MODULUS OF CONTINUITY WITH RESPECT TO SEMIGROUPS OF ANALYTIC FUNCTIONS AND APPLICATIONS

O. BLASCO

ABSTRACT. Given a complex Banach space E, a semigroup of analytic functions (φ_t) and an analytic function $F : \mathbb{D} \to E$ we introduce the modulus $w_\varphi(F,t) = \sup_{|z|<1} \|F(\varphi_t(z)) - F(z)\|$. We show that if $0 < \alpha \leq 1$ and F belongs to the vector-valued disc algebra $A(\mathbb{D}, E)$, the Lipschitz condition $M_\infty(F', r) = O((1-r)^{1-\alpha})$ as $r \to 1$ is equivalent to $w_\varphi(F,t) = O(t^\alpha)$ as $t \to 0$ for any semigroup of analytic functions (φ_t), with $\varphi_t(0) = 0$ and infinitesimal generator $\mathcal{G}$, satisfying that φ_t' and $\mathcal{G}$ belong to $H^\infty(\mathbb{D})$, and in particular is equivalent to the condition $\|F - F_r\|_{A(\mathbb{D},E)} = O((1-r)^\alpha)$ as $r \to 1$. We apply this result to particular semigroups (φ_t) and particular spaces of analytic functions E, such as Hardy or Bergman spaces, to recover several known results about Lipschitz type functions.

1. INTRODUCTION

Let $\mathcal{H}(\mathbb{D})$ be the Fréchet space of all analytic functions in the unit disk endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$. A Banach space $(X, \|\cdot\|)$ such that $X \subset \mathcal{H}(\mathbb{D})$ with continuous inclusion will be called a *Banach space of analytic functions*. We shall say that X is an *homogeneous Banach space of analytic functions* (see [2, 9]) if it satisfies the following properties

$$A(\mathbb{D}) \subset X \subset \mathcal{H}(\mathbb{D}) \tag{1}$$

with continuous inclusions,

$$f \in X \Longrightarrow f_\xi \in X \text{ and } \|f_\xi\| = \|f\| \text{ for any } |\xi| = 1, \tag{2}$$

and there exists $C > 0$ such that

$$f \in X \Longrightarrow f_r \in X \text{ and } \|f_r\| \leq C\|f\| \text{ for any } 0 \leq r < 1, \tag{3}$$

where $f_z(w) = f(zw)$ for $z \in \overline{\mathbb{D}}$ and $A(\mathbb{D}) = C(\overline{\mathbb{D}}) \cap H^\infty$ stands for the disc algebra, that is the closed subspace of bounded analytic functions with continuous extension to the boundary.

Date: March 15, 2015.

2000 *Mathematics Subject Classification.* Primary 30H05, 32A37, 47B33, 47D06; Secondary 46E15.

Key words and phrases. semigroups of analytic functions, Lipschitz spaces; Hardy spaces; Bergman spaces.

The author is partially supported by the Project MTM2011-23164(MECC. Spain).

Particular examples are, for $1 \leq p \leq \infty$, the Hardy spaces H^p with the norm

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(f,r)$$

where $M_p(f,r) = (\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi})^{1/p}$ (or $M_\infty(f,r) = \sup_{|\xi|=1} |f(r\xi)|$) and the Bergman spaces A^p with the norm

$$\|f\|_{A^p} = (\int_{\mathbb{D}} |f(w)|^p dA(w))^{1/p}$$

where $dA(w)$ stands for the normalized Lebesgue measure in $\mathbb{D}$.

If $(X, \|\cdot\|)$ is an homogeneous Banach space of analytic functions and $f \in X$ we shall denote

$$w_X(f,t) = \sup_{|\theta|\leq t} \|f_{e^{i\theta}} - f\| \text{ and } M_X(f,r) = \sup_{0\leq\delta\leq r} \|f_\delta\|.$$

We keep the classical notations $w_p(f,t) = w_{H^p}(f,t)$ and $M_p(f,r) = M_{H^p}(f,r)$ for $f \in H^p$. It is easy to see (due to the density of analytic polynomials in the spaces $A(\mathbb{D})$, H^p and A^p) that $w_p(f,t) \to 0$ and $w_{A^p}(f,t) \to 0$ as $t \to 0^+$ for any f in the corresponding space. It is also well known (see for instance [5, 6, 13]) that $\|f - f_r\|_{H^p} \to 0$ and $\|f - f_r\|_{A^p} \to 0$ as $r \to 1$ for $1 \leq p < \infty$. It goes back to the work of Hardy and Littlewood and further extensions (see for instance [5, 8, 12]) that for each $1 \leq p \leq \infty$, $0 < \alpha \leq 1$ and $f \in H^p$ the following conditions are equivalent:

(a) $w_p(f,t) = O(t^\alpha), \quad t \to 0.$

(b) $\|f - f_r\|_{H^p} = O((1-r)^\alpha), \quad r \to 1.$

(c) $M_p(f',r) = O((1-r)^{\alpha-1}), \quad r \to 1.$

In a recent paper Galanopoulos, Siskakis and Stylogiannis (see [7, Theorem 4.1]) have shown the analogue of this result for Bergman spaces, namely for $1 \leq p < \infty$, $0 < \alpha \leq 1$ and $f \in A^p$ the following are equivalent

(i) $w_{A^p}(f,t) = O(t^\alpha), \quad t \to 0^+$

(ii) $\|f - f_r\|_{A^p} = O((1-r)^\alpha), \quad r \to 1$

(iii) $A_p(f',r) = O((1-r)^{\alpha-1}), \quad r \to 1$ where $A_p(f,r) = \|f_r\|_{A^p}$.

There are three goals in the paper: First to exhibit that the equivalences between (a), (b) and (c) and between (i),(ii) and (iii) for Hardy spaces H^p and Bergman spaces A^p in the case $1 \leq p < \infty$ actually follow from the case $p = \infty$ with the use of vector-valued functions in the disc algebra, second to show that they hold true not only for Hardy and Bergman spaces but also for any homogeneous Banach space of analytic functions and third to add other equivalent formulations in terms of

$$w_\varphi(F,t) = \sup_{|z|<1} \|F(\varphi_t(z)) - F(z)\|$$

for a class of semigroups of analytic functions including the translation and dilation semigroups $\tau_t(z) = e^{it}z$ and $D_t(z) = e^{-t}z$.

The paper is organized as follows. In the first section we introduce the modulus of continuity $w_\varphi(F,t)$ of a vector-valued analytic function $F : \mathbb{D} \to E$ with respect a semigroup (φ_t) and discuss on semigroups of analytic functions which are strongly continuous on the vector-valued disc algebra. In the second section we consider functions whose modulus $w_\varphi(F,t)$ behaves like t^α and connect with the behaviour of the derivative of F' and the generator $\mathcal{G}$. In particular we show that for semigroups (φ_t) with $\varphi_t' \in H^\infty(\mathbb{D})$, $\varphi_t(0) = 0$ and $\mathcal{G} \in H^\infty$ we have that the condition $M_\infty(F', r) = O((1-r)^{\alpha-1})$ for $0 < \alpha \le 1$ is equivalent to $w_\varphi(F,t) = O(t^\alpha)$. Finally we apply this result to prove the equivalences mentioned above for Hardy and Bergman spaces.

2. Semigroups on the vector-valued disc algebra

A (one-parameter) semigroup of analytic functions is any continuous homomorphism $\Phi : t \mapsto \Phi(t) = \varphi_t$ from the additive semigroup of nonnegative real numbers into the composition semigroup of all analytic functions which map $\mathbb{D}$ into $\mathbb{D}$. In other words, $\Phi = (\varphi_t)$ consists of analytic functions on $\mathbb{D}$ with $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ and for which the following three conditions hold:

- φ_0 is the identity in $\mathbb{D}$,
- $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \ge 0$,
- $\varphi_t \to \varphi_0$, as $t \to 0$, uniformly on compact subsets of $\mathbb{D}$.

If (φ_t) is a semigroup, then each map φ_t is univalent. The infinitesimal generator of (φ_t) is the function

$$\mathcal{G}(z) = \lim_{t\to 0^+} \frac{\varphi_t(z) - z}{t}, \; z \in \mathbb{D}.$$

This convergence holds uniformly on compact subsets of $\mathbb{D}$ so $\mathcal{G} \in \mathcal{H}(\mathbb{D})$. Moreover $\mathcal{G}$ satisfies

$$\mathcal{G}(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial t} = \mathcal{G}(z)\varphi_t'(z), \; z \in \mathbb{D}, \; t \ge 0. \tag{4}$$

Further $\mathcal{G}$ has a representation

$$\mathcal{G}(z) = (\bar{b}z - 1)(z - b)P(z), \; z \in \mathbb{D} \tag{5}$$

where $b \in \overline{\mathbb{D}}$ and $P \in \mathcal{H}(\mathbb{D})$ with $\operatorname{Re} P(z) \ge 0$ for all $z \in \mathbb{D}$. If G is not identically null, the couple (b, P) is uniquely determined from (φ_t) and the point b is called the Denjoy-Wolff point of the semigroup. We refer to [1, 11] for the basic facts and references on semigroups of analytic functions.

We shall be interested only in those such that $\varphi_t(0) = 0$. Some examples to have in mind are the following:

(E1) $\tau_t(z) = e^{it}z$ where $\tau_t'(z) = e^{it}$ and $\mathcal{G}(z) = iz$.
(E2) $D_t(z) = e^{-t}z$ where $D_t'(z) = e^{-t}$ and $\mathcal{G}(z) = -z$.
(E3) $\varphi_t(z) = \frac{e^{-t}z}{(e^{-t}-1)z+1}$ where $\varphi_t'(z) = \frac{e^{-t}}{(1-(1-e^{-t})z)^2}$ and $\mathcal{G}(z) = -z(1-z)$.

(E4) $\varphi_t(z) = 1 - (1-z)^{e^{-t}}$ where $\varphi_t'(z) = \frac{e^{-t}}{(1-z)^{1-e^{-t}}}$ and with generator $G(z) = -(1-z)\log\frac{1}{1-z}$.

Each semigroup of analytic functions gives rise to a semigroup of operatots (C_t) consisting of composition operators on $\mathcal{H}(\mathbb{D})$ defined by $C_t(f) := f \circ \varphi_t$ for each $f \in \mathcal{H}(\mathbb{D})$.

If X is a Banach space of analytic functions and (φ_t) is a semigroup of analytic functions we say that it *generates a semigroup on* X if (C_t) defines a strongly continuous semigroup of operators on X, that is to say

$$\|f \circ \varphi_t - f\| \to 0, \quad t \to 0^+, \forall f \in X. \tag{6}$$

In particular it was shown in [3] that if (φ_t) is a semigroup with generator $\mathcal{G}$ and X is a Banach space of analytic functions such that (C_t) is strongly continuous on X then the infinitesimal generator Γ of (C_t) is given by $\Gamma(f)(z) = \mathcal{G}(z)f'(z)$ with domain $D(\Gamma) = \{f \in X : Gf' \in X\}$.

Classical choices of X treated in the literature are the Hardy spaces H^p, the disk algebra $A(\mathbb{D})$, the Bergman spaces A^p, the Dirichlet space $\mathcal{D}$, $BMOA$, Bloch as well as their "little oh" analogues. We just mention here that any semigroup of analytic functions generates a semigroup of operators on X in the case of Hardy spaces H^p $(1 \le p < \infty)$, Bergman spaces A^p $(1 \le p < \infty)$, the Dirichlet space, and on the spaces VMOA and little Bloch $\mathcal{B}_0$. However there is no non-trivial semigroup which generates a semigroup of operators in the space H^∞ or in Bloch space $\mathcal{B}$ (see [3, Thm 3]). In the case $X = A(\mathbb{D})$ it was shown by Contreras and Diaz (see [4, Thm 1.2]) that φ_t generates a semigroup on $A(\mathbb{D})$ if and only if $\varphi_t \in A(\mathbb{D})$ for any $t \ge 0$.

DEFINITION 2.1. *Let E be a complex Banach space, let (φ_t) be a semigroup of analytic functions and let $F : \mathbb{D} \to E$ be analytic and bounded. For each $t > 0$ we define*

$$w_\varphi(F,t) = \sup_{|z|<1} \|F(\varphi_t(z)) - F(z)\|. \tag{7}$$

For $\tau_t(z) = e^{it}z$ we write $w(F,t) = \sup_{|z|<1} \|F(e^{it}z)) - F(z)\|$.

We shall be interested in the action of semigroups of analytic functions on the disc algebra of vector-valued functions. Let E be a complex Banach space and denote $A(\mathbb{D}, E)$ the space of holomorphic and bounded functions $F : \mathbb{D} \to E$ with continuous extension to the boundary. We keep the notation F for the function in $C(\overline{\mathbb{D}}, E)$. Of course we have

$$\|F\|_{A(\mathbb{D},E)} = \sup_{|z|<1} \|F(z)\| = \sup_{|\xi|=1} \|F(\xi)\|.$$

We use the notation $F_r(z) = F(rz)$ for each $0 < r < 1$ and $M_\infty(F,r) = \|F_r\|_{A(\mathbb{D},E)}$. We also denote $\mathcal{P}(E)$ for the E-valued analytic polynomials, that is $P(z) = \sum_{k=0}^N x_k z^k$ where $x_k \in E$ and $N \in \mathbb{N}$.

As above we say that a semigroup of analytic functions (φ_t) *generates a semigroup on* $A(\mathbb{D}, E)$ if $C_t^E : A(\mathbb{D}, E) \to A(\mathbb{D}, E)$ given by

$$C_t^E(F)(z) = F(\varphi_t(z))$$

defines a strongly continuous semigroup of operators on $A(\mathbb{D}, E)$, that is to say

$$w_\varphi(F, t) \to 0, \quad t \to 0^+, \forall F \in A(\mathbb{D}, E). \tag{8}$$

Hence if $F \in A(\mathbb{D}, E)$ then $w(F, t) = \sup_{|\xi|=1} \|F(e^{it}\xi) - F(\xi)\|$ and, due to its continuity on $\mathbb{T}$, we have that τ_t generates a semigroup on $A(\mathbb{D}, E)$.

We first remark the following elementary fact:

PROPOSITION 2.1. *Let E be any complex Banach space and let (φ_t) be semigroup in $\mathcal{H}(\mathbb{D})$. Then the following statements are equivalent:*
(i) (φ_t) generates a semigroup on $A(\mathbb{D}, E)$.
(ii) (φ_t) generates a semigroup on $A(\mathbb{D})$.
(iii) $\varphi_t \in A(\mathbb{D})$ for any $t \geq 0$.

Proof. (i) $\Longrightarrow$ (ii). Let $x \in E$ and $f \in A(\mathbb{D})$ and denote $f \otimes x(z) = f(z)x \in A(\mathbb{D}, E)$. The result follows using that

$$C_t(f)(z)x = C_t^E(f \otimes x)(z).$$

(ii) $\Longrightarrow$ (i) Let $F \in A(\mathbb{D}, E)$ and $\varepsilon > 0$. Select $P(z) = \sum_{k=0}^N x_k z^k \in \mathcal{P}(E)$ such that $\|F(z) - P(z)\| < \varepsilon$ for all $z \in \mathbb{D}$. Now

$$\begin{aligned} \|F \circ \varphi_t(z) - F(z)\| &\leq \|P \circ \varphi_t(z) - P(z)\| + 2\varepsilon \\ &\leq \|\sum_{k=0}^N x_k \Big((\varphi_t(z))^k - z^k\Big)\| + 2\varepsilon/3 \\ &\leq \max_{0 \leq k \leq N} |(\varphi_t(z))^k - z^k| \Big(\sum_{k=0}^N \|x_k\|\Big) + 2\varepsilon. \end{aligned}$$

Since $\|p \circ \varphi_t - p\|_\infty \to 0$ as $t \to 0$ for each analytic polynomial p we obtain (i).

(ii) $\Longleftrightarrow$ (iii) It is shown in [4, Theorem 1.2]. □

In particular all the examples (E1)-(E4) generate semigroups in $A(\mathbb{D}, E)$.

PROPOSITION 2.2. *Let $F : \mathbb{D} \to E$ be a bounded analytic function. The following statements are equivalent:*
(i) $F \in A(\mathbb{D}, E)$.
(ii) $\lim_{t \to 0^+} w_\varphi(F, t) = 0$ for any a semigroup of analytic functions (φ_t) such that $\varphi_t \in A(\mathbb{D})$ for each $t \geq 0$.
(iii) $\lim_{r \to 1} \|F_r - F\|_{A(\mathbb{D}, E)} = 0$.

Proof. (i) $\Longrightarrow$ (ii) It follows from Proposition 2.1.

(ii) $\Longrightarrow$ (iii) It follows using the dilation semigroup $D_t(z) = e^{-t}z$ with the change of notation $r = e^{-t}$.

(iii) $\Longrightarrow$ (i) It follows trivially since $F_r \in A(\mathbb{D}, E)$ for all $0 < r < 1$. □

3. Lipschitz type functions with respect to semigroups

Let us first introduce the Lipschitz type and Bloch type spaces to be used in this section.

DEFINITION 3.1. *Let E, (φ_t) and $W : [0,1] \to R^+$ be a complex Banach space, an analytic semigroup and a continuous function with $W(0) = 0$ respectively. We define $\Lambda_{(\varphi_t),W}(\mathbb{D}, E)$ the set of bounded analytic functions $F : \mathbb{D} \to E$ such that*

$$w_\varphi(F,t) = O(W(t)), \quad t \to 0.$$

In the case $W(t) = t^\alpha$ for some $0 < \alpha \le 1$ we simply write $\Lambda_{(\varphi_t),\alpha}(\mathbb{D}, E)$ and $\Lambda_\alpha(\mathbb{D}, E)$ whenever $W(t) = t^\alpha$ and $\varphi_t(z) = e^{it}z$.

DEFINITION 3.2. *Let $F : \mathbb{D} \to E$ be an analytic function and let $w : \mathbb{D} \to \mathbb{R}^+$ be a bounded continuous function. We say that $F \in \mathcal{B}_w(\mathbb{D}, E)$ whenever*

$$\sup_{|z|<1} w(z)\|F'(z)\| < \infty.$$

As usual we denote $\mathcal{B}_\beta(\mathbb{D}, E)$ the case $w(z) = (1 - |z|)^\beta$ for $0 \le \beta \le 1$.

We rephrase the Hardy and Littlewood theorem(see [5, Theorem 5.1]) as $\Lambda_\alpha(\mathbb{D}, \mathbb{C}) = \mathcal{B}_{1-\alpha}(\mathbb{D}, \mathbb{C})$ for $0 < \alpha \le 1$. Similarly Storozhenko's result (see [12]) which estates that $M_p(f', r) = O((1-r)^{\alpha-1})$ iff $\|f_r - f\|_{H^p} = O((1-r)^\alpha)$ corresponds for $p = \infty$ to $\Lambda_{(D_t),\alpha}(\mathbb{D}, \mathbb{C}) = \mathcal{B}_{1-\alpha}(\mathbb{D}, \mathbb{C})$ where $D_t(z) = e^{-t}z$ for $t > 0$ and $z \in \mathbb{D}$.

We would like to extend these results to more general semigroups. Let us first estimate the modulus $w_\varphi(F, t)$ in terms of $\mathcal{G}F'$.

LEMMA 3.1. *Let E be a complex Banach space, let (φ_t) be a semigroup of analytic functions such that $\varphi_t \in A(\mathbb{D})$ and $\varphi_t(0) = 0$ for all $t > 0$. If F belong to $A(\mathbb{D}, E)$ then there exists $C > 0$ such that*

$$w_\varphi(F,t) \le \int_0^t M_\infty(\mathcal{G}F', M_\infty(\varphi_s, r))ds + C\int_r^1 M_\infty(F', \rho)M_\infty(\varphi_t', \rho)d\rho \tag{9}$$

for each $0 < t \le 1/2 \le r < 1$.

Proof. Let $0 < r < 1$. We use triangular inequality to estimate

$$\begin{aligned} w_\varphi(F,t) &\le \sup_{|\xi|=1} \|F(\varphi_t(\xi)) - F(\varphi_t(r\xi))\| \\ &+ \sup_{|\xi|=1} \|F(\varphi_t(r\xi)) - F(r\xi)\| \\ &+ \sup_{|\xi|=1} \|F(r\xi) - F(\xi)\|. \end{aligned}$$

On the one hand

$$F(\varphi_t(\xi)) - F(\varphi_t(r\xi)) = \int_r^1 F'(\varphi_t(\rho\xi))\varphi_t'(\rho\xi)\xi d\rho.$$

Therefore, since $|\varphi_t(z)| \le |z|$ and then $M_\infty(\varphi_t, \rho) \le \rho$, we can estimate

$$\|F(\varphi_t(\xi)) - F(\varphi_t(r\xi))\| \le \int_r^1 M_\infty(F', \rho) M_\infty(\varphi_t', \rho) d\rho. \tag{10}$$

On the other hand, we consider $g(t) = F \circ \varphi_t(r\xi)$. Invoking (4) we obtain

$$\frac{\partial g}{\partial t}(t) = F'(\varphi_t(r\xi)) \mathcal{G}(\varphi_t(r\xi)).$$

This together with the fundamental theorem of calculus gives

$$F(\varphi_t(r\xi)) - F(r\xi) = \int_0^t F'(\varphi_s(r\xi)) \mathcal{G}(\varphi_s(r\xi)) ds.$$

Hence we have

$$\|F(\varphi_t(r\xi)) - F(r\xi)\| \le \int_0^t M_\infty(\mathcal{G}F', M_\infty(\varphi_s, r)) ds. \tag{11}$$

Finally use that for $\rho \ge r \ge 1/2$ we have

$$M_\infty(\varphi_t', \rho) \ge |\varphi_t'(1/2)| \ge \frac{\min_{t\in[0,1/2]} |\mathcal{G}(\varphi_t(1/2))|}{|\mathcal{G}(1/2)|} = C_1$$

and $F(r\xi)) - F(\xi) = -\int_r^1 F'(\rho\xi)\xi d\rho$ to conclude

$$\|F(r\xi)) - F(\xi)\| \le C_1^{-1} \int_r^1 M_\infty(F', \rho) M_\infty(\varphi_t', \rho) d\rho. \tag{12}$$

The result follows from (11), (10) and (12). □

COROLLARY 3.1. *Let E be a complex Banach space and (φ_t) be a semigroup of analytic functions such that $\varphi_t \in A(\mathbb{D})$ and $\varphi_t(0) = 0$ for all $t > 0$ and $\mathcal{G} \in H^\infty(\mathbb{D})$. If F belong to $A(\mathbb{D}, E)$ then there exist $C > 0$ such that*

$$w_\varphi(F, t) \le C \int_{1-t}^1 M_\infty(F', \rho) M_\infty(\varphi_t', \rho) d\rho, \quad 0 < t < 1/2. \tag{13}$$

Proof. Choose $r = 1 - t$ in (9) and note that $M_\infty(\varphi_s, 1-t) \le 1 - t$ and therefore

$$\begin{aligned} \int_0^t M_\infty(F'\mathcal{G}, M_\infty(\varphi_s, 1-t)) ds &\le tM_\infty(F'\mathcal{G}, 1-t) \\ &\le t\|\mathcal{G}\|_\infty M_\infty(F', 1-t) \\ &\le C \int_{1-t}^1 M_\infty(F', \rho) M_\infty(\varphi_t', \rho) d\rho. \end{aligned}$$

□

Let us now give the converse estimates, that is $|\mathcal{G}(z)F'(z)|$ in terms of $w_\varphi(F, t)$.

THEOREM 3.1. *Let (φ_t) a semigroup of analytic functions with infinitesimal generator $\mathcal{G}$. If $F \in A(\mathbb{D}, E)$ and $0 < t, r < 1$ then*

$$M_\infty(\mathcal{G}F', r) \leq \frac{w_\varphi(F,t)}{t} + \frac{M_\infty(\mathcal{G}, r)}{(1-r)t}\int_0^t w_\varphi(F,s)ds. \tag{14}$$

Proof. Using the formula (4) we observe that

$$\mathcal{G}(z)(F\circ\varphi_s)'(z) = \frac{\partial F\circ\varphi_s}{\partial t}(z).$$

Hence

$$F(\varphi_t(z)) - F(z) = \mathcal{G}(z)\int_0^t (F\circ\varphi_s - F)'(z)ds + t\mathcal{G}(z)F'(z).$$

Now using Cauchy formula we have

$$\begin{aligned} t\mathcal{G}(z)F'(z) &= F(\varphi_t(z)) - F(z) + \mathcal{G}(z)\int_0^t (F - F\circ\varphi_s)'(z)ds \\ &= F(\varphi_t(z)) - F(z) + \mathcal{G}(z)\int_0^t \Big(\int_0^{2\pi} \frac{(F - F\circ\varphi_s)(e^{i\theta})}{(1-e^{-i\theta}z)^2}\frac{d\theta}{2\pi}\Big)ds \\ &= F(\varphi_t(z)) - F(z) + \mathcal{G}(z)\int_0^{2\pi} \frac{\int_0^t (F - F\circ\varphi_s)(e^{i\theta})ds}{(1-e^{-i\theta}z)^2}\frac{d\theta}{2\pi}. \end{aligned}$$

Therefore

$$\begin{aligned} \|\mathcal{G}(z)F'(z)\| &\leq \frac{\|F(\varphi_t(z)) - F(z)\|}{t} \\ &+ |\mathcal{G}(z)|\frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{1}{t}\int_0^t \big(\|F(\varphi_s(e^{i\theta})) - F(e^{i\theta})\|\big)ds\Big)\frac{d\theta}{|1-e^{-i\theta}z|^2} \\ &\leq \frac{w_\varphi(F,t)}{t} \\ &+ \Big(\frac{1}{t}\int_0^t w_\varphi(F,s)ds\Big)\frac{|\mathcal{G}(z)|}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-e^{-i\theta}z|^2} \\ &\leq \frac{w_\varphi(F,t)}{t} + \frac{M_\infty(\mathcal{G}, |z|)}{(1-|z|^2)t}\int_0^t w_\varphi(F,s)ds. \end{aligned}$$

This gives the result. □

COROLLARY 3.2. *Let (φ_t) a semigroup of analytic functions, with infinitesimal generator $\mathcal{G} \in H^\infty$, $0 \leq \alpha < 1$ and set $w_\alpha(z) = |\mathcal{G}(z)|(1-|z|)^{1-\alpha}$. Then*

$$\Lambda_{(\varphi_t),\alpha}(\mathbb{D}, E) \subseteq \mathcal{B}_{w_\alpha}(\mathbb{D}, E).$$

Proof. Let $F \in \Lambda_{(\varphi_t),\alpha}(\mathbb{D}, E)$. Choosing $t = 1 - r$ in (14) one gets

$$M_\infty(\mathcal{G}F', r) \leq \frac{1 + \|\mathcal{G}\|_\infty}{1-r}\sup_{s\leq 1-r} w_\varphi(F,s) \leq C(1-r)^{\alpha-1}.$$

This gives $\sup_{|z|<1} w_\alpha(z)|F'(z)| < \infty$. □

COROLLARY 3.3. *Let E be a complex Banach space and $0 \leq \alpha_1, \alpha_2 < 1$ with $\alpha_1 + \alpha_2 < 1$. If (φ_t) be a semigroup of analytic functions such that $\varphi_t \in \mathcal{B}_{\alpha_2}(\mathbb{D})$ and $\varphi_t(0) = 0$ for all $t > 0$ and*

$$\sup_{0 \leq t, \rho < 1} (1-\rho)^{\alpha_2} M_\infty(\varphi_t', \rho) < \infty \tag{15}$$

then

$$\mathcal{B}_{\alpha_1}(\mathbb{D}, E) \cap \mathcal{B}_{w_\alpha}(\mathbb{D}, E) \subset \Lambda_{(\varphi_t),\alpha}(\mathbb{D}, E)$$

where $\alpha = 1 - (\alpha_1 + \alpha_2)$ and $w_\alpha(z) = |\mathcal{G}(z)|(1-|z|)^{1-\alpha}$.

Proof. Let $F \in \mathcal{B}_{\alpha_1}(\mathbb{D}, E) \cap \mathcal{B}_{w_\alpha}(\mathbb{D}, E)$. From (9) for $r = 1 - t$ and (15) we have

$$\begin{aligned} w_\varphi(F,t) &\leq \int_0^t M_\infty(F'\mathcal{G}, M_\infty(\varphi_s, 1-t))ds + C\int_{1-t}^1 M_\infty(F', \rho)(1-\rho)^{-\alpha_2} d\rho \\ &\leq tM_\infty(F'\mathcal{G}, 1-t) + C\int_{1-t}^1 (1-\rho)^{-(\alpha_1+\alpha_2)} d\rho \\ &\leq C_1 t^\alpha + C_2 t^{1-(\alpha_1+\alpha_2)} = C_3 t^\alpha. \end{aligned}$$

□

THEOREM 3.2. *Let E be a complex Banach space, let $F \in A(\mathbb{D}, E)$ and $0 < \alpha \leq 1$. The following are equivalent:*

(i) $F \in \Lambda_\alpha(\mathbb{D}, E)$.

(ii) $M_\infty(F', r) = O(1-r)^{\alpha-1}$ as $r \to 1$.

(iii) $w_\varphi(F,t) = O(t^\alpha)$ as $t \to 0$ for any semigroup of analytic functions (φ_t) with $\varphi_t(0) = 0$ for each $t \geq 0$ and satisfying that $\mathcal{G} \in H^\infty(\mathbb{D})$ and $\varphi_t' \in H^\infty(\mathbb{D})$ with $\sup_{0 \leq t \leq 1} \|\varphi_t'\|_\infty < \infty$.

(iv) $\|F - F_r\|_{A(\mathbb{D},E)} = O((1-r)^\alpha)$, as $r \to 1$.

Proof. (i) $\Longleftrightarrow$ (ii) It follows combining the scalar-valued case, due to Hardy-Littlewood result (see [5]) and the facts

$$w(F,t) = \sup_{\|x^*\|=1} w(\langle F, x^*\rangle, t), \quad M_\infty(F', r) = \sup_{\|x^*\|=1} M_\infty(\langle F', x^*\rangle, r)$$

where $\langle F, x^*\rangle(z) = \langle F(z), x^*\rangle \in A(\mathbb{D})$ for any $x^* \in E^*$ and $\langle F, x^*\rangle'(z) = \langle F'(z), x^*\rangle$.

(ii) $\Longrightarrow$ (iii) It follows from Corollary 3.1 because

$$w_\varphi(F,t) \leq C\int_{1-t}^1 M_\infty(F', s)ds \leq Ct^\alpha.$$

(iii) $\Longrightarrow$ (iv) Select $\varphi_t(z) = D_t(z) = e^{-t}z$ and notice that $F_r(z) = F(\varphi_t(z))$ where $r = e^{-t}$ and $1 - e^{-t} \approx t$ as $t \to 0$. Hence that

$$\|F - F_r\|_{A(\mathbb{D},E)} = w_\varphi(F,t) = O(t^\alpha) = O((1-r)^\alpha).$$

(iv) $\Longrightarrow$ (ii) It follows from Corollary 3.2 applied to $\varphi_t = D_t$. Note that the assumption means that $w_\varphi(F,t) = O(t^\alpha)$ and then

$$M_\infty(F', r) = \frac{1}{r} M_\infty(\mathcal{G}F', r) \leq \frac{2}{r} \frac{\sup_{s<1-r} w_\varphi(F,s)}{1-r} \leq C(1-r)^{\alpha-1}.$$

□

4. Applications

Let $(X, \|\cdot\|)$ be a homogeneous Banach space of analytic functions. For each $f \in X$, say $f(w) = \sum_{n=0}^\infty a_n w^n$, we define the X-valued function given by

$$F_f(z) = f_z = D_s f_{e^{i\theta}} = \sum_{n=0}^\infty a_n u_n z^n, \quad z = e^{-s} e^{i\theta}.$$

where $u_n(w) = w^n$ for each $n \in \mathbb{N} \cup \{0\}$.

PROPOSITION 4.1. *Let $(X, \|\cdot\|)$ be a homogeneous Banach space of analytic function such that the translation and dilation semigroups $\tau_t(z) = e^{it}z$ and $D_s(z) = e^{-s}z$ generate semigroups on X.*

Then the mapping $f \to F_f$ is an isometric inclusion from X into $A(\mathbb{D}, X)$.

Proof. We first observe that (1) implies that there exists a constant $C > 0$ such that $\|u_n\| \leq C$ for each n. Therefore the series $\sum_{n=0}^\infty a_n u_n z^n$ converges absolutely in X for each $z \in \mathbb{D}$. Moreover, due to (2) and (3), if $f \in X$ then $F_f(z) = f_z \in X$ for each $z \in \overline{\mathbb{D}}$. Hence $F_f \in H^\infty(\mathbb{D}, X)$ with $\sup_{|z|<1} \|F_f(z)\| \leq C\|f\|$. Let us show first that $F_f \in A(\mathbb{D}, X)$. Let $z = e^{-s}e^{i\theta}$ and $z' = e^{-s'}e^{i\theta'}$. Therefore

$$\begin{aligned}
\|f_z - f_{z'}\| &= \|D_s f_\theta - D_{s'} f_{\theta'}\| \\
&\leq \|D_s f_\theta - D_{s'} f_\theta\| + \|D_{s'}(f_\theta - f_{\theta'})\| \\
&\leq C\|D_{|s-s'|} f_\theta - f_\theta\| + C\|\tau_{|\theta-\theta'|} f - f\|.
\end{aligned}$$

Since $z' \to z$ in particular gives that $|s - s'| \to 0$ and $|\theta' - \theta| \to 0$, the continuity of $\tau_t f$ and $D_s f$ gives that $F_f \in A(\mathbb{D}, X)$. Finally observe that from (2) we get

$$\|F_f\|_{A(\mathbb{D},X)} = \sup_{|\xi|=1} \|F_f(\xi)\| = \sup_{|\xi|=1} \|f_\xi\| = \|f\|.$$

□

Making use of this proposition and the results for vector-valued analytic functions in the disc algebra we can recover the equivalences mentioned in the introduction.

COROLLARY 4.1. *Let $1 \leq p < \infty$ and $0 < \alpha \leq 1$. Let X^p denote either the Hardy space H^p or the Bergman space A^p and let $f \in X^p$. The following are equivalent*

(i) $\sup_{|\theta|\leq t} \|f_{e^{i\theta}} - f\|_{X^p} = O(t^\alpha), t \to 0.$

(ii) $\|f - f_r\|_{X^p} = O((1-r)^\alpha), r \to 1.$
(iii) $\|f_r'\|_{X^p} = O((1-r)^{\alpha-1}), r \to 1.$

Proof. Notice that both H^p and A^p for $1 \le p < \infty$ satisfy the assumption in Proposition 4.1. Hence the function $F_f \in A(\mathbb{D}, X^p)$ and $\|F_f\|_{A(\mathbb{D},X^p)} = \|f\|_{X^p}$. In particular, since $F_f - (F_f)_r = F_{f-f_r}$ we have $\|F_f - (F_f)_r\|_{A(\mathbb{D},X^p)} = \|f - f_r\|_{X^p}$. On the other hand

$$w(F_f, t) = \sup_{|\xi|=1} \|F_f(e^{it}\xi) - F_f(\xi)\|_{X^p} = \sup_{|\theta|\le t} \|f_{e^{i\theta}} - f\|_{X^p}.$$

Hence (i) and (ii) in this corollary correspond to conditions (i) and (iv) in Theorem 3.2 and turn out to be equivalent to $M_\infty(F_f', r) = O((1-r)^{\alpha-1}), r \to 1$. Finally observe that for $z \in \mathbb{D}$ we have

$$(F_f)'(z) = \sum_{n=0}^{\infty} n a_n u_n z^{n-1} = S\Big((f')_z\Big)$$

where $S(f)(w) = wf(w)$. Therefore for $X^p = H^p$ we have $M_\infty((F_f)', r) = M_p(f', r) = \|f_r'\|_{H^p}$ and for $X^p = A^p$ we have

$$M_\infty((F_f)', r) = \sup_{|z|=r} \Big(\int_{\mathbb{D}} |w|^p |f'(wz)|^p dA(w)\Big)^{1/p} = r\|f_r'\|_{A^p}. \tag{16}$$

In both cases we recover (iii).

□

REFERENCES

[1] E. Berkson and H. Porta, *Semigroups of analytic functions and composition operators*, Michigan Math. J. **25** (1978) 101–115.

[2] O. Blasco and M. Pavolvic *Coefficient multipliers on Banach spaces of analytic functions.* Rev. Mat. Iberoam., **27**(2)(2011) 415447.

[3] O. Blasco, M. D. Contreras, S. Díaz-Madrigal,J. Martinez, M. Papadimitriakis and A. G. Siskakis*Semigroups of composition operators and integral operators in spaces of analytic functions*, Ann. Acad. Scien. Fennicae Math.**38** (2013), 1–23.

[4] M.D. Contreras and S. Díaz-Madrigal, *Fractional iteration in the disk algebra: prime ends and composition operators*, Revista Mat. Iberoamericana **21** (2005) 911–928.

[5] P. L. Duren, Theory of H^p spaces, Academic Press, New York and London, 1970.

[6] J. B. Garnett, Bounded Analytic Functions, Pure and Applied Mathematics, Vol. **96** Academic Press, New York, 1981.

[7] P. Galanopoulos, A.G. Siskakis and G. Stylogiannis, *On the mean Lipschitz spaces*, J. Math. Anal. Appl. (to appear) .

[8] M. Pavlovic, *Lipschitz conditions on the modulus of a harmonic function*, Rev. Mat. Iberoamericana **23** (2007) 831–845.

[9] M. Pavlovic, Function spaces on the unit disc, An introduction. De gruyter Studies in Math., Vol 52 . De Gruyter, Berlin, 2014.

[10] A. G. Siskakis, *Semigroups of composition operators on the Dirichlet space*, Results Math. **30** (1996), 165–173.

[11] A. G. Siskakis, *Semigroups of composition operators on spaces of analytic functions, a review*, Contemp. Math. **213** (1998) 229–252.

[12] Eh. A. Storozhenko, *On a problem of Hardy-Littlewood*, Math. USSR. Sb. **47** (1984), 557–577; translation from Mat. Sb., Nov. Ser.**119** (1982), 564–583.

[13] K. Zhu, Operator Theory in Function spaces, Marcel Dekker, Inc., New York and Basel, 1990.

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE VALENCIA, 46100 BURJASSOT, VALENCIA, SPAIN.
E-mail address: `Oscar.Blasco@uv.es`