NOTES ON THE SPACES OF BILINEAR MULTIPLIERS

OSCAR BLASCO

Abstract. A locally integrable function $m(\xi,\eta)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$ is said to be a bilinear multiplier on $\mathbb{R}^n$ of type (p_1,p_2,p_3) if

$$B_m(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat f(\xi)\hat g(\eta)m(\xi,\eta)e^{2\pi i(\langle\xi+\eta,x\rangle}d\xi d\eta$$

defines a bounded bilinear operator from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$. The study of the basic properties of such spaces is investigated and several methods of constructing examples of bilinear multipliers are provided. The special case where $m(\xi,\eta)=M(\xi-\eta)$ for a given M defined on $\mathbb{R}^n$ is also addressed.

1. Introduction.

Throughout the paper $C_{00}(\mathbb{R}^n)$ denotes the space of continuous functions defined in $\mathbb{R}^n$ with compact support, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class on $\mathbb{R}^n$, i.e. $f:\mathbb{R}^n\to\mathbb{C}$ such that $f\in C^\infty(\mathbb{R}^n)$ and $x^\alpha\frac{\partial^{|\beta|}f(x)}{\partial x_1^{\beta_1}...\partial x_n^{\beta_n}}$ is bounded for any $\beta=(\beta_1,...,\beta_n)$ and $\alpha=(\alpha_1,...,\alpha_n)$ where $x^\alpha=x_1^{\alpha_1}...x_n^{\alpha_n}$ and $|\beta|=\beta_1+...+\beta_n$ and $\mathcal{P}(\mathbb{R}^n)$ stands for the set of functions in $\mathcal{S}(\mathbb{R}^n)$ such that $\hat f\in C_{00}(\mathbb{R}^n)$ where $\hat f(\xi)=\int_{\mathbb{R}^n}f(x)e^{-2\pi i\langle x,\xi\rangle}dx$.

We shall use the notation $\mathcal{M}_{p,q}(\mathbb{R}^n)$ (respect. $\tilde{\mathcal{M}}_{p,q}(\mathbb{R}^n)$), for $1\le p,q\le\infty$, for the space of distributions $u\in\mathcal{S}'(\mathbb{R}^n)$ such that $u*\phi\in L^q(\mathbb{R}^n)$ for all $\phi\in L^p(\mathbb{R}^n)$ (respect. for the space of bounded functions m such that T_m defines a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $\widehat{T_m(\phi)}(\xi)=m(\xi)\hat f(\xi)$.) We endow the space $\tilde{\mathcal{M}}_{p,q}(\mathbb{R}^n)$ with the "norm" of the operator T_m, that is $\|m\|_{p,q}=\|T_m\|$.

Let us start off by mentioning some well known properties of the space of linear multipliers (see [1, 14]): $\mathcal{M}_{p,q}(\mathbb{R}^n)=\{0\}$ whenever $q<p$, $\mathcal{M}_{p,q}(\mathbb{R}^n)=\mathcal{M}_{q',p'}(\mathbb{R}^n)$ for $1<p\le q<\infty$ and for $1\le p\le 2$, $\mathcal{M}_{1,1}(\mathbb{R}^n)\subset\mathcal{M}_{p,p}(\mathbb{R}^n)\subset\mathcal{M}_{2,2}(\mathbb{R}^n)$. We also have the identifications

$$\tilde{\mathcal{M}}_{2,2}(\mathbb{R}^n)=L^\infty(\mathbb{R}^n),$$
$$\mathcal{M}_{1,q}(\mathbb{R}^n)=\{u\in\mathcal{S}'(\mathbb{R}^n):u\in L^q(\mathbb{R}^n)\},1<q<\infty,$$
$$\mathcal{M}_{1,1}(\mathbb{R}^n)=\{u\in\mathcal{S}'(\mathbb{R}^n):u=\mu\in M(\mathbb{R}^n)\}.$$

In this paper we shall be dealing with their bilinear analogues.

Definition 1.1. *Let $1\le p_1,p_2\le\infty$ and $0<p_3\le\infty$ and let $m(\xi,\eta)$ be a locally integrable function on $\mathbb{R}^n\times\mathbb{R}^n$. Define*

$$B_m(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat f(\xi)\hat g(\eta)m(\xi,\eta)e^{2\pi i(\langle\xi+\eta,x\rangle}d\xi d\eta$$

Key words and phrases. spaces of bilinear multipliers, bilinear Hilbert transform, bilinear fractional transform.

Partially supported by Proyecto MTM2008-04594/MTM.

for $f, g \in \mathcal{P}(\mathbb{R}^n)$.

m is said to be a bilinear multiplier on $\mathbb{R}^n$ *of type* (p_1, p_2, p_3) *if there exists* $C > 0$ *such that*

$$\|B_m(f,g)\|_{p_3} \leq C\|f\|_{p_1}\|g\|_{p_2}$$

for any $f, g \in \mathcal{P}(\mathbb{R}^n)$, *i.e.* B_m *extends to a bounded bilinear operator from* $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ *to* $L^{p_3}(\mathbb{R}^n)$ *(where we replace* $L^\infty(\mathbb{R}^n)$ *for* $C_0(\mathbb{R}^n)$ *in the case* $p_i = \infty$ *for* $i = 1, 2$*).*

We write $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R}^n)$ *for the space of bilinear multipliers of type* (p_1, p_2, p_3) *and* $\|m\|_{p_1,p_2,p_3} = \|B_m\|$.

The study of bilinear multipliers for smooth symbols (where $m(\xi, \eta)$ is a "nice" regular function) goes back to the work by R.R. Coifman and Y. Meyer in [6].

Particularly simple examples are the following bilinear convolution-type operators: For a given $K \in L^1_{loc}(\mathbb{R}^n)$ we define

$$C_K(f,g)(x) = \int_{\mathbb{R}} f(x-y)g(x+y)K(y)dy \tag{1}$$

for f and g belonging to $C_{00}(\mathbb{R}^n)$.

If $K \in L^1(\mathbb{R}^n)$ then $m(\xi, \eta) = \hat{K}(\xi - \eta)$ defines a multiplier in $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R}^n)$ for $1/p_1 + 1/p_2 = 1/p_3$ if $p_3 \geq 1$ and $\|m\|_{p_1,p_2,p_3} \leq \|K\|_1$.

Indeed, for f and $g \in \mathcal{S}(\mathbb{R})$, one has $f(x-y) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i\langle x-y,\xi\rangle}d\xi$ and $g(x+y) = \int_{\mathbb{R}^n} \hat{g}(\eta)e^{2\pi i\langle x+y,\eta\rangle}d\eta$. Hence we have

$$\begin{aligned}
C_K(f,g)(x) &= \int_{\mathbb{R}^n} f(x-y)g(x+y)K(y)dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)K(y)e^{2\pi i\langle x-y,\xi\rangle}e^{2\pi i\langle x+y,\eta\rangle}d\xi d\eta dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}^n} K(y)e^{-2\pi i\langle \xi-\eta,y\rangle}dy)e^{2\pi i\langle \xi+\eta,x\rangle}d\xi d\eta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{g}(\eta)\hat{f}(\xi)\hat{K}(\xi-\eta)e^{2\pi i\langle \xi+\eta,x\rangle}d\xi d\eta.
\end{aligned}$$

This motivates the introduction of the following class of multipliers.

Definition 1.2. *Let* $1 \leq p_1, p_2 \leq \infty$ *and* $0 < p_3 \leq \infty$. *We denote by* $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R}^n)$ *the space of measurable functions* $M : \mathbb{R}^n \to \mathbb{C}$ *such that* $m(\xi, \eta) = M(\xi - \eta) \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$, *that is to say*

$$B_M(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)M(\xi-\eta)e^{2\pi i\langle \xi+\eta,x\rangle}d\xi d\eta$$

extends to a bounded bilinear map from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ *into* $L^{p_3}(\mathbb{R}^n)$. *We keep the notation* $\|M\|_{p_1,p_2,p_3} = \|B_M\|$.

It was only in the last decade that the cases $M_0(x) = \frac{1}{|x|^{1-\alpha}}$ were shown to define bilinear multipliers of type (p_1, p_2, p_3) for $1/p_3 = 1/p_1 + 1/p_2 - \alpha$ for $1 < p_1, p_2 < \infty$ and $0 < \alpha < 1/p_1 + 1/p_2$ (see (3) in Theorem 1.3) and, in the case $n = 1$, $M_1(x) = -isign(x)$ was shown to define a bilinear multiplier of type (p_1, p_2, p_3) for $1/p_3 = 1/p_1 + 1/p_2$ for $1 < p_1, p_2 < \infty$ and $p_3 > 2/3$ (see (2) in Theorem 1.3). These two main examples correspond to the following bilinear operators: the *bilinear fractional integral* defined by

$$I_\alpha(f,g)(x) = \int_{\mathbb{R}} \frac{f(x-y)g(x+y)}{|y|^{1-\alpha}}dy, \quad 0 < \alpha < 1$$

and the *bilinear Hilbert transform* defined by

$$H(f,g)(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi}\int_{|y|>\varepsilon} \frac{f(x-y)g(x+y)}{y}dy$$

respectively.

Let us collect the results about their boundedness which are known nowadays.

Theorem 1.3. *Let $1 < p_1, p_2 < \infty$, $0 < \alpha < 1/p_1 + 1/p_2$, $1/q = 1/p_1 + 1/p_2 - \alpha$, $1/p_3 = 1/p_1 + 1/p_2$ and $2/3 < p_3 < \infty$. Then there exist constants A and B such that*

(2) $$\|H(f,g)\|_{p_3} \le A\|f\|_{p_1}\|g\|_{p_2} \text{ (Lacey-Thiele, [12, 13]),}$$

(3) $$\|I_\alpha(f,g)\|_q \le B\|f\|_{p_1}\|g\|_{p_2}. \text{ (Kenig-Stein [11], Grafakos-Kalton [10]).}$$

Our objective is to study the basic properties of the classes $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\mathcal{M}_{p_1,p_2,p_3}(\mathbb{R})$, to find examples of bilinear multipliers in these classes and to get methods to produce new ones.

As usual, if $f \in L^1(\mathbb{R}^n)$ we denote by τ_x, M_x and D_t^p the translation $\tau_x f(y) = f(y-x)$ for $x \in \mathbb{R}^n$, the modulation $M_x f(y) = e^{2\pi i\langle x,y\rangle} f(y)$ and the dilation $D_t^p f(x) = t^{-n/p} f(\frac{x}{t})$ for $0 < p, t < \infty$.

With this notation out of the way one has, for $1 \le p \le \infty$ and $1/p + 1/p' = 1$,

(4) $$\widehat{(\tau_x f)}(\xi) = M_{-x}\hat{f}(\xi), \quad \widehat{(M_x f)}(\xi) = \tau_x \hat{f}(\xi), \quad \widehat{(D_t^p f)}(\xi) = D_{t^{-1}}^{p'} \hat{f}(\xi).$$

Clearly τ_x, M_x and D_t^p are isometries on $L^p(\mathbb{R}^n)$ for any $0 < p \le \infty$.

Although most of the results presented in what follows have a formulation in $n \ge 1$ we shall restrict ourselves to the case $n = 1$ for simplicity. The reader is referred to [2, 3, 4, 5, 7] for several similar results on other groups, and to find same methods of transference.

2. Bilinear multipliers: The basics

Let us start by pointing out a characterization, for $p_3 \ge 1$, in terms of the duality, whose elementary proof is left to the reader.

Proposition 2.1. *Let $1 \le p_3 \le \infty$. Then $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ if and only if there exists $C > 0$ such that*

$$\left|\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)d\xi d\eta\right| \le C\|f\|_{p_1}\|g\|_{p_2}\|h\|_{p_3'}$$

for all $f, g, h \in \mathcal{P}(\mathbb{R})$.

We now present a basic example of a bilinear multiplier. For a Borel regular measure in $\mathbb{R}$ μ we denote $\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} d\mu(x)$ its Fourier transform.

Proposition 2.2. *Let $p_3 \ge 1$ and $1/p_1 + 1/p_2 = 1/p_3$ and let $m(\xi,\eta) = \hat{\mu}(\alpha\xi + \beta\eta)$ where μ is a Borel regular measure in $\mathbb{R}$ and $(\alpha,\beta) \in \mathbb{R}^2$. Then $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\|m\|_{p_1,p_2,p_3} \le \|\mu\|_1$.*

Proof. Let us first rewrite the value $B_m(f,g)$ as follows:

$$\begin{aligned} B_m(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\hat{\mu}(\alpha\xi+\beta\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}} e^{-2\pi i(\alpha\xi+\beta\eta)t}d\mu(t))e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}} (\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(x-\alpha t)\xi}e^{2\pi i(x-\beta t)\eta}d\xi d\eta)d\mu(t) \\ &= \int_{\mathbb{R}} f(x-\alpha t)g(x-\beta t)d\mu(t). \end{aligned}$$

Hence, using Minkowski's inequality, one has

$$\begin{aligned} \|B_m(f,g)\|_{p_3} &\leq \int_{\mathbb{R}} \|f(\cdot-\alpha t)g(\cdot-\beta t)\|_{p_3}d|\mu|(t) \\ &\leq \int_{\mathbb{R}} \|f(\cdot-\alpha t)\|_{p_1}\|g(\cdot-\beta t)\|_{p_2}d|\mu|(t) \\ &= \|f\|_{p_1}\|g\|_{p_2}\int_{\mathbb{R}} d|\mu|(t) = \|\mu\|_1\|f\|_{p_1}\|g\|_{p_2}. \end{aligned}$$

■

Let us start with some elementary properties of the bilinear multipliers when composing with translations, modulations and dilations.

Proposition 2.3. *Let $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$.*

(a) *If $m_1 \in \tilde{\mathcal{M}}_{s_1,p_1}(\mathbb{R})$ and $m_2 \in \tilde{\mathcal{M}}_{s_2,p_2}(\mathbb{R})$ then $m_1(\xi)m(\xi,\eta)m_2(\eta) \in \mathcal{BM}_{(s_1,s_2,p_3)}(\mathbb{R})$. Moreover*

$$\|m_1 m m_2\|_{s_1,s_2,p_3} \leq \|m_1\|_{s_1,p_1}\|m\|_{p_1,p_2,p_3}\|m_2\|_{s_2,p_2}$$

(b) *$\tau_{(\xi_0,\eta_0)}m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ for each $(\xi_0,\eta_0) \in \mathbb{R}^2$ and*

$$\|\tau_{(\xi_0,\eta_0)}m\|_{p_1,p_2,p_3} = \|m\|_{p_1,p_2,p_3}.$$

(c) *$M_{(\xi_0,\eta_0)}m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ for each $(\xi_0,\eta_0) \in \mathbb{R}^2$ and*

$$\|M_{(\xi_0,\eta_0)}m\|_{p_1,p_2,p_3} = \|m\|_{p_1,p_2,p_3}$$

(d) *If $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$ and $0 < t < \infty$ then $D_t^q m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ and*

$$\|D_t^q m\|_{p_1,p_2,p_3} = \|m\|_{p_1,p_2,p_3}.$$

Proof. Use (4) to deduce the following formulas

$$B_{m_1 m m_2}(f,g) = B_m(T_{m_1}f, T_{m_2}g). \tag{5}$$

$$B_{\tau_{(\xi_0,\eta_0)}m}(f,g) = M_{\xi_0+\eta_0}B_m(M_{-\xi_0}f, M_{-\eta_0}g). \tag{6}$$

$$B_{M_{(\xi_0,\eta_0)}m}(f,g) = B_m(\tau_{-\xi_0}f, \tau_{-\eta_0}g). \tag{7}$$

$$B_m(D_t^{p_1}f, D_t^{p_2}g) = D_t^{p_3}B_{D_t^q m}(f,g). \tag{8}$$

Let us check only the validity of last one. The other ones follow easily from the previous facts.

$$\begin{aligned} B_m(D_t^{p_1}f, D_t^{p_2}g)(x) &= \int_{\mathbb{R}^2} t^{\frac{1}{p_1'}}\hat{f}(t\xi)t^{\frac{1}{p_2'}}\hat{g}(t\eta)m(\xi,\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}^2} t^{\frac{1}{p_1'}}\hat{f}(\xi)t^{\frac{1}{p_2'}}\hat{g}(\eta)m(\frac{\xi}{t},\frac{\eta}{t})e^{2\pi i(\xi+\eta)\frac{x}{t}}t^{-2}d\xi d\eta \\ &= t^{-\frac{1}{p_3}}\int_{\mathbb{R}^2}\hat{f}(\xi)\hat{g}(\eta)t^{-\frac{1}{p_1}-\frac{1}{p_2}+\frac{1}{p_3}}m(\frac{\xi}{t},\frac{\eta}{t})e^{2\pi i(\xi+\eta)\frac{x}{t}}d\xi d\eta \\ &= D_t^{p_3}B_{D_t^q m}(f,g)(x). \end{aligned}$$

■

From (8) we can see that the condition $1/p_1 + 1/p_2 = 1/p_3$ is also connected to the homogeneity of the symbol.

Proposition 2.4. *Let $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ such that $m(t\xi, t\eta) = m(\xi,\eta)$ for any $t > 0$. Then $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$.*

Proof. From assumption $D_t^\infty m = m$. Using (8) we have

$$B_m(D_t^{p_1}f, D_t^{p_2}g) = t^{1/p_3-(1/p_1+1/p_2)}D_t^{p_3}B_m(f,g)$$

and therefore

$$\begin{aligned} \|B_m(f,g)\|_{p_3} &= \|D_t^{p_3}B_m(f,g)\|_{p_3} \\ &= t^{-1/p_3+(1/p_1+1/p_2)}\|B_m(D^{p_1}f, D_t^{p_2}g)\|_{p_3} \\ &\leq t^{-1/p_3+(1/p_1+1/p_2)}\|B_m\|\|f\|_{p_1}\|g\|_{p_2}. \end{aligned}$$

For this to hold for any $0 < t < \infty$ one needs $1/p_1 + 1/p_2 = 1/p_3$. ■

Let us combine the previous results to get new bilinear multipliers from a given one.

Proposition 2.5. *Let $p_3 \geq 1$ and $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$.*

- (a) *If $Q = [a,b] \times [c,d]$ and $1 < p_1, p_2 < \infty$ then $m\chi_Q \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\|m\chi_Q\|_{p_1,p_2,p_3} \leq C\|m\|_{p_1,p_2,p_3}$.*
- (b) *If $\Phi \in L^1(\mathbb{R}^2)$ then $\Phi * m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\|\Phi * m\|_{p_1,p_2,p_3} \leq \|\Phi\|_1\|m\|_{p_1,p_2,p_3}$.*
- (c) *If $\Phi \in L^1(\mathbb{R}^2)$ then $\hat{\Phi}m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\|\hat{\Phi}m\|_{p_1,p_2,p_3} \leq \|\Phi\|_1\|m\|_{p_1,p_2,p_3}$.*
- (d) *If $\psi \in L^1(\mathbb{R}^+, t^{\frac{1}{p_3}-(\frac{1}{p_1}+\frac{1}{p_2})})$ then $m_\psi(\xi,\eta) = \int_0^\infty m(t\xi,t\eta)\psi(t)dt \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$. Moreover $\|m_\psi\|_{p_1,p_2,p_3} \leq \|\psi\|_1\|m\|_{p_1,p_2,p_3}$.*

Proof. (a) Use that $\chi_{[a,b]} \in \tilde{\mathcal{M}}_{p_1,p_1}$ for $1 < p_1 < \infty$ and $\chi_{[c,d]} \in \tilde{\mathcal{M}}_{p_2,p_2}$ for $1 < p_2 < \infty$ together with Proposition 2.3 part (a).

(b) Note that

$$\begin{aligned} B_{\Phi*m}(f,g)(x) &= \int_{\mathbb{R}^2}\hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}^2} m(\xi-u,\eta-v)\Phi(u,v)dudv)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\hat{f}(\xi)\hat{g}(\eta)m(\xi-u,\eta-v)e^{2\pi i(\xi+\eta)x}d\xi d\eta\Big)\Phi(u,v)dudv \\ &= \int_{\mathbb{R}^2} B_{\tau_{(u,v)}m}(f,g)(x)\Phi(u,v)dudv. \end{aligned}$$

From the vector-valued Minkowski inequality and Proposition 2.3 part (b), we have

$$\begin{aligned} \|B_{\Phi * m}(f,g)\|_{p_3} &\leq \int_{\mathbb{R}^2} \|B_{\tau_{(u,v)}m}(f,g)\|_{p_3} |\Phi(u,v)| du dv \\ &\leq \|m\|_{p_1,p_2,p_3} \|f\|_{p_1} \|g\|_{p_2} \|\Phi\|_1. \end{aligned}$$

(c) Observe that

$$\begin{aligned} B_{\hat{\Phi}m}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}^2} M_{(-u,-v)}m(\xi,\eta)\Phi(u,v)dudv)e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \int_{\mathbb{R}^2} B_{M_{(-u,-v)}m}(f,g)(x)\Phi(u,v)dudv. \end{aligned}$$

Argue as above, using now Proposition 2.3 part (c), to conclude the result.
(d) Use now Proposition 2.3 part (d), for $\frac{1}{p_3} - (\frac{1}{p_1} + \frac{1}{p_2}) = -\frac{2}{q}$,

$$\begin{aligned} B_{m_\psi}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_0^\infty D^q_{t^{-1}} m(\xi,\eta) t^{-2/q}\psi(t)dt)e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \int_0^\infty B_{D^q_{t^{-1}}m}(f,g)(x) t^{-2/q}\psi(t)dt. \end{aligned}$$

■

With all these procedures we have several useful methods to produce multipliers in $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$. Let us mention one application of each of them.

Example 2.6. (1) *If* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$, $m_1 \in \tilde{\mathcal{M}}_{(p_1,p_1)}$ *and* $m_2 \in \tilde{\mathcal{M}}_{(p_2,p_2)}$ *then* $m(\xi,\eta) = m_1(\xi)m_2(\eta) \in \mathcal{BM}_{p_1,p_2,p_3}$.

(2) *If* $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$, $p_3 \geq 1$ *and* Q_1, Q_2 *are bounded measurable sets in* $\mathbb{R}$ *then*

$$\frac{1}{|Q_1||Q_2|} \int_{Q_1 \times Q_2} m(\xi+u, \eta+v) du dv \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R}).$$

(3) *If* $\Phi \in L^1(\mathbb{R}^2)$ *then* $\hat{\Phi} \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ *for* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$, $p_3 \geq 1$.

(4) *If* $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$, $|\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}| < 1$ *then*

$$m_1(\xi,\eta) = \int_0^\infty m(t\xi, t\eta) \frac{dt}{1+t^2} \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R}).$$

A combination of the previous results gives the following examples of bilinear multipliers in $\mathcal{B}M_{(1,1,p_3)}(\mathbb{R})$ whose proof is left to the reader.

Corollary 2.7. *Let* $\Phi \in L^1(\mathbb{R}^2)$, $\psi_1 \in L^{p_1}(\mathbb{R})$ *and* $\psi_2 \in L^{p_2}(\mathbb{R})$ *and* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} \leq 1$ *then*

$$m(\xi,\eta) = \hat{\psi}_1(\xi)\hat{\Phi}(\xi,\eta)\hat{\psi}_2(\eta) \in \mathcal{B}M_{(1,1,p_3)}(\mathbb{R}).$$

Let us use Proposition 2.1 and interpolation to get a sufficient integrability condition to guarantee that $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$.

Theorem 2.8. *Let* $1 \leq p_1, p_2 \leq p \leq 2$ *and* $p_3 \geq p'$ *such that* $\frac{1}{p_1} + \frac{1}{p_2} - \frac{2}{p} = \frac{1}{p_3}$. *If* $m \in L^p(\mathbb{R}^2)$ *then* $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$.

Proof. Let us show first that $m \in \mathcal{BM}_{(p,p,\infty)}(\mathbb{R})$. Let $f, g \in L^p(\mathbb{R})$ and $h \in L^1(\mathbb{R})$. Using Hölder and Hausdorff-Young's inequalities one gets

$$\begin{aligned} |\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)d\xi d\eta| &\leq \|m\|_{L^p(\mathbb{R}^2)}\|\hat{h}\|_\infty \|\hat{f}\|_{p'}\|\hat{g}\|_{p'} \\ &\leq \|m\|_{L^p(\mathbb{R}^2)}\|h\|_1\|f\|_p\|g\|_p. \end{aligned}$$

Similarly, changing the variables $\xi + \eta = u$, $\xi = -v$, one has

$$\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)d\xi d\eta = \int_{\mathbb{R}^2} \hat{f}(-v)\hat{g}(u+v)\hat{h}(u)m(-v,u+v)dvdu.$$

An argument as above gives also the estimate

$$|\int_{\mathbb{R}^2} \hat{f}(-v)\hat{g}(u+v)\hat{h}(u)m(-v,u+v)dvdu| \leq \|m\|_{L^p(\mathbb{R}^2)}\|g\|_1\|f\|_p\|h\|_p.$$

This shows that $m \in \mathcal{BM}_{(p,1,p')}(\mathbb{R})$. A similar argument shows also that $m \in \mathcal{BM}_{(1,p,p')}(\mathbb{R})$.

Given $1 \leq \tilde{p}_1 \leq p$ and $p' \leq \tilde{p}_3 \leq \infty$ with $\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_3} = \frac{1}{p}$ we have $0 \leq \theta \leq 1$ such that $\frac{1}{\tilde{p}_1} = \frac{1-\theta}{p} + \frac{\theta}{1}$ and $\frac{1}{\tilde{p}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{p'}$. Hence, by interpolation, $m \in \mathcal{BM}_{(\tilde{p}_1,p,\tilde{p}_3)}(\mathbb{R})$.

Similarly $m \in \mathcal{BM}_{(p,\tilde{p}_2,\tilde{q}_3)}(\mathbb{R})$ whenever $1 \leq \tilde{p}_2 \leq p$ and $p' \leq \tilde{q}_3 \leq \infty$ with $\frac{1}{\tilde{p}_2} - \frac{1}{\tilde{q}_3} = \frac{1}{p}$.

To finish the proof we observe that if $1 < p_1 < p$ and $1 < p_2 < p$ then for each $0 < \theta < 1$ there exist $1 \leq \tilde{p}_1 \leq p_1 < p$ and $1 \leq \tilde{p}_2 \leq p_2 < p$ such that

$$\frac{1}{p_1} - \frac{1}{p} = (1-\theta)(\frac{1}{\tilde{p}_1} - \frac{1}{p}), \quad \frac{1}{p_2} - \frac{1}{p} = \theta(\frac{1}{\tilde{p}_2} - \frac{1}{p}).$$

Denoting $\tilde{p}_3, \tilde{q}_3$ the values such that $\frac{1}{\tilde{p}_2} - \frac{1}{p} = \frac{1}{\tilde{p}_3}$ and $\frac{1}{\tilde{p}_2} - \frac{1}{p} = \frac{1}{\tilde{q}_3}$ one obtains that

$$\frac{1}{p_1} = \frac{(1-\theta)}{\tilde{p}_1} + \frac{\theta}{p}, \quad \frac{1}{p_2} = \frac{(1-\theta)}{p} + \frac{\theta}{\tilde{p}_1}, \quad \frac{1}{p_3} = \frac{(1-\theta)}{\tilde{p}_3} + \frac{\theta}{\tilde{q}_3}.$$

Hence the result follows again from interpolation between the last ones. ■

3. Bilinear multipliers defined by functions in one variable

Let us restrict ourselves to a smaller family of multipliers where $m(\xi,\eta) = M(\xi - \eta)$ for some M defined in $\mathbb{R}$. These multipliers satisfy

$$B_m(M_x f, M_x g) = M_{2x} B_m(f,g). \tag{9}$$

As in the introduction we use the notation $\tilde{\mathcal{M}}_{p_1,p_2,p_3}(\mathbb{R})$ for the space of functions $M : \mathbb{R} \to \mathbb{C}$ such that $m(\xi,\eta) = M(\xi-\eta) \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$, that is to say

$$B_M(f,g)(x) = \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)M(\xi-\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta,$$

defined for $\hat{f}$ and $\hat{g}$ compactly supported, extends to a bounded bilinear map from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$. We keep the notation $\|M\|_{p_1,p_2,p_3} = \|B_M\|$.

The reader should be aware that the starting assumption on the function M is only relevant for the definition of the bilinear mapping to make sense when acting on certain classes of "nice" functions. Then a density argument allows to extend functions belonging to Lebesgue spaces. We would like to point out the following observation.

Remark 3.1. *If* $M_n \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ *are functions such that* $M_n(x) \to M(x)$ *a.e and* $\sup_n \|M_n\| < \infty$ *then* $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ *and* $\|M\|_{p_1,p_2,p_3} \leq \sup_n \|M_n\|_{p_1,p_2,p_3}$*.*

Indeed, this fact follows from Fatou's lemma, since

$$\|B_M(f,g)\|_{p_3} \leq \liminf \|B_{M_n}(f,g)\|_{p_3} \leq \sup_n \|M_n\|_{p_1,p_2,p_3} \|f\|_{p_1} \|g\|_{p_2}.$$

Remark 3.2. *The case* $M(x) = \frac{1}{|x|^{1-\alpha}}$ *(and even the n-dimensional case) corresponds to the bilinear fractional integral. This was first shown by C. Kenig and E. Stein in* [11] *to belong to* $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ *for any* $1 < p_1, p_2 < \infty$, $0 < \alpha < 1/p_1 + 1/p_2$ *and* $1/p_1 + 1/p_2 = 1/p_3 - \alpha$*. Another very important and non trivial example is the bilinear Hilbert transform, given by* $M(x) = -i\,sign(x)$*, which was shown by M. Lacey and C.Thiele in* [12, 13, **?**] *to belong to* $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ *for any* $1 < p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/p_3$ *and* $p_3 > 2/3$*. These results were extended to other cases in* [10] *and* [8, 9] *respectively.*

We start reformulating the definition of this class of bilinear multipliers.

Proposition 3.3. *Let* $M \in L^1_{loc}(\mathbb{R})$, $f, g \in \mathcal{P}(\mathbb{R})$*. Then*

$$B_M(f,g)(x) = \frac{1}{2}\int_{\mathbb{R}^2} \hat{f}(\frac{u+v}{2})\hat{g}(\frac{u-v}{2})M(v)e^{2\pi iux}dudv \tag{10}$$

$$B_M(f,g)(-x) = \int_{\mathbb{R}} (\widehat{\tau_x g} * M)(\xi)\widehat{\tau_x f}(\xi)d\xi. \tag{11}$$

$$\widehat{B_M(f,g)}(x) = \frac{1}{2}C_M(\widehat{D^1_{1/2}f}, \widehat{D^1_{1/2}g})(x). \tag{12}$$

Proof. (10) follows changing variables.

To show (11) observe that

$$\begin{aligned} B_M(f,g)(-x) &= \int_{\mathbb{R}^2} \widehat{\tau_x f}(\xi)\widehat{\tau_x g}(\eta)M(\xi-\eta)d\xi d\eta \\ &= \int_{\mathbb{R}} (\int_{\mathbb{R}} \widehat{\tau_x g}(\eta)M(\xi-\eta)d\eta)\widehat{\tau_x f}(\xi)d\xi \\ &= \int_{\mathbb{R}} (\widehat{\tau_x g} * M)(\xi)\widehat{\tau_x f}(\xi)d\xi \end{aligned}$$

Finally, using (10), we have

$$\begin{aligned} B_M(f,g)(x) &= \frac{1}{2}\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \hat{f}(\frac{u+v}{2})\hat{g}(\frac{u-v}{2})M(v)dv\Big)e^{2\pi iux}dv \\ &= \frac{1}{2}\int_{\mathbb{R}} C_M(D^\infty_{1/2}\hat{f}, D^\infty_{1/2}\hat{g})(u)e^{2\pi iux}du. \end{aligned}$$

This implies (12). ■

For symbols M which are integrable we can write B_M in terms C_K for some kernel K.

Proposition 3.4. *Let* $M \in L^1(\mathbb{R})$ *and set* $K(t) = \hat{M}(-t)$*. Then* $B_M = C_K$*, i.e*

$$B_M(f,g) = \int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt$$

Proof.

$$\begin{aligned} C_K(f,g)(x) &= \int_{\mathbb{R}} f(x-t)g(x+t)\hat{M}(-t)dt \\ &= \int_{\mathbb{R}} (\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(x-t)\xi}e^{2\pi i(x+t)\eta}d\xi d\eta)\hat{M}(-t)dt \\ &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}} \hat{M}(t)e^{2\pi i(\xi-\eta)t}dt)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= B_M(f,g)(x). \end{aligned}$$

■

This class does have much richer properties than $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$. As above use the notation $f_t(x) = D_t^1 f(x) = \frac{1}{t}f(\frac{x}{t})$ for a function f defined in $\mathbb{R}$. The following facts are immediate.

$$\tau_y B_M(f,g) = B_M(\tau_y f, \tau_y g), y \in \mathbb{R}. \tag{13}$$

$$M_{2y} B_M(f,g) = B_M(M_y f, M_y g), y \in \mathbb{R}. \tag{14}$$

$$(B_M(f,g))_t = B_{D^1_{t^{-1}}M}(f_t, g_t), t > 0. \tag{15}$$

When specializing the properties obtained for $m(\xi,\eta)$ to the case $M(\xi-\eta)$ we get the following facts:

$$B_M(\tau_{-y}f, \tau_y g) = B_{M_y M}(f,g), y \in \mathbb{R}. \tag{16}$$

$$B_M(M_y f, M_{-y} g) = B_{\tau_{2y}M}(f,g), y \in \mathbb{R}. \tag{17}$$

For $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$ we have

$$B_M(D_t^{p_1} f, D_t^{p_2} g) = D_t^{p_3} B_{D_t^q M}(f,g), t > 0. \tag{18}$$

As in the previous section we can generate new multipliers in $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$.

Proposition 3.5. *Let $p_3 \geq 1$, $\phi \in L^1(\mathbb{R})$ and $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. Then*

(a) *$\phi * M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\|\phi * M\|_{p_1,p_2,p_3} \leq \|\phi\|_1 \|M\|_{p_1,p_2,p_3}$.*
(b) *$\hat{\phi}M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\|\hat{\phi}M\|_{p_1,p_2,p_3} \leq \|\phi\|_1 \|M\|_{p_1,p_2,p_3}$.*
(c) *If $\psi \in L^1(\mathbb{R}^+, t^{\frac{1}{p_3}-(\frac{1}{p_1}+\frac{1}{p_2})})$ then $M_\psi(\xi) = \int_0^\infty M(t\xi)\psi(t)dt \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. Moreover $\|M_\psi\|_{p_1,p_2,p_3} \leq \|\psi\|_1 \|M\|_{p_1,p_2,p_3}$.*

Proof. (a) Apply Minkowski's inequality to the following fact:

$$\begin{aligned} B_{\phi * M}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}} M(\xi-\eta-u)\phi(u)du)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}} (\int_{\mathbb{R}^2} \widehat{M_{-u}f}(\xi)\hat{g}(\eta)M(\xi-\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta)e^{2\pi iux}\phi(u)du \\ &= \int_{\mathbb{R}} M_u B_M(M_{-u}f,g)(x)\phi(u)du. \end{aligned}$$

(b) Observe that

$$\begin{aligned} B_{\hat{\phi}m}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}} (M_{-u}m)(\xi-\eta)\phi(u)du)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}^2} B_{M_{-u}m}(f,g)(x)\phi(u)du. \end{aligned}$$

Use now Minkowski's again and (16).
(c) Write $\frac{1}{p_3} - (\frac{1}{p_1} + \frac{1}{p_2}) = -\frac{1}{q}$,

$$\begin{aligned} B_{M_\psi}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_0^\infty D^q_{t^{-1}}M(\xi)t^{-1/q}\psi(t)dt)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_0^\infty B_{D^q_{t^{-1}}M}(f,g)(x)t^{-1/q}\psi(t)dt. \end{aligned}$$

The result follows from (18) and Minkowski's again. ■

Proposition 3.6. *Let $p_3 \geq 1$, $\phi \in L^1(\mathbb{R})$ and $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. Then $m(\xi,\eta) = M(\xi-\eta)\hat{\phi}(\xi+\eta) \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\|m\|_{p_1,p_2,p_3} \leq \|\phi\|_1\|M\|_{p_1,p_2,p_3}$.*

Proof. Apply Young's inequality to the following fact:

$$\begin{aligned} B_m(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)M(\xi-\eta)(\int_{\mathbb{R}} \phi(y)e^{-2\pi i(\xi+\eta)y}dy)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}}(\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)M(\xi-\eta)e^{2\pi i(\xi+\eta)(x-y)}d\xi d\eta)\phi(y)dy \\ &= \phi * B_M(f,g)(x). \end{aligned}$$

■

Let us show that the classes $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ are reduced to $\{0\}$ for some values of the parameters.

Theorem 3.7. *Let $p_3 \geq 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{p_3}$. Then $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R}) = \{0\}$.*

Proof. Let $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. Using Proposition 3.5 we have that $\phi * M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ for any ϕ continuous with compact support. Hence we may assume that $M \in L^1(\mathbb{R})$. Using Proposition 3.4 one has that

$$B_M(f,g)(x) = \int_{(x+B_R)\cap(-x+B_R)} f(x-t)g(x+t)\hat{M}(-t)dt$$

for any f and g continuous functions supported in a ball $B_R = \{|x| \leq R\}$. Therefore one concludes that $supp(B_M(f,g)) \subset B_{2R}$ in such a case. On the other hand for any compactly supported function h, $0 < p < \infty$ and y big enough one can say that $\|h \pm \tau_y f\|_p = 2^{1/p}\|f\|_p$.

Consider $\{r_k\}$ the Rademacher system in $[0,1]$ and observe that, for each $N \in \mathbb{N}$ and $y \in \mathbb{R}$, the orthonormality of the system gives

$$\int_0^1 B_M(\sum_{k=0}^N r_k(t)\tau_{ky}f, \sum_{k=0}^N r_k(t)\tau_{ky}f)dt = \sum_{k=0}^N B_M(\tau_{ky}f, \tau_{ky}g)$$

Therefore, since $\sum_{k=0}^N B_M(\tau_{ky}f,\tau_{ky}g) = \sum_{k=0}^N \tau_{ky}B_M(f,g)$, we conclude that for y big enough

$$\|\sum_{k=0}^N \tau_{ky}B_M(f,g)\|_{p_3}^{p_3} = (N+1)\|B_M(f,g)\|_{p_3}^{p_3}.$$

On the other hand, for $p_3 \geq 1$,

$$\begin{aligned}
&\|\int_0^1 B_M(\sum_{k=0}^N r_k(t)\tau_{ky}f, \sum_{k=0}^N r_k(t)\tau_{ky}g)dt\|_{p_3}\\
\leq\ & \int_0^1 \|B_M(\sum_{k=0}^N r_k(t)\tau_{ky}f, \sum_{k=0}^N r_k(t)\tau_{ky}g)\|_{p_3}dt\\
\leq\ & \int \|B_M\|\|\sum_{k=0}^N r_k(t)\tau_{ky}f\|_{p_1}\|\sum_{k=0}^N r_k(t)\tau_{ky}g)\|_{p_2}dt\\
\leq\ & \|B_M\| \sup_{0<t<1}\|\sum_{k=0}^N r_k(t)\tau_{ky}f\|_{p_1} \sup_{0<t<1}\|\sum_{k=0}^N r_k(t)\tau_{ky}g\|_{p_2}\\
\leq\ & \|B_M\|(N+1)^{1/p_1}\|f\|_{p_1}(N+1)^{1/p_2}\|g\|_{p_2}.
\end{aligned}$$

This implies that $(N+1)^{1/p_3}\|B_M(f,g)\|^{p_3} \leq C(N+1)^{1/p_1+1/p_2}\|f\|_{p_1}\|g\|_{p_2}$. Hence $1/p_1 + 1/p_2 \geq 1/p_3$. ■

The following elementary lemma is quite useful to get necessary conditions on multipliers.

Lemma 3.8. *Let $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. If $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$ then there exists $C > 0$ such that*

$$|\int_{\mathbb{R}} e^{-\lambda^2\xi^2}M(\xi)d\xi| \leq C\|M\|_{p_1,p_2,p_3}\lambda^{\frac{1}{q}-1}$$

for any $\lambda > 0$.

Proof. Let $\lambda > 0$ and denote G_λ such that $\hat{G}_\lambda(\xi) = e^{-2\lambda^2\xi^2}$. Using (10) one concludes that

$$\begin{aligned}
B_M(G_\lambda, G_\lambda)(x) &= \frac{1}{2}\int_{\mathbb{R}^2} e^{-\lambda^2v^2}e^{-\lambda^2u^2}M(v)e^{2\pi iux}dudv\\
&= \frac{1}{2}(\int_{\mathbb{R}} e^{-\lambda^2v^2}M(v)dv)(\frac{1}{\lambda}\int_{\mathbb{R}} e^{-u^2}e^{2\pi iu\frac{x}{\lambda}}du)\\
&= C\frac{1}{\lambda}e^{-\pi^2\frac{x^2}{\lambda^2}}(\int_{\mathbb{R}} e^{-\lambda^2v^2}M(v)dv).
\end{aligned}$$

Since $\|G_\lambda\|_p = C_p\lambda^{\frac{1}{p}-1}$ and $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ one gets that

$$\|B_M(G_\lambda, G_\lambda)\|_{p_3} = C|\int_{\mathbb{R}} e^{-\lambda^2v^2}M(v)dv|\lambda^{\frac{1}{p_3}-1} \leq C\|M\|_{p_1,p_2,p_3}\lambda^{\frac{1}{p_1}-1}\lambda^{\frac{1}{p_2}-1}.$$

Therefore $|\int_{\mathbb{R}} e^{-\lambda^2\xi^2}M(\xi)d\xi| \leq C\|M\|_{p_1,p_2,p_3}\lambda^{\frac{1}{q}-1}$. ■

Theorem 3.9. *If there exists a non-zero continuous and integrable function M belonging to $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ then*

$$\frac{1}{p_3} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{p_3} + 1.$$

Proof. Assume first that $\frac{1}{p_1}+\frac{1}{p_2}<\frac{1}{p_3}$. Use Lemma 3.8 applied to $\tau_{-2y}M$ for any $y\in\mathbb{R}$ together with (16) to obtain

$$|\lambda\int_{\mathbb{R}} e^{-\lambda^2\xi^2}M(\xi+2y)d\xi|\le C\|M\|_{p_1,p_2,p_3}\lambda^{\frac{1}{q}}.$$

Therefore, using the continuity of M and $q<0$ one gets

$$\lim_{\lambda\to\infty}|\lambda\int_{\mathbb{R}} e^{-\lambda^2\xi^2}M(\xi+2y)d\xi|=|M(2y)|=0.$$

Hence $M=0$.

Assume now that $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}>1$. Using again Lemma 3.8, applied to M_yM, together with (17) we obtain

$$|\int_{\mathbb{R}} e^{-\lambda^2\xi^2}M(\xi)e^{2\pi iy\xi}d\xi|\le C\|M\|_{p_1,p_2,p_3}\lambda^{\frac{1}{q}-1}.$$

Therefore, taking limits again as $\lambda\to 0$, since $1/q-1>0$ we get $|\hat{M}(y)|=0$. Hence $M=0$. ■

Corollary 3.10. *(see [16, Prop 3.1]) Let $p_3\ge 1$ such that $\frac{1}{p_1}+\frac{1}{p_2}<\frac{1}{p_3}$ or $\frac{1}{p_1}+\frac{1}{p_2}>\frac{1}{p_3}+1$. Then $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})=\{0\}$.*

Proof. Let $M\in\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. From Proposition 3.5 we have that $\phi*M\in\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ for any ϕ compactly supported and continuous. Now use Theorem 3.9 to conclude that $\phi*M=0$ for any compactly supported and continuous ϕ. This implies that $M=0$. ■

Let us now use some interpolation methods to get more examples of multipliers in $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. First note that, selecting $\alpha=1$ and $\beta=-1$ in Proposition 2.2 we obtain the following simple example.

Proposition 3.11. *If $\mu\in M(\mathbb{R})$ then $M=\hat{\mu}\in\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ for $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}\le 1$ and $\|M\|\le\|\mu\|_1$.*

Theorem 3.12. *Let $\frac{1}{p_3}\le\frac{1}{p_1}+\frac{1}{p_2}\le\min\{2,\frac{1}{p_3}+1\}$. If $M\in L^1(\mathbb{R})$ and $M=\hat{K}$ for some $K\in L^q(\mathbb{R})$ where $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=1-\frac{1}{q}$ then $M\in\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ with $\|M\|_{p_1,p_2,p_3}\le C\|K\|_q$.*

Proof. Consider the trilinear form

$$T(K,f,g)=\int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt.$$

From Proposition 3.4 we have $B_M(f,g)=T(K,f,g)$ for $M=\hat{K}$. Now use Proposition 3.11 to conclude that T is bounded in $L^1(\mathbb{R})\times L^{q_1}(\mathbb{R})\times L^{q_2}(\mathbb{R})\to L^{s_1}(\mathbb{R})$ where $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{s_1}\le 1$ and it has norm bounded by 1.

Assume first that $\frac{1}{p_1}+\frac{1}{p_2}\le 1$. Hence T is bounded in $L^1(\mathbb{R})\times L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\to L^p(\mathbb{R})$ for $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$.

On the other hand, using Hölder's inequality

$$\sup_x|\int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt|\le\|f\|_{p_1}\|g\|_{p_2}\|K\|_{p'}.$$

This shows that T is also bounded in $L^{p'}(\mathbb{R})\times L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\to L^\infty(\mathbb{R})$. Therefore, by interpolation, selecting $0<\theta<1$ such that $\frac{1}{p_3}=\frac{1-\theta}{p}$, one obtains that T is bounded in $L^q(\mathbb{R})\times L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\to L^{p_3}(\mathbb{R})$ for $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=1-\frac{1}{q}$.

Assume now that $1 < \frac{1}{p_1} + \frac{1}{p_2} \leq 2$.

Using that $\int_{\mathbb{R}} f(x-t)g(x+t)dt = f * g(2x)$, Young's inequality implies that

$$\| \int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt\|_{r_3} \leq \|K\|_\infty \|D^\infty_{1/2}(|f| * |g|)\|_{r_3} \leq C\|f\|_{r_1}\|g\|_{r_2}\|K\|_\infty$$

whenever $\frac{1}{r_1} + \frac{1}{r_2} \geq 1$ and $\frac{1}{r_1} + \frac{1}{r_2} - 1 = \frac{1}{r_3}$.

Hence T is bounded in $L^\infty(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^p(\mathbb{R})$ where $\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p} \leq 1$.

Using duality, $\langle T(K, f, g), h\rangle = \langle T(h, \bar{f}, g), K\rangle$, where $\bar{f}(x) = f(-x$, that is

$$\int_{\mathbb{R}^2} f(x-t)g(x+t)K(t)h(x)dtdx = \int_{\mathbb{R}} (\int_{\mathbb{R}} \bar{f}(t-x)g(x+t)h(x)dx)K(t)dt.$$

Therefore T is also bounded in $L^{p'}(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^1(\mathbb{R})$.

Select $0 \leq \theta \leq 1$ such that $\frac{1}{p_3} = \frac{1}{p} + \frac{\theta}{p'}$. Now using interpolation T will be bounded in $L^q(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_3}(\mathbb{R})$ for $\frac{1}{q} = \frac{\theta}{p'} = \frac{1}{p_3} - \frac{1}{p} = \frac{1}{p_3} - \frac{1}{p_1} - \frac{1}{p_2} + 1$. ∎

References

[1] Bergh, J., Löfström, J., *Interpolation spaces.* Springer Verlag [1970].

[2] Blasco,O. *Bilinear multipliers and transference.* Int. J. Math. Math. Sci. **2005**(4) [2005], 545-554.

[3] Berkson, E., Blasco, O., Carro, M. and Gillespie, A. *Discretization versus transference for bilinear operators.* Banach Spaces and their Applications in Analysis, 11-30 , De Gruyter [2007].

[4] Blasco, O., Carro, M., and Gillespie, A. *Bilinear Hilbert transform on measure spaces.* J. Fourier Anal. and Appl.**11** [2005], 459-470.

[5] Blasco, O., Villarroya, F., *Transference of bilinear multipliers on Lorentz spaces.* Illinois J. Math. 47(4),[2005], 1327-1343

[6] Coifman R.R., Meyer, Y. *Fourier Analysis of multilinear convolution, Calderón theorem and analysis of Lipschitz curves.* Euclidean Harmonic Analysis (Proc. Sem. Univ. Maryland, College Univ., Md) Lecture Notes in Maths. **779**, [1979], 104-122.

[7] Fan, D., Sato, S., *Transference of certain multilinear multipliers operators.* J. Austral. Math. Soc. **70**, [2001], pp. 37-55.

[8] Gilbert, J, Nahmod, A.,*Bilinear operators with non-smooth symbols,* J. Fourier Anal. Appl., **7**[2001], pp. 435-467.

[9] Gilbert, J, Nahmod, A.,*Boundedness Bilinear operators with non-smooth symbols,* Mat. Res. Letters, **7**[2000], pp. 767-778.

[10] Grafakos, L., Kalton, N., *Some remarks on multilinear maps and interpolation.* Math. Annalen **319** [2001], pp. 151-180

[11] Kenig, C. E., Stein, E.M., *Multilinear estimates and fractional integration.* Math. Res. Lett. **6** [1999], pp. 1-15.

[12] Lacey M., Thiele C., *L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$* Annals Math. 146, [1997], pp. 693-724.

[13] Lacey, M., Thiele, C., *On Calderón's conjecture.* Ann. Math. **149** 2 [1999] pp. 475-496.

[14] Stein, E. M., Weiss, G., *Introduction to Fourier Analysis on Euclidean spaces.* Princeton Univ. Press [1971].

[15] Villarroya, P., *La transformada de Hilbert bilineal.* Doctoral Dissertation. Univ. of Valencia [2002].

[16] Villarroya, P., *Bilinear multipliers on Lorentz spaces.* Czechoslovac Math. J. (to appear) [2007].

Department of Mathematics, Universitat de Valencia, Burjassot 46100 (Valencia) Spain

E-mail address: oscar.blasco@uv.es