathcal{F} = \{F_J : J \in \mathcal{J}''_{big}\}$.

We now use Lemma 3.15 together with Theorem 3.11 to obtain the equivalence with (17), (18) and (19) and $(\lambda_n)_{n\notin\Lambda_{\mathcal{H}}} \in (\ell^{\mathcal{I}''}(r,s), \ell^{\mathcal{J}''}(u,v))$.

□

4 An application

Let $\rho : [0,1) \to [0,\infty)$ be a non-decreasing function such that $\rho(0) = 0$ and $\rho(t)/t \in L^1([0,1))$ we define the weighted Bergman-Besov space $B^1(\rho)$ as those analytic functions F in the unit disk such that

$$\int_{\mathbb{D}} |F'(z)| \frac{\rho(1-|z|)}{1-|z|} dA(z) < \infty.$$

An analytic function F is called lacunary if $F(z) = \sum_{n\in\Lambda_{\mathcal{L}}} a_n z^n$ where $\mathcal{L} = \{\{n_k\} : k \in \mathbb{N}_0\}$ for some (n_k) such that $\inf_k n_{k+1}/n_k > 1$.

Recently weights with the following condition had been considered in [KD-SAA]: There exist $C_1, C_2 > 0$ such that

$$C_1 \int_0^1 r^{2^n - 1} \frac{\rho(1-r)}{1-r} dr \leq K(n,\rho) \leq C_2 \int_{1-2^{-n}}^{1-2^{-(n+1)}} r^{2^{n+1}-1} \frac{\rho(1-r)}{1-r} dr \qquad (22)$$

and the following result has been shown.

Theorem 4.1. *(see [KD-SAA]) Let $F(z) = \sum_{n\in\Lambda_{\mathcal{L}}} a_n z^n$ be a lacunary function and let ρ be a weight satisfying (22). Then F belongs to $B^1(\rho)$ if and only if*

$$\sum_{k=0}^{\infty} (\sum_{n\in J_k} |a_n|^2)^{1/2} 2^k K(k,\rho) < \infty \qquad (23)$$

where $J_k = \{n : 2^k - 1 \leq k < 2^{k+1} - 1\}$.

We shall extend the previous result for more general classes of weight functions and families of intervals $\mathcal{J}$.

Definition 4.2. Let $0 < q < \infty$, $\mathcal{J}$ be a collection of disjoint intervals in $\mathbb{N}_0$, say $J_k = \mathbb{N}_0 \cap [m_k, m_{k+1})$ where $m_0 = 0$ and (m_k) is some increasing sequence in $\mathbb{N}_0$. and let $\rho : [0,1) \to [0,\infty)$ be a measurable function such that $\rho(t)/t \in L^1([0,1))$.

We say that ρ is q-adapted to $\mathcal{J}$ whenever there exists $C > 0$ such that

$$\int_0^1 r^{qm_n} \frac{\rho(1-r)}{1-r} dr \leq C \int_{A_n} r^{qm_{n+1}} \frac{\rho(1-r)}{1-r} dr \tag{24}$$

for all $n \geq 0$ where $A_0 = [0, 1 - \frac{1}{m_1})$ and $A_n = [1 - \frac{1}{m_n}, 1 - \frac{1}{m_{n+1}})$ for $n \geq 1$.

We denote

$$\mu_\rho(s) = \int_0^1 r^s \frac{\rho(1-r)}{1-r} dr, s \geq 0. \tag{25}$$

In particular, from condition (24) if ρ is q-adapted to $\mathcal{J}$ we get that

$$\mu_\rho(qm_n) \approx \mu_\rho(qm_{n+1}) \tag{26}$$

Note also that condition (22) means that ρ is 1/2-adapted for $\mathcal{J}$ where $m_n = 2^n - 1$.

Proposition 4.3. *Let $\rho_\alpha(t) = t^\alpha$ with $\alpha > 0$ and $\mathcal{J} = \{[m_n, m_{n+1}) \cap \mathbb{N}_0 : n \in \mathbb{N}_0\}$. The following statements are equivalent:*

(i) ρ_α is q-adapted to $\mathcal{J}$ for all $q > 0$.

(ii) ρ_α is q-adapted to $\mathcal{J}$ for some $q > 0$.

(iii) $\sup_n m_{n+1}/m_n < \infty$.

Proof. $(i) \Longrightarrow (ii)$ Obvious.

$(ii) \Longrightarrow (iii)$ It is well known that $B(n+1, \alpha) = \int_0^1 r^n (1-r)^{\alpha-1} dr \approx n^{-\alpha}$ and therefore $\mu_{\rho_\alpha}(qm_n) \approx m_n^{-\alpha}$.

Hence it follows from (26) that $m_{n+1} \approx m_n$. therefore $\sup m_{n+1}/m_n < \infty$.

$(iii) \Longrightarrow (i)$ Let $\sup m_{n+1}/m_n = \delta$ and take $q > 0$. Now observe that

$$\begin{aligned}
\int_{1-\frac{1}{m_n}}^{1-\frac{1}{m_{n+1}}} r^{qm_{n+1}} (1-r)^{\alpha-1} dr &\geq (1 - \frac{1}{m_n})^{qm_{n+1}} \int_{\frac{1}{m_{n+1}}}^{\frac{1}{m_n}} s^{\alpha-1} ds \\
&\geq \frac{1}{\alpha} (1 - \frac{1}{m_n})^{qm_{n+1}} m_n^{-\alpha} (1 - (\frac{m_n}{m_{n+1}})^\alpha) \\
&\geq \frac{1}{\alpha} ((1 - \frac{1}{m_n})^{m_n})^{\delta q} m_n^{-\alpha} (1 - (\frac{1}{\delta})^\alpha) \\
&\geq C \mu_{\rho_\alpha}(qm_n).
\end{aligned}$$

□

We now modify the proof of Lemma 3 in [B2] to obtain the following result.

Lemma 4.4. *Let $0 < q \le 1$, let $\mathcal{J}$ be a collection of disjoint intervals in $\mathbb{N}_0$ and assume ρ is a weight q-adapted to $\mathcal{J}$. If $(\alpha_n) \ge 0$ then*

$$\int_0^1 (\sum_{n=0}^{\infty} \alpha_n r^n)^q \frac{\rho(1-r)}{1-r} dr \approx \sum_{n=0}^{\infty} (\sum_{k \in J_n} \alpha_k)^q \mu_\rho(qm_n)$$

where $J_n = \{k : m_n \le k < m_{n+1}\}$

Proof. As above $A_0 = [0, 1 - \frac{1}{m_1})$ and $A_n = [1 - \frac{1}{m_n}, 1 - \frac{1}{m_{n+1}})$ for $n \ge 1$. Then

$$\begin{aligned}
\int_0^1 (\sum_{n=0}^{\infty} \alpha_n r^n)^q \frac{\rho(1-r)}{1-r} dr &= \sum_{n=0}^{\infty} \int_{A_n} (\sum_{n=0}^{\infty} \alpha_n r^n)^q \frac{\rho(1-r)}{1-r} dr \\
&\ge \sum_{n=0}^{\infty} \int_{A_n} (\sum_{k \in J_n} \alpha_k r^k)^q \frac{\rho(1-r)}{1-r} dr \\
&\ge \sum_{n=0}^{\infty} \int_{A_n} (\sum_{k \in J_n} \alpha_k)^q r^{qm_{n+1}} \frac{\rho(1-r)}{1-r} dr \\
&\ge C^{-1} \sum_{n=0}^{\infty} (\sum_{k \in J_n} \alpha_k)^q \mu_\rho(qm_n).
\end{aligned}$$

Conversely, since $q \le 1$,

$$\begin{aligned}
\int_0^1 (\sum_{n=0}^{\infty} \alpha_n r^n)^q \frac{\rho(1-r)}{1-r} dr &\le \int_0^1 \sum_{n=0}^{\infty} (\sum_{k \in J_n} \alpha_k r^k)^q \frac{\rho(1-r)}{1-r} dr \\
&\le \sum_{n=0}^{\infty} (\sum_{k \in J_n} \alpha_k)^q (\int_0^1 r^{qm_n} \frac{\rho(1-r)}{1-r} dr) \\
&\le \sum_{n=0}^{\infty} (\sum_{k \in J_n} \alpha_k)^q \mu_\rho(qm_n).
\end{aligned}$$

$\square$

We first note that for lacunary functions F and $0 < p < \infty$ we have (see [Z])

$$M_p(F, r) = (\int_0^{2\pi} |F(re^{i\theta})|^p \frac{d\theta}{2\pi})^{1/p} \approx M_2(F, r) = (\int_0^{2\pi} |F(re^{i\theta})|^2 \frac{d\theta}{2\pi})^{1/2}.$$

Therefore for lacunary functions F one has that $F \in B^1(\rho)$ if and only if

$$\int_0^1 M_2(F', r)\frac{\rho(1-r)}{1-r}dr < \infty.$$

Therefore invoking Plancherel's theorem and Lemma 4.4 we recover Theorem 4.1.

Recall that an analytic function $F : \mathbb{D} \to \mathbb{C}$ with $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is said to belong to $H(p, q, \rho)$ (see [B2, Definition 2]) whenever

$$\|F\|_{H(p,q,\rho)} = (\int_0^1 M_p^q(F, r)\frac{\rho(1-r^2)}{1-r^2}rdr)^{1/q} < \infty.$$

We use the notation $H(p, q, \alpha)$ if $\rho(t) = t^\alpha$.

A consequence of Lemma 4.4 is the following result.

Corollary 4.5. *Let $0 < q \leq 2$, let $\mathcal{J}$ be a collection of disjoint intervals in $\mathbb{N}_0$ and ρ be a weight $q/2$-adapted to $\mathcal{J}$. Then*

$$\|F\|_{H(2,q,\rho)} \approx \Big(\sum_{n=0}^{\infty}(\sum_{k\in J_n} |a_k|^2)^{q/2}\mu_\rho((qm_n)/2)\Big)^{1/q}.$$

Moreover if F is lacunary and $0 < p < \infty$ then

$$\|F\|_{H(p,q,\rho)} \approx \Big(\sum_{n=0}^{\infty}(\sum_{k\in J_n\cap\Lambda_{\mathcal{L}}} |a_k|^2)^{q/2}\mu_\rho((qm_n)/2)\Big)^{1/q}.$$

Theorem 4.6. *Let $0 < q < \infty$, let $\mathcal{J}$ be a collection of disjoint intervals in $\mathbb{N}_0$ and assume ρ is a weight q-adapted to $\mathcal{J}$. Define $\lambda = (\lambda_k)_k$ such that*

$$\lambda_k = (\int_0^1 r^{qm_n}\frac{\rho(1-r)}{1-r})^{1/q}, k \in J_n$$

and $\lambda_k = 0$ otherwise. Then $(\lambda_k)_k \in (H(1, q, \rho), \ell^{\mathcal{J}}(\infty, q))$.

Proof. We shall show that

$$\Big(\sum_{n=0}^{\infty}(\sup_{k\in J_n} |a_k|)^q\mu_\rho(qm_n)\Big)^{1/q} \leq C\|F\|_{H(1,q,\rho)}.$$

Recall that

$$\sup_{k\in J_{n-1}} |a_k|r^k \leq M_1(F, r)$$

and therefore, if $A_0 = [0, 1 - \frac{1}{m_1})$ and $A_n = [1 - \frac{1}{m_n}, 1 - \frac{1}{m_{n+1}})$ for $n \geq 1$ then

$$\begin{aligned}
\sum_{n=0}^{\infty} (\sup_{k \in J_n} |a_k|)^q \mu_\rho(qm_n) &\leq C \sum_{n=0}^{\infty} (\sup_{k \in J_n} |a_k|)^q \int_{A_n} r^{qm_{n+1}} \frac{\rho(1-r)}{1-r} dr \\
&\leq C \sum_{n=0}^{\infty} \int_{A_n} (\sup_{k \in J_n} |a_k| r^k)^q \frac{\rho(1-r)}{1-r} dr \\
&\leq C \sum_{n=0}^{\infty} \int_{A_n} M_1^q(F, r) \frac{\rho(1-r)}{1-r} dr \\
&= C \|F\|^q_{H(1,q,\rho)}.
\end{aligned}$$

□

Theorem 4.7. *Let $1 \leq q_2 < q_1 \leq 2$ and let $\mathcal{J}$ and $\mathcal{I}$ be collections of disjoint intervals in $\mathbb{N}_0$, generated by sequences m_k and n_k respectively, such that $\mathcal{I} \leq \mathcal{J}$. Assume that ρ_1 is a weight $q_1/2$-adapted to $\mathcal{I}$ and ρ_2 is a weight $q_2/2$-adapted to $\mathcal{J}$. Denote*

$$\mu_{\rho_1,\rho_2}(k) = \Big((\mu_{\rho_2}((q_2 m_k)/2))^{1/q_2} (\mu_{\rho_1}((q_1 n_k)/2))^{-1/q_1} \Big)^{1/q_2 \ominus q_1}$$

Then

$$(H(2, q_1, \rho_1), H(2, q_2, \rho_2)) = \{(\lambda_n)_n; (\sup_{k \in I_n} \mu_{\rho_1,\rho_2}(k)|\lambda_k|) \in \ell^{\mathcal{J}/\mathcal{I}}(\infty, q_2 \ominus q_1)\}.$$

Proof. Let

$$F_\mathcal{I}(z) = \sum_{k=0}^{\infty} (\mu_{\rho_1}(q_1 n_k/2))^{1/q_1} (\sum_{j \in I_k} z^j),$$

$$\tilde{F}_\mathcal{I}(z) = \sum_{k=0}^{\infty} (\mu_{\rho_1}(q_1 n_k/2))^{-1/q_1} (\sum_{j \in I_k} z^j),$$

and

$$G_\mathcal{J}(z) = \sum_{k=0}^{\infty} (\mu_{\rho_2}(q_2 m_k/2))^{1/q_2} (\sum_{j \in J_k} z^j)$$

Using Corollary 4.5 one has that $f \in H(2, q_1, \rho_1)$ if and only if $f * F_\mathcal{I} \in \ell^\mathcal{I}(2, q_1)$ and $g \in H(2, q_2, \rho_2)$ if and only if $g * G_\mathcal{J} \in \ell^\mathcal{J}(2, q_2)$

We use that $\lambda \in (H(2, q_1, \rho_1), H(2, q_2, \rho_2))$ is equivalent to $\lambda * G_\mathcal{J} \in (H(2, q_1, \rho_1), \ell^\mathcal{J}(2, q_2))$ and also equivalent to $\lambda * G_\mathcal{J} * \tilde{F}_\mathcal{I} \in (\ell^\mathcal{I}(2, q_1), \ell^\mathcal{J}(2, q_2))$.

Making use of Theorem 3.8 we have

$$(\ell^\mathcal{I}(2, q_1), \ell^\mathcal{J}(2, q_2)) = \{(\gamma_n)_n; (\sup_{k \in I_n} |\gamma_k|)_n \in \ell^{\mathcal{J}/\mathcal{I}}(\infty, q_2 \ominus q_1)\}.$$

This concludes the result. □

Let us finish by observing some examples to apply the above results.

Example 4.8. Let $\lambda > 1$ and denote $m_0(\lambda) = 0$ and $m_k(\lambda) = [\lambda^k]$ for $k \in \mathbb{N}_0$ and $\mathcal{J}(\lambda)$ the partition of intervals $J_k(\lambda) = [m_k(\lambda), m_{k+1}(\lambda)) \cap \mathbb{N}_0$. In this case $\mu_{\rho_\alpha}(qm_n) \approx \lambda^{-\alpha n}$, and then, from Proposition 4.3, ρ_α is q-adapted to $\mathcal{J}(\lambda)$ for any value of $q > 0$.

Let $\lambda > \gamma > 1$ with $\lambda = \gamma^N$ with $N \in \mathbb{N}_0$. Then $\mathcal{J}(\gamma) \leq \mathcal{J}(\lambda)$ because

$$m_k(\lambda) = [\lambda^k] = [\gamma^{Nk}] = m_{Nk}(\gamma)$$

and therefore

$$J_k(\lambda) = \cup_{l \in F_k} J_l(\gamma)$$

where $F_k = \{l : Nk \leq l < Nk + N\}$. Hence $\mathcal{J}(\lambda)/\mathcal{J}(\gamma) = \mathcal{I}$ where $I_k = [Nk, N(k+1)) \cap \mathbb{N}_0$, that is $m_k(\mathcal{I}) = Nk$.

References

[B1] O. Blasco *Operators on weighted Bergman spaces* $(0 < p \leq 1)$ *and applications*, Duke Math. J. 66, 3 (1992) 443-467.

[B2] O. Blasco *Multipliers on weighted Besov spaces of analytic functions*, Contemporary Mathematics 144,(1993) 23-33.

[B3] O. Blasco *Multipliers on spaces of analytic functions*, Canadian J. Math. 20, (1994) 1–21.

[CL] D.M. Campbell, R.J. Leach *A survey of Hp multipliers as related to classical function theory*, Complex Variables 3 (1984), 85-111.

[F1] T.M. Flett *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. and Appl. 38 (1972), 746-765.

[F2] T.M. Flett *Lipschitz spaces of functions on the circle and the disc*, J. Math. Anal. and Appl. 39 (1972), 125-158.

[F3] T.M. Flett *On the rate of growth of mean valeus of holomorphic and harmonic functions*, Proc. London Math. Soc. 20 (1970), 749-768.

[HL1] G.H. Hardy, J.E. Littlewood *Some properties of fractional integrals II*, Math. Z. 34 (1932), 403-439.

[HL2] G.H. Hardy, J.E. Littlewood *Theorems concerning mean values of analytic or harmonic functions*, Quart. J. Math. 12 (1941), 221-256.

[K] C. N. Kellogg, *An extension of the Hausdorff-Young theorem* , Michigan Math. J. 18, 2 (1971) 121-127.

[KD-SAA] E. Kwessi, G. De Souza, A. Abebe, R. Aulaskari *Characterization of lacunary functions in weighted Bergman-Besov-Lipschitz spaces*, Complex Variables and Elliptic equations 58, 2 (2013) 157-162.

[O] B. Osikiewicz *Multipliers of Hardy spaces*, Quaestiones Math. 27 (2004), 57-73.

[P1] M. Pavlovic *Mixed norm spaces of analtytic and harmonic functions I*, Publ. Inst. Math. (Beograd) (N.S.) 40(54) (1986), 117-141 .

[P2] M. Pavlovic *Mixed norm spaces of analtytic and harmonic functions II*, Publ. Inst. Math. (Beograd) (N.S.) 41(55) (1987), 97-110 .

[S1] W. Sledd *Some results about spaces of analytic functions introduced by Hardy and Littlewood*, J. London Math. Soc. 2 (1974), 328-336.

[S2] W. Sledd *On multipliers of H^p spaces*, Indiana Univ. Math. J. 27 (1978), 797-803.

[Z] A. Zygmund, *Trigonometric series, 3rd Ed* , Vols I and II. Cambrigde Math. Library, Cambrigde, UK (2002)

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