THE p-BOHR RADIUS OF A BANACH SPACE

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Abstract. Following the scalar-valued case considered by Djakow and Ramanujan in [20] we introduce, for each complex Banach space X and each $1 \leq p < \infty$, the p-Bohr radius of X as the value

$$r_p(X) = \sup\{r \geq 0 : \sum_{n=0}^{\infty} \|x_n\|^p r^{np} \leq \sup_{|z|<1} \|f(z)\|^p\}$$

where $x_n \in X$ for each $n \in \mathbb{N} \cup \{0\}$ and $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^\infty(\mathbb{D}, X)$. We show that for a complex (possibly infinite dimensional) Banach space X the condition $r_p(X) > 0$ for some $p \geq 2$ and is equivalent to X being p-uniformly $\mathbb{C}$-convex. We analyze the p-Bohr radius in the cases $X = L^q(\mu)$ for different values of p and q showing that for $p < 2$ and $dim(L^q(\mu)) \geq 2$ one has $r_p(L^q(\mu)) = 0$ while for $p \geq 2$ one has $r_p(L^q(\mu)) = 1$ whenever $p' \leq q \leq p$. We also provide some lower estimates for $r_2(L^q(\mu))$ for $1 \leq q < 2$.

1. Introduction and preliminaries

Let us start by recalling the remarkable discovery of H. Bohr of a universal constant $r_1 = \frac{1}{3}$ (denoted the Bohr radius) satisfying

$$\sum_{n=0}^{\infty} |a_n| (\frac{1}{3})^n \leq \|f\|_\infty, \tag{1.1}$$

for any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^\infty(\mathbb{D}, \mathbb{C})$. The reader is referred to the paper by H. Bohr [10] which includes Wiener's proof showing that $r_1 = \frac{1}{3}$ is sharp. A bit later some other proofs of such inequality were obtained (see [22, 26]).

Throughout the decades several variations of Bohr's inequality (1.1) have appeared. Djakov and Ramanujan in [20] (see also [4] for further considerations replacing the H^∞-norm by the H^p-norm) studied, for each $1 \leq p < \infty$, the best constant r_p such that

$$\Big(\sum_{n=0}^{\infty} |a_n|^p (r_p)^{np}\Big)^{1/p} \leq \|f\|_{H^\infty}, \tag{1.2}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Notice that although Bohr's result establishes that $r_1 = 1/3$ and clearly $r_p = 1$ for $p \geq 2$ due to Haussdorf-Young's inequality, however computing the precise value of r_p for $1 < p < 2$ seems to be rather complicated. As far as we know the best

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known estimates were obtained in [20, Theorem 3] and are given by

$$\Big(1+(\frac{2}{p})^{\frac{1}{2-p}}\Big)^{\frac{p-2}{p}} \leq r_p \leq \inf_{0\leq a<1} \frac{(1-a^p)^{1/p}}{((1-a^2)^p+a^p(1-a^p))^{1/p}}. \tag{1.3}$$

A bit later V. Paulsen, G. Popescu, D. Singh [22, Corollary 2.7] gave the following modification of (1.1)

$$|a_0|^2+\sum_{n=1}^{\infty}|a_n|(\frac{1}{2})^n \leq 1 \tag{1.4}$$

whenever $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $\|f\|_{H^\infty}\leq 1$. Also the value $1/2$ is sharp.

More recently the author (see [6, Proposition 2.4]) extended such a result to $1\leq p\leq 2$ showing that if $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $\|f\|_{H^\infty}\leq 1$ one has

$$|a_0|^p+\sum_{n=1}^{\infty}|a_n|(\frac{p}{2+p})^n \leq 1. \tag{1.5}$$

Also the value $\frac{p}{2+p}$ is sharp.

Several authors (see [4, 11, 12, 19, 20, 28]) have found some other extensions of Bohr's estimate in different directions. For instance, after the paper by Dineen and Timoney [18] some multi-dimensional analogues of Bohr's inequality where the disc $\mathbb{D}$ is replaced by a domain $\Omega\subset\mathbb{C}^m$ were considered in [9]. Since then several applications and connections with local Banach space theory and other topics were shown by different authors (see for instance [1, 2, 3, 14, 15, 16]).

In this paper we are interested in the vector-valued analogue of (1.2) and to show its possible connection with Banach space theory. Let us fix our notation and give some definitions first. Throughout this paper $H^\infty(\mathbb{D},X)$ stands for the space of bounded holomorphic functions from the unit disc $\mathbb{D}$ into a complex Banach space X and we write $\|f\|_{H^\infty(\mathbb{D},X)}=\sup_{|z|<1}\|f(z)\|$. As usual for $1\leq p<\infty$, $H^p(\mathbb{D},X)$ stands for the space of holomorphic functions from $\mathbb{D}$ into X such that

$$\|f\|_{H^p(\mathbb{D},X)}=\sup_{0<r<1}(\int_0^{2\pi}\|f(re^{it})\|^p\frac{dt}{2\pi})^{1/p}<\infty.$$

In [6] the author defined the Bohr's radius of a Banach space X as the value

$$R(X)=\sup\{r\geq 0:\sum_{n=0}^{\infty}\|x_n\|r^n\leq\|f\|_{H^\infty(\mathbb{D},X)}\}.$$

Embedding $\mathbb{C}$ into X we obtain the trivial upper bound $R(X)\leq\frac{1}{3}$ for any Banach space X. However it was shown that $R(X)=0$ for $X=\mathbb{C}_p^m$ whenever $m\geq 2$, where $\mathbb{C}_p^m$, for $1\leq p\leq\infty$, stands for the space $\mathbb{C}^m$ endowed with the norm $\|w\|_p=(\sum_{i=1}^m|w_i|^p)^{1/p}$ for $1\leq p<\infty$ and $\|w\|_\infty=\sup_{i=1}^m|w_i|$.

This fact led the author to consider the vector-valued analogue of (1.4) and to introduce for a given Banach space X and parameters $0<p,q<\infty$, the quantities (see [6, Definition 1.3])

$$R_{p,q}(f,X)=\sup\{r\geq 0:\|x_0\|^p+(\sum_{n=1}^{\infty}\|x_n\|r^n)^q\leq 1\} \tag{1.6}$$

where $f(z)=\sum_{n=0}^{\infty}x_nz^n$ with $\|f\|_{H^\infty(\mathbb{D},X)}\leq 1$, and

$$R_{p,q}(X)=\inf\{R_{p,q}(f,X):\|f\|_{H^\infty(\mathbb{D},X)}\leq 1\}. \tag{1.7}$$

Several results concerning the values $R_{p,q}(X)$ for $X=\mathbb{C}$, $X=\mathbb{C}_p^m$ and $X=L^p$ were analyzed.

Let us consider now the vector-valued analogue of the approach due to Djakov and Ramanujan.

Definition 1.1. Let $1\le p<\infty$ and let X be a complex Banach space. We write

$$r_p(f,X)=\sup\{r\ge 0:\sum_{n=0}^{\infty}\|x_n\|^p r^{np}\le 1\} \tag{1.8}$$

where $f(z)=\sum_{n=0}^{\infty}x_nz^n$ with $\|f\|_{H^\infty(\mathbb{D},X)}\le 1$ and define the *p-Bohr radius of X*

$$\begin{aligned} r_p(X) &= \inf\{r_p(f,X):\|f\|_{H^\infty(\mathbb{D},X)}\le 1\}\\ &= \sup\{r>0:\Big(\sum_{n=0}^{\infty}\|x_n\|^p r^{np}\Big)^{1/p}\le \|f\|_{H^\infty(\mathbb{D},X)}\}.\end{aligned}$$

Of course the quantities $r_p(X)$ and $R_{p,q}(X)$ are related. Actually for $1\le p<\infty$ and $1/p+1/p'=1$ we have that

$$R_{p,p}(X)\le r_p(X)\le 2^{1/p'}R_{p,p}(X) \tag{1.9}$$

Indeed, clearly $R_{p,p}(X)\le r_p(X)$ using the estimate $\sum_{n=1}^{\infty}\|x_n\|^p r^{np}\le(\sum_{n=1}^{\infty}\|x_n\|r^n)^p$. On the other hand, if $f\in H^\infty(\mathbb{D},X)$ and we denote $r_p(f,X)=r$ then for each $0<s<1$ one has

$$\begin{aligned}\|x_0\|^p+(\sum_{n=1}^{\infty}\|x_n\|r^ns^n)^p &\le \|x_0\|^p+(\sum_{n=1}^{\infty}\|x_n\|^p r^{np})(\frac{s^{p'}}{1-s^{p'}})^{p-1}\\ &\le \|x_0\|^p+(1-\|x_0\|^p)(\frac{s^{p'}}{1-s^{p'}})^{p-1}.\end{aligned}$$

Choosing $s=2^{-1/p'}$ one gets $r_p(X)\le 2^{1/p'}R_{p,p}(X)$.

In particular using (1.9) and (1.5) we obtain a lower estimate for r_p, namely

$$r_p\ge R_{p,p}\ge R_{p,1}=\frac{p}{2+p}. \tag{1.10}$$

The reader should notice that (1.10) is sharp for $p=1$ while for $p=2$ one only gets $r_2\ge\frac{1}{2}$.

Remark 1.2. For any Banach space X the function $p\to r_p^p(X)$ is increasing, that is

$$r_{p_1}^{p_1}(X)\le r_{p_2}^{p_2}(X),\quad p_1\le p_2. \tag{1.11}$$

Indeed, first recall that $\|x_n\|\le\|f\|_{H^\infty(\mathbb{D},X)}$ for all n (this can be seen composing with functionals $x^*\in X^*$ and using the scalar-valued case). Hence if $f(z)=\sum_{n=0}^{\infty}x_nz^n\in H^\infty(\mathbb{D},X)$ has norm 1 and $p_1\le p_2$ then

$$\sum_{n=0}^{\infty}\|x_n\|^{p_2}(r_{p_1}(f,X)^{p_1/p_2})^{np_2}\le\sum_{n=0}^{\infty}\|x_n\|^{p_1}r_{p_1}(f,X)^{np_1}\le 1.$$

This gives $r_{p_1}^{p_1}(f,X)\le r_{p_2}^{p_2}(f,X)$ and we obtain (1.11).

The estimate (1.11) can be easily improved using interpolation as the following lemma shows.

Proposition 1.3. *Let X be a complex Banach space, $1 \leq p_1 < p < p_2 < \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then*

$$r_p(X) \geq r_{p_1}(X)^{1-\theta} r_{p_2}(X)^{\theta}. \tag{1.12}$$

Proof. Let us show that for each $f \in H^\infty(\mathbb{D}, X)$ with norm 1 we have that

$$r_p(f,X) \geq r_{p_1}(f,X)^{1-\theta} r_{p_2}(f,X)^{\theta}.$$

Denote $r_1 = r_{p_1}(f,X)$ and $r_2 = r_{p_1}(f,X)$. Hence $(\sum_{n=0}^\infty \|x_n\|^{p_1} r_1^{np_1})^{1/p_1} \leq 1$ and $(\sum_{n=0}^\infty \|x_n\|^{p_2} r_2^{np_2})^{1/p_2} \leq 1$. Now setting $\frac{p_1}{(1-\theta)p} = q_1$, $\frac{p_2}{\theta p} = q_2$ and $r = r_1^{(1-\theta)} r_2^{\theta}$ we can use Hölder's inequality to obtain

$$\begin{aligned} (\sum_{n=0}^\infty \|x_n\|^p r^{np})^{1/p} &= (\sum_{n=0}^\infty \|x_n\|^{p(1-\theta)} r_1^{n(1-\theta)p} \|x_n\|^{p\theta} r_2^{np\theta})^{1/p} \\ &\leq (\sum_{n=0}^\infty \|x_n\|^{p_1} r_1^{np_1})^{(1-\theta)/p_1} (\sum_{n=0}^\infty \|x_n\|^{p_2} r_2^{np_2})^{\theta/p_2} \leq 1. \end{aligned}$$

This shows that $r_p(f,X) \geq r_1^{(1-\theta)} r_2^{\theta}$. □

As a consequence of (1.12) one gets the lower estimate

$$r_p \geq (\frac{1}{3})^{2/p-1}, \quad 1 < p < 2. \tag{1.13}$$

Note that since $3^y \leq 2 + y$ for $0 \leq y \leq 1$, choosing $y = 2/p - 1$, we obtain $\frac{p}{2+p} \leq (\frac{1}{3})^{2/p-1}$ and then (1.13) improves (1.10). One may wonder whether $r_p = (\frac{1}{3})^{2/p-1}$ for $1 < p < 2$. However the already known lower estimate given in (1.3) is better than the one obtained in (1.13). Indeed, the inequality $2^x < x + 1$ for $0 < x < 1$ implies, choosing $x = p - 1$, that $(\frac{2}{p})^{\frac{1}{2-p}} < 2$ and then

$$(1 + (\frac{2}{p})^{\frac{1}{2-p}})^{\frac{p-2}{p}} > (\frac{1}{3})^{\frac{2-p}{p}}, 1 < p < 2.$$

Of course $0 \leq r_p(X) \leq 1$ for any Banach space (just take $f(z) = xz$ with $\|x\| = 1$) and $r_p(X) \leq r_p$ for any complex Banach space X. Due to (1.9) and [6, Theorem 2.2] one can not expect $r_p(X) > 0$ for $dim(X) \geq 2$. However it is not difficult to find examples with $r_p(X) > 0$ or even $r_p(X) = 1$ for values $p \geq 2$.

We would like to mention two well-known properties which appear naturally when considering the p-Bohr radius and which allow us to see that if $1 < q < \infty$ then $r_p(L^q(\mu)) = 1$ for certain values of $p \geq 2$ and $r_p(L^\infty(\mu)) = 0$ for any $p \geq 1$.

We first recall the notion of Fourier type p first introduced by J. Peetre

Definition 1.4. (see [25]) Let $1 < p \leq 2$. A complex Banach space X is said to have *Fourier type* p if there exists $F_p(X) > 0$ such that for any $f \in L^p(\mathbb{T}, X)$

$$(\sum_{n=-\infty}^{\infty} \|\hat{f}(n)\|^{p'})^{1/p'} \leq F_p(X) \|f\|_{L^p(\mathbb{T},X)}$$

where $1/p + 1/p' = 1$.

Proposition 1.5. *Let $1 < p \leq 2$ and $1/p + 1/p' = 1$. If X has Fourier type $p > 1$ with $F_p(X) = 1$ then $r_{p'}(X) = 1$.*

In particular

$$r_{p'}(L^q(\mu)) = 1, \quad 1 < q < \infty, \quad p' \geq \max\{q, q'\}. \tag{1.14}$$

Proof. The inequality

$$(\sum_{n=0}^{\infty}\|x_n\|^{p'})^{1/p'} \le \|f\|_{H^p(\mathbb{D},X)} \le \|f\|_{H^\infty(\mathbb{D},X)}$$

for any $f=\sum_{n=0}^\infty x_n z^n \in H^\infty(\mathbb{D},X)$, gives that $r_{p'}(f,X)\ge 1$.

(1.14) follows using that $L^q(\mu)$ has Fourier type $p=\min\{q,q'\}$ and $F_p(L^q(\mu))=1$ (see [25]). □

Besides the notion of Fourier type bigger than one, there is another geometrical property of the Banach space that plays some important role to have $r_p(X)>0$.

Definition 1.6. (see [27]) A complex Banach space X is said to satisfy the *strong maximum modulus theorem* if $\|f(z)\|$ has no maximum in $\mathbb{D}$ for any non-constant bounded analytic function $f:\mathbb{D}\to X$.

Proposition 1.7. *If $r_p(X)>0$ for some $1\le p<\infty$ then X does satisfy the strong maximum modulus theorem.*

In particular $X=\mathbb{C}^m_\infty$ for $m\ge 2$, $X=c_0$ and $X=C([0,1])$ satisfy that $r_p(X)=0$ for any $1\le p<\infty$.

Proof. Assume that there exists a non constant $f\in H^\infty(\mathbb{D},X)$ of norm 1 and $z_0\in\mathbb{D}$ such that $\|f(z_0)\|=1$. Using a Moebius transformation we may assume that $z_0=0$. Hence $r_p(f,X)=0$ and therefore $r_p(X)=0$ for any $p\ge 1$.

To finish the proof it suffices to recall that C^m_∞ for $m\ge 2$, c_0 and $C([0,1])$ do not satisfy the strong maximum modulus theorem (see [27]). □

The strong maximum modulus theorem is related with the strict c-convexity (see [21, 27]). Let us mention certain notions on $\mathbb{C}$-convexity that are particularly interesting in our situation.

Definition 1.8. (see [21, 13]) Let $2\le p<\infty$. A complex Banach space X is called p-uniformly $\mathbb{C}$-convex if there exists a constant $\lambda>0$ such that

$$(\|x\|^p+\lambda\|y\|^p)^{1/p}\le \max_\theta \|x+e^{i\theta}y\| \tag{1.15}$$

for all $x,y\in X$. Denote $A_p(X)$ the supremum of the constants λ satisfying (1.15).

There are some equivalent formulations of such a concept. One is the so-called p-uniformly PL-convexity (we refer to [17, 21, 23, 24] for information on that) where the $\max_\theta \|x+e^{i\theta}y\|$ is replaced by $(\int_0^{2\pi}\|x+e^{i\theta}y\|^q\frac{d\theta}{2\pi})^{1/q}$ for some $1\le q<\infty$ and another one is given in terms of Littlewood-Paley inequalities (see [7]).

Our main theorem gives another interesting characterization of such a convexity property.

Theorem 1.9. *Let X be a complex Banach space and $p\ge 2$. X is p-uniformly $\mathbb{C}$-convex if and only if the p-Bohr radius $r_p(X)>0$.*

This result is closely related to another description of p-uniformly $\mathbb{C}$-convexity achieved in [7, Proposition 2.1] which states the existence of a constant $\lambda>0$ such that

$$(\|f(0)\|^p+\lambda\|f'(0)\|^p)^{1/p}\le \|f\|_{H^\infty(\mathbb{D},X)} \tag{1.16}$$

for any $f\in H^\infty(\mathbb{D},X)$.

The paper is divided into two sections. In the first one we prove Theorem 1.9 and in the second one we study $r_p(L^q(\mu)$ for different values of $1 \leq p, q < \infty$. From (1.14) we know that for $1 < q < 2$ one has that $r_{q'}(L^q(\mu)) = 1$ and since $L^q(\mu)$ is 2-uniformly $\mathbb{C}$-convex (see [21, 13]) for $1 \leq p \leq 2$ Theorem 1.9 gives that actually $r_2(L^q(\mu)) > 0$. We shall get some lower estimates of $r_p(L^q(\mu)$ whenever $1 \leq q < p' \leq 2 \leq p < \infty$.

2. Geometrical characterizations

Let is introduce the following variation of the p-Bohr radius motivated by (1.16).

Definition 2.1. Let X be a complex Banach space and $1 \leq p < \infty$. We denote

$$\tilde{r}_p(X) = \sup\{r > 0 : \|f(0)\|^p + r^p\|f'(0)\|^p \leq \|f\|^p_{H^\infty(\mathbb{D},X)}\} \tag{2.1}$$

According to the result mentioned in the introduction p-uniformly $\mathbb{C}$-convexity means $\tilde{r}_p(X) > 0$ for $p \geq 2$. Since $r_p(X) \leq \tilde{r}_p(X)$ one has that spaces with positive p-Bohr radius are always p-uniformly $\mathbb{C}$-convex. However we shall present an independent proof of Theorem 1.9 and get the characterization in (1.16) as a consequence.

Let us first mention that contrary to the situation for $r_p(\mathbb{C})$ a precise value of $\tilde{r}_p(\mathbb{C})$ can be computed for all values of p.

Proposition 2.2. *Let $1 \leq p < \infty$ and define*

$$\gamma_p = \inf_{0<a<1} \frac{(1-a^p)^{1/p}}{1-a^2}. \tag{2.2}$$

Then $\tilde{r}_p(\mathbb{C}) = \gamma_p$.

Proof. From Schwarz-Pick lemma we have that $|f'(0)| \leq (1 - |f(0)|^2)$ for any $f \in H^\infty(\mathbb{D})$ with norm 1. Hence for $f(z) = \sum_{n=0}^\infty a_n z^n$ and $\|f\|_{H^\infty} \leq 1$ we have

$$|a_0|^p + |a_1|^p\gamma_p^p \leq |a_0|^p + (1 - |a_0|^2)^p\gamma_p^p \leq 1.$$

Therefore we obtain $\tilde{r}_p(\mathbb{C}) \geq \gamma_p$.

To see the other inequality consider $\phi_a(z) = \frac{z-a}{1-az} = -a + \frac{1-a^2}{a}\sum_{n=1}^\infty a^n z^n$ which belongs to the unit ball of H^∞. Then

$$|a|^p + \tilde{r}_p(\mathbb{C})^p(1 - |a|^2)^p \leq \|\phi_a\|^p_{H^\infty} = 1, \quad |a| < 1$$

This shows that

$$\tilde{r}_p(\mathbb{C})^p \leq \frac{1-a^p}{(1-a^2)^p}, \quad 0 < a < 1.$$

Hence $\tilde{r}_p(\mathbb{C}) \leq \gamma_p$. □

The value of γ_p is given in the following formula.

Proposition 2.3. *Let $p \geq 1$. Then $\tilde{r}_1(\mathbb{C}) = \frac{1}{2}$, $\tilde{r}_p(\mathbb{C}) = 1$ for $p \geq 2$ and*

$$\tilde{r}_p(\mathbb{C}) = \frac{x_p^{-1/p'} + x_p^{1/p}}{2}, \quad 1 < p < 2$$

where $(1 - x_p)^{2/p-1} = \frac{1}{1+x_p}$ and $0 < x_p < 1$.

Proof. The case $p = 1$ and $p \geq 2$ are immediate from Proposition 2.2. Let us consider $1 < p < 2$ and denote

$$h_p(x) = \frac{(1-(1-x)^{2/p})^p}{x}. \tag{2.3}$$

Clearly $\gamma_p^{-p} = \sup_{0<x<1} h_p(x)$.

One observes that $h_p(1) = 1$, $\lim_{x\to 0^+} h_p(x) = 0$, $h_p'(0^+) = \lim_{x\to 0} \frac{h_p(x)}{x} = (\frac{2}{p})^p \lim_{x\to 0} \frac{x^p}{x^2} = \infty$ and $h_p'(1^-) = -1$. Since for $0 < x < 1$ one has

$$h_p'(x) = \frac{1}{x^2}(1-(1-x)^{2/p})^{p-1}\Big((1-x)^{2/p-1}(1+x) - 1\Big)$$

the function h_p attains it maximum at $0 < x_p < 1$ such that $h_p'(x_p) = 0$ (that is to say $(1-x_p)^{2/p-1} = \frac{1}{1+x_p}$). Moreover

$$\gamma_p^{-p} = h_p(x_p) = \frac{2^p x_p^{p-1}}{(1+x_p)^p} \tag{2.4}$$

and the result is achieved. □

Theorem 2.4. *Let $p \geq 1$. Then*

$$\frac{\tilde{r}_p(X)}{(\tilde{r}_p^p(X)+1)^{1/p}} \leq r_p(X) \leq \tilde{r}_p(X). \tag{2.5}$$

$$\frac{A_p(X)^{1/p}}{2} \leq \tilde{r}_p(X) \leq A_p(X)^{1/p}, \quad p \geq 2. \tag{2.6}$$

Proof. Let $f \in H^\infty(\mathbb{D}, X)$ with norm 1 and $f(z) = \sum_{n=0}^\infty x_n z^n$. Let $n \in \mathbb{N}$ and consider $\xi = e^{\frac{2\pi i}{n}}$ and define $g(z) = \frac{1}{n}\sum_{j=1}^n f(\xi^j z)$. Using that $\sum_{j=1}^n \xi^j = 0$ we obtain

$$g(z) = x_0 + x_n z^n + x_{2n} z^{2n} + \cdots.$$

Since $g \in H^\infty(\mathbb{D}, X)$ with norm $\|g\|_{H^\infty(\mathbb{D},X)} \leq 1$ we have that

$$\|g'(0)\|^p \leq \tilde{r}_p(X)^{-p}(1 - \|g(0)\|^p).$$

This gives the estimate

$$\|x_n\|^p \leq \tilde{r}_p(X)^{-p}(1 - \|x_0\|^p), \quad n \geq 1.$$

Therefore

$$\begin{aligned} \|x_0\|^p + \sum_{n=1}^\infty \|x_n\|^p r^{pn} &\leq \|x_0\|^p + \tilde{r}_p(X)^{-p}(1 - \|x_0\|^p)(\sum_{n=1}^\infty r^{pn}) \\ &\leq \|x_0\|^p + \tilde{r}_p(X)^{-p}(1 - \|x_0\|^p)\frac{r^p}{1-r^p} \\ &\leq \max\{1, \tilde{r}_p(X)^{-p}\frac{r^p}{1-r^p}\}. \end{aligned}$$

Now choosing r such that $\tilde{r}_p(X)^p = \frac{r^p}{1-r^p}$ we obtain (2.5).

Let us now see (2.6). Let $f \in H^\infty(\mathbb{D}, X)$ with $\|f\|_{H^\infty(\mathbb{D},X)} \leq 1$ and let ξ be in the unit ball of X^*. Since $\langle \xi, f\rangle \in H^\infty(\mathbb{D}, \mathbb{C})$ with $\|\langle \xi, f\rangle\|_{H^\infty(\mathbb{D},\mathbb{C})} \leq 1$ then Schwarz-Pick lemma gives

$$|\langle \xi, f'(0)\rangle| \leq 1 - |\langle \xi, f(0)\rangle|^2 \leq 2(1 - |\langle \xi, f(0)\rangle|).$$

This shows that, for any ξ in the unit ball of X^*,

$$|\langle \xi, f(0)\rangle| + \frac{1}{2}|\langle \xi, f'(0)\rangle| \leq 1.$$

Therefore, for any $\theta \in [0, 2\pi)$,

$$\|f(0) + \frac{e^{i\theta}}{2} f'(0)\| = \sup_{\|\xi\|=1} |\langle \xi, f(0)\rangle + \frac{e^{i\theta}}{2}\langle \xi, f'(0)\rangle| \leq 1.$$

Hence

$$\|f(0)\|^p + \frac{A_p(X)}{2^p}\|f'(0)\|^p \leq 1.$$

This gives $\frac{A_p(X)^{1/p}}{2} \leq \tilde{r}_p(X)$. □

Proof of Theorem 1.9

Applying Theorem 2.4 one obtains

$$\frac{A_p^{1/p}(X)}{(A_p(X)+2^p)^{1/p}} \leq r_p(X) \leq A_p^{1/p}(X). \tag{2.7}$$

Taking into account that p-uniformly $\mathbb{C}$-convexity means $A_p(X) > 0$ the proof is complete. □

Combining (2.5) in Theorem 2.4 and Proposition 2.2 one gets the following lower estimate for r_p.

Corollary 2.5. *Let $1 < p < 2$. Then*

$$r_p \geq (1 + \gamma_p^{-p})^{-1/p}. \tag{2.8}$$

Combining (2.6) and (2.5) we also obtain the following lower estimate.

Corollary 2.6. *If X is 2-uniformly $\mathbb{C}$-convex then $r_2(X) \geq \sqrt{\frac{A_2(X)}{A_2(X)+4}}$.*

3. p-Bohr radius of L^q-spaces

In this section $X = L^q(\mu)$ where μ is a measure space and $1 \leq q \leq \infty$. It was shown (see [6]) that $r_1(L^q(mu)) = 0$ for $1 \leq q \leq \infty$ whenever $dim(L^q(\mu)) \geq 2$.

We shall see that $r_p(L^q(\mu)) = 0$ for $1 < p < 2$ while $r_p(L^q(\mu)) > 0$ for $p \geq 2$ and $1 \leq q \leq p$.

Next result follows closely the ideas in [6] and it is included for sake of completeness.

Theorem 3.1. *Let (Ω, Σ, μ) a measure space such that there exists a couple of disjoint measurable sets $A, B \in \Sigma$ with $0 < \mu(A), \mu(B) < \infty$. Then*

$$\tilde{r}_p(L^q(\mu)) = 0, \quad 1 \leq p < q < \infty \text{ or } 1 \leq q \leq p < 2.$$

In particular, if $1 \leq p < 2$ then $r_p(\mathbb{C}_q^m) = 0$ for $m \geq 2$ and $r_p(L^q(\mathbb{T})) = 0$.

Proof. Assume first $p < q$. Since $\lim_{y\to\infty} y^{p/q} - (y-1)^{p/q} = 0$, one has that for each $\varepsilon > 0$ we can find $0 < \gamma < 1$ such that

$$(\gamma^{-1})^{p/q} - (\gamma^{-1} - 1)^{p/q} < \varepsilon^p.$$

Equivalently

$$(1-\gamma)^{p/q} + \varepsilon^p \gamma^{p/q} > 1. \tag{3.1}$$

Now define $x_0 = (1-\gamma)^{1/q}\frac{\chi_A}{\mu(A)^{1/q}}$ and $x_1 = \gamma^{1/q}\frac{\chi_B}{\mu(B)^{1/q}}$ and set

$$f(z) = x_0 + x_1 z.$$

Clearly $\sup_{|z|<1} \|f(z)\|_{L^q(\mu)} = 1$ and

$$\|x_0\|^p + \|x_1\|^p \varepsilon^p > 1.$$

Hence $\tilde{r}_p(f, L^q(\mu)) \leq \varepsilon$ and then $\tilde{r}_p(X) = 0$.

Assume now $p < 2$ and $q \leq 2$. We argue as above choosing now for each $\varepsilon > 0$ a value $0 < \gamma < 1$ satisfying

$$(1-\gamma)^{p/2} + \varepsilon^p \gamma^{p/2} > 1. \tag{3.2}$$

We now define

$$x_0 = 2^{-1/q}(1-\gamma)^{1/2}\Big(\frac{\chi_A}{\mu(A)^{1/q}} + \frac{\chi_B}{\mu(B)^{1/q}}\Big),$$
$$x_1 = 2^{-1/q}\gamma^{1/2}\Big(\frac{\chi_A}{\mu(A)^{1/q}} - \frac{\chi_B}{\mu(B)^{1/q}}\Big)$$

and set

$$f(z) = x_0 + x_1 z = 2^{-1/q}(\sqrt{1-\gamma} + \sqrt{\gamma}z)\frac{\chi_A}{\mu(A)^{1/q}} + 2^{-1/q}(\sqrt{1-\gamma} - \sqrt{\gamma}z)\frac{\chi_B}{\mu(B)^{1/q}}.$$

Observe that

$$\begin{aligned}\|f(z)\| &= 2^{-1/q}\Big(|\sqrt{1-\gamma} + \sqrt{\gamma}z|^q + |\sqrt{1-\gamma} - \sqrt{\gamma}z|^q\Big)^{1/q} \\ &\leq 2^{-1/2}\Big(|\sqrt{1-\gamma} + \sqrt{\gamma}z|^2 + |\sqrt{1-\gamma} - \sqrt{\gamma}z|^2\Big)^{1/2} \leq 1.\end{aligned}$$

On the other hand, $\|x_0\| = \sqrt{1-\gamma}$ and $\|x_1\| = \sqrt{\gamma}$. The proof is finished using(3.2) and arguing as in the previous case. □

Let us now analyze the case $p \geq 2$ and $q \leq p$.

Theorem 3.2. *Let $2 \leq p < \infty$ and $p' \leq q \leq p$. Then*

$$r_p(L^q(\mu)) = 1. \tag{3.3}$$

Proof. We first consider $p = 2$ (hence $q = 2$). We can use Plancherel's theorem to obtain

$$(\sum_{n=0}^{\infty} \|x_n\|^2_{L^2(\mu)})^{1/2} = \|f\|_{H^2(\mathbb{D}, L^2(\mu))} \tag{3.4}$$

and, then $r_2(L^2(\mu)) \geq 1$. The other inequality is always true.

Assume $2 < p < \infty$ and $p' \leq q \leq p$. Select $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{2}$ and $1 \leq \beta \leq \infty$ so that $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{\beta}$. Observe first that

$$\max_n \|x_n\|_{L^\beta(\mu)} \leq \|f\|_{H^1(\mathbb{D}, L^\beta(\mu))}. \tag{3.5}$$

Now we can use complex interpolation, and apply the result (see [8])

$$[H^{p_1}(\mathbb{D}, L^\alpha(\mu)), H^{p_2}(\mathbb{D}, L^\beta(\mu))]_\theta = H^{p_3}(\mathbb{D}, L^\gamma(\mu)),$$

for $\frac{1}{p_3} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $\frac{1}{\gamma} = \frac{1-\theta}{\alpha} + \frac{\theta}{\beta}$. We then conclude, due to (3.4) and (3.5) by considering $p_1 = 2$, $p_2 = 1$ and $\alpha = 2$ (and hence $p_3 = \frac{2}{1+\theta}$ and $\gamma = q$)

$$(\sum_{n=0}^{\infty} \|x_n\|^p_{L^q(\mu)})^{1/p} \leq \|f\|_{H^{p_3}(\mathbb{D}, L^q(\mu))}.$$

This shows that for each $f(z)=\sum_{n=0}^\infty x_n z^n$ belonging to the unit ball of $H^\infty(\mathbb{D}, L^q(\mu))$ we have $(\sum_{n=0}^\infty \|x_n\|^p_{L^q(\mu)})^{1/p} \leq 1$. Hence $r_p(L^q(\mu)) \geq 1$. $\square$

The only case that is left is $1 \leq q < p' \leq 2 \leq p$. Using Corollary 2.6 and Remark 1.11 one gets that $r_p(L^q(\mu)) \geq r_2^{2/q}(L^q(\mu)) > 0$. Let us see that in general also $r_p(L^q(\mu)) < 1$.

Proposition 3.3. *Let $X = L^q(\mathbb{T})$ and $1 \leq q < 2$. Then $0 < r_2(X) < 1$.*

Proof. As mentioned above the fact that $r_2(X) > 0$ follows from Corollary 2.6.

We shall show that there exists f in the unit ball of $H^\infty(\mathbb{D}, L^q(\mathbb{T}))$ with $\sum_{n=0}^\infty \|x_n\|^2 = \infty$. Therefore $r_2(X) < 1$.

It suffices to select $F \in H^q(\mathbb{T})\backslash H^2(\mathbb{T})$, say $F(w) = \sum_{n=0}^\infty a_n w^n$, with $\|F\|_{H^q(\mathbb{T})} = 1$, that is $\sup_{0<r<1} \|F_r\|_{L^q(\mathbb{T})} \leq 1$ where $F_r(e^{it}) = F(re^{it})$ and consider the $L^q(\mathbb{T})$-valued function

$$f(z)(e^{it}) = F(ze^{it}) = \sum_{n=0}^\infty a_n e^{int} z^n.$$

Hence $x_n(e^{it}) = a_n e^{int}$ and $\|f(z)\|_{L^q(\mathbb{T})} = \|F_{|z|}\|_{H^q} \leq 1$ for all $0 < |z| < 1$ and

$$\sum_{n=0}^\infty \|x_n\|^2 = \sum_{n=0}^\infty |a_n|^2.$$

$\square$

Our aim is now to find lower estimates for $r_p(L^q(\mu)$ in the case $1 \leq q < p' \leq 2 \leq p$. We shall need the following lemma.

Lemma 3.4. *Let $1 \leq q \leq 2$ and $F \in H^q(\mathbb{D})$. Then*

$$\Big(|F(0)|^2 + (\frac{1}{2})^{\frac{2-q}{q}} |F'(0)|^2\Big)^{1/2} \leq \|F\|_{H^q(\mathbb{D})} \tag{3.6}$$

Proof. Let us first show that

$$\Big(|F(0)|^2 + \frac{1}{2}|F'(0)|^2\Big)^{1/2} \leq \|F\|_{H^1(\mathbb{D})}.$$

Assume that $\|F\|_{H^1(\mathbb{D})} = 1$. Now, using factorization to write $F = gh$ with $g, h \in H^2(\mathbb{D})$ and $\|h\|_{H^2(\mathbb{D})} = \|g\|_{H^2(\mathbb{D})} = 1$. Hence $F(0) = g(0)h(0)$,

$$F'(0) = h(0)g'(0) + h'(0)g(0)$$

and

$$|g(0)|^2 + |g'(0)|^2 \leq 1, \quad |h(0)|^2 + |h'(0)|^2 \leq 1.$$

Using that $2xy \leq x^2 + y^2$ we have

$$\begin{aligned} |F(0)|^2 + \frac{1}{2}|F'(0)|^2 &\leq |g(0)|^2|h(0)|^2 + |g(0)|^2|h'(0)|^2 + |g'(0)|^2|h(0)|^2 \\ &= |g(0)|^2(|h(0)|^2 + |h'(0)|^2) + |g'(0)|^2|h(0)|^2 \\ &\leq (|g(0)|^2 + |g'(0)|^2)(|h(0)|^2 + |h'(0)|^2) \leq 1. \end{aligned}$$

Therefore

$$\Big(|F(0)|^2 + \frac{1}{2}|F'(0)|^2\Big)^{1/2} \leq \|F\|_{H^1(\mathbb{D})}$$

which combined with

$$\Big(|F(0)|^2 + |F'(0)|^2\Big)^{1/2} \leq \|F\|_{H^2(\mathbb{D})}$$

gives also the case $1 < q < 2$ in (3.6) by invoking interpolation (see [5, Theorem 5.6.2]). $\square$

Theorem 3.5. *Let* (Ω, Σ, μ) *a measure space and* $1 \leq q \leq 2$. *Then* $\tilde{r}_2(L^q(\mu)) \geq 2^{\frac{1}{2}-\frac{1}{q}}$.

Proof. Let $f : \mathbb{D} \to L^q(\mu)$ be a bounded holomorphic function, say $f(z) = \sum_{n=0}^{\infty} x_n z^n$. We use Fubini's theorem and Minkowski's inequality, together with Lemma 3.4, to get the following estimates

$$\begin{aligned}
\|f\|^q_{H^\infty(\mathbb{D},L^q(\mu))} &\geq \|f\|^q_{H^q(\mathbb{D},L^q(\mu))} \\
&= \sup_{0<s<1} \int_0^{2\pi} \|f(se^{i\theta})\|^q_{L^q(\mu)} \frac{d\theta}{2\pi} \\
&= \sup_{0<s<1} \int_\Omega \Big(\int_0^{2\pi} |f(se^{i\theta})(w)|^q \frac{d\theta}{2\pi} \Big) d\mu(w) \\
&\geq \sup_{0<s<1} \int_\Omega \Big(|x_0(w)|^2 + (\frac{1}{2})^{\frac{2-q}{q}} |x_1(w)|^2 s^2 \Big)^{q/2} d\mu(w) \\
&= \sup_{0<s<1} \int_\Omega \|\big(|x_0(w)|^q, (\frac{1}{2})^{\frac{2-q}{2}} |x_1(w)|^q s^q\big)\|_{\mathbb{C}^2_{2/q}} d\mu(w) \\
&\geq \sup_{0<s<1} \Big\|\big(\int_\Omega |x_0(w)|^q d\mu(w), (\frac{1}{2})^{\frac{2-q}{2}} \int_\Omega |x_1(w)|^q s^q d\mu(w)\big)\Big\|_{\mathbb{C}^2_{2/q}} \\
&= \sup_{0<s<1} (\|x_0\|^2_{L^q(\mu)} + (\frac{1}{2})^{\frac{2-q}{q}} \|x_1\|^2_{L^q(\mu)} s^q)^{q/2} \\
&= \Big(\|x_0\|^2_{L^q(\mu)} + (\frac{1}{2})^{\frac{2-q}{q}} \|x_1\|^2_{L^q(\mu)}\Big)^{q/2}.
\end{aligned}$$

Hence $\tilde{r}_2(L^q(\mu)) \geq (\frac{1}{2})^{\frac{1}{q}-\frac{1}{2}}$. $\square$

Corollary 3.6. *Let* (Ω, Σ, μ) *a measure space and* $1 \leq q < p' \leq 2 \leq p < \infty$. *Then* $r_p(L^q(\mu)) \geq (1 + 2^{\frac{2}{q}-1})^{-1/p}$.

In particular $r_p(L^1(\mu)) \geq (1/3)^{1/p}$.

Proof. Note that $\phi_p(t) = \frac{t}{(t^p+1)^{1/p}}$ is increasing. Combining now Theorem 2.4 and Theorem 3.5 one obtains

$$r_2(L^q(\mu)) \geq \phi_2(\tilde{r}_2(L^q(\mu))) \geq \phi_2(2^{\frac{1}{2}-\frac{1}{q}}) = (1 + 2^{\frac{2}{q}-1})^{-1/2}.$$

Finally use $r_p(X) \geq r_2(X)^{2/p}$ to complete the result.

$\square$

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