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Oscar Blasco*, Departamento de Análisis Matemático, Universidad de Valencia, 46100 Burjassot, Valencia, Spain, (e-mail: oscar.blasco@uv.es)

Pablo Gregori*, Departamento de Análisis Matemático, Universidad de Valencia, 46100 Burjassot, Valencia, Spain, (e-mail: pablo.gregori@uv.es)

Jose M. Calabuig*, Departamento de Análisis Matemático, Universidad de Valencia, 46100 Burjassot, Valencia, Spain, (e-mail: jose.m.calabuig@uv.es)

FINITE SEMIVARIATION AND REGULATED FUNCTIONS BY MEANS OF BILINEAR MAPS

Abstract

It is shown that Z does not contain a copy of c_0 if and only if for every bilinear map $B : X \times Y \to Z$, every function $f : [0,1] \to X$ of bounded $(\mathcal{B})$-variation is $(\mathcal{B})$-regulated.

1 Introduction

In the setting of Riemann-Stieltjes type integrals, and more precisely in the Henstock-Kurzweil or Dushnik or Young integrals, the concepts of 'function of bounded variation' and 'regulated function' defined by means of a particular bilinear form play an important role. In a recent paper [1], L. Barbanti showed the relation between both concepts under an assumption of uniform convexity for some particular bilinear forms. In this note we set the assumption which characterizes this relation.

We shall work with functions f defined over the unit interval $[0,1]$ and having values in a Banach space X, and we use the following notation: D is the set of all finite partitions of $[0,1]$, d represents any element of D, and $|d|$ is the length of d, that is $d = \{t_0, t_1, \ldots, t_{|d|}\}$ where $0 = t_0 < t_1 < \ldots < t_{|d|} = 1$.

First of all we write down the well-known definitions of the (total) variation and semivariation of a function

$$\text{var}(f) = \sup_{d \in D} \sum_{k=1}^{|d|} \|[f(t_k) - f(t_{k-1})]\|_X$$

$$\text{semivar}(f) = \sup_{d \in D} \| \sum_{k=1}^{|d|} \epsilon_k [f(t_k) - f(t_{k-1})] \|_X$$

$$= \sup_{\substack{d \in D \\ \|x^*\| \leq 1}} \sum_{k=1}^{|d|} |\langle f(t_k) - f(t_{k-1}), x^* \rangle|$$

where $x^* \in X^*$.

A function $f : [0,1] \to X$ is said to have *finite variation* when $\text{var}(f) < \infty$, *finite semivariation* if $\text{semivar}(f) < \infty$ and to be *regulated* if it has at most discontinuities of first kind at every $t \in [0,1]$.

Let us consider now $\mathcal{B} = (B; X, Y, Z)$ formed by three Banach spaces X, Y and Z, and a continuous bilinear map $B : X \times Y \to Z$.

We can extend now the concepts of variation and regulated functions as follows. A function $f : [0,1] \to X$ is said to have *finite $(\mathcal{B})$-variation* if

$$(\mathcal{B})\text{var}(f) := \sup_{\substack{d \in D \\ \|y_k\| \leq 1 \\ 1 \leq k \leq |d|}} \| \sum_{k=1}^{|d|} B(f(t_k) - f(t_{k-1}), y_k) \|_Z$$

is finite, and it is called *$(\mathcal{B})$-regulated* if the function $f_{B,y} : [0,1] \to Z$ mapping $t \mapsto B(f(t), y)$ is regulated for every y in the unit ball of Y. Let us mention the following three examples:

- $\mathcal{B}_1 = (B_1; X, \mathbb{K}, X)$ with $B_1(x, \lambda) = \lambda x$. Here $(\mathcal{B})\text{var}(f) = \text{semivar}(f)$.

- $\mathcal{B}_2 = (B_2; X, X^*, \mathbb{K})$ with $B_2(x, x^*) = \langle x, x^* \rangle$. Here $(\mathcal{B})\text{var}(f) = \text{var}(f)$.

- $\mathcal{B}_3 = (B_3; L(X, Y), X, Y)$ with $B_3(T, x) = Tx$. Here the value of $(\mathcal{B})\text{var}(f)$ depends on the spaces X and Y.

A look at the definitions of the different 'variations' shows that

$$\text{semivar}(f_{B,y}) \leq (\mathcal{B})\text{var}(f) \leq \text{var}(f)$$

for every $y \in Y$ with $\|y\|_Y \leq 1$ (when $\|B\| \leq 1$). In the special case where a Banach space X can be considered isometrically embedded in the Banach

space $L(Y, Z)$ where Y and Z are also Banach spaces, it is not difficult to prove (see [3]) that

$$\text{semivar}(f) \leq (\mathcal{B})\text{var}(f) \leq \text{var}(f)$$

In [1], dealing with Riemann-Stieltjes integrals and $\mathcal{B} = (B; L(X), X, X)$ and $B(T, x) = Tx$, a question arises: "Is every $L(X)$-valued function with finite $(\mathcal{B})$-variation a $(\mathcal{B})$-regulated function?" The answer to this question is given in

Theorem 1. ([1]) *Let $\mathcal{B} = (B; L(X), X, X)$ and $B(T, x) = Tx$. If X is a uniformly convex space then, every $L(X)$-valued function f with $(\mathcal{B})\text{var}(f) < \infty$ is $(\mathcal{B})$-regulated.*

It is clear that every function of bounded variation is $(\mathcal{B})$-regulated for any bilinear map. However, bounded semivariation is not enough to ensure the assertion.

Our purpose is to improve the result of [1]. The following example has been useful in order to guide our steps:

We take the bilinear form $B : \ell^\infty \times \ell^\infty \to \ell^\infty$ mapping $B((\alpha_n)_n, (\beta_n)_n) = (\alpha_n \beta_n)_n$ and the ℓ^∞-valued function

$$f = 0\chi_{\{0\}} + \sum_{k=1}^{\infty} e_k \chi_{(\frac{1}{k+1}, \frac{1}{k}]}.$$

This function is not $(\mathcal{B})$-regulated but of finite $(\mathcal{B})$-variation.

2 The main theorem

The reader can deduce from the proofs of our statements that, in spite of using functions, they actually depend upon the behaviour of certain telescopic series. In fact, the famous theorem of Bessaga and Pelczinsky is the cornerstone of our results.

Theorem 2 (Bessaga-Pelczynski). (see [2]) *The Banach space X contains no copy of c_0 if and only if all series $\sum_n x_n$ in X, with $\sum_n |\langle x_n, x^* \rangle| < \infty$ for all $x^* \in X^*$, are unconditionally convergent in norm.*

Then if X is a Banach space containing a copy of c_0, we can always find a sequence $(x_n)_{n=1}^{\infty}$ in X and constants $C_1, C_2 > 0$ such that for every $N \in \mathbb{N}$

$$C_1 \sup_{1 \leq n \leq N} |\lambda_n| \leq \| \sum_{n=1}^{N} \lambda_n x_n \|_X \leq C_2 \sup_{1 \leq n \leq N} |\lambda_n|. \tag{1}$$

Moreover, for such a sequence we have that $\sup_{\|x^*\|\leq 1} \sum_{n=1}^{\infty} |\langle x_n, x^* \rangle| < \infty$ and $\sum_{n=1}^{\infty} x_n$ is not norm convergent.

Then, the main theorem is

Theorem 3. *Let Z be a Banach space. The following assertions are equivalent:*

(1) Z does not contain copy of c_0.

(2) Every function $f : [0, 1] \to Z$ with finite semivariation is regulated.

Proof. (I) Let f be a function of finite semivariation. In order to prove that it is regulated at any $t \in [0, 1]$, it suffices to show the existence of $\lim_n f(t_n)$ for any sequence $(t_n)_n \subset [0, 1]$ increasing or decreasing to t. We shall restrict ourselves to work the case $(t_n)_n$ increasing.

Let (t_n) be a sequence increasing to t. For $N \in \mathbb{N}$ let $d_N = \{0, t_1, t_2, \ldots, t_N, 1\}$ be a partition of $[0, 1]$. From the assumptions

$$\sup_{\|z^*\|\leq 1} \sum_{k=1}^{N} |\langle f(t_k) - f(t_{k-1}), z^* \rangle| \leq \mathrm{semivar}(f) < \infty$$

and therefore $\sup_{\|z^*\|\leq 1} \sum_{k=1}^{\infty} |\langle f(t_k) - f(t_{k-1}), z^* \rangle| < \infty$. Making use of the theorem of Bessaga and Pelczynski we should have that the telescopic series $\sum_{k=1}^{\infty} [f(t_k) - f(t_{k-1})]$ converges to an element $z \in Z$.

It is easy to prove that the element z is independent of the sequence $(t_n)_n$. For that aim, just choose another sequence $(s_n)_n$ increasing to t . As above $\sum_{k=1}^{\infty} [f(s_k) - f(s_{k-1})]$ converges to an element $y \in Z$. Therefore $\lim_n f(t_n) = z + f(0)$ and $\lim_n f(s_n) = y + f(0)$. Finally, consider the sequence $(r_n)_n$ made by ordering nondecreasingly the set of points $\{t_n : n \in \mathbb{N}\} \cup \{s_n : n \in \mathbb{N}\}$. With this new sequence we have that $\sum_{k=1}^{\infty} [f(r_k) - f(r_{k-1})]$ converges to an element $x \in Z$, but as long as it is telescopic, we find in the partial sums of this series, alternatively, subsequences of $(f(t_n))_n$ and $(f(s_n))_n$. Then $x = y = z$ and f is regulated.

(II) Let us assume that Z does contain a copy of c_0. The theorem of Bessaga and Pelczynski provides us with a sequence $(z_n)_n \subset Z$ as the one described in (1).

The function $f = 0\chi_{\{0\}} + \sum_{k=1}^{\infty} z_n \chi_{(\frac{1}{k+1}, \frac{1}{k}]}$ defined in $[0, 1]$ is not regulated at the origin. Indeed

$$\|f(\tfrac{1}{k}) - f(\tfrac{1}{k-1})\| = \|z_k - z_{k-1}\| \geq C_1$$

for some $C_1 > 0$, and then the discontinuity of f at $t = 0$ is not of first kind.

However, if $d \in D$ and $z^* \in Z^*$

$$\sum_{k=1}^{|d|} |\langle f(t_k) - f(t_{k-1}), z^* \rangle| = \sum_{k=1}^{|d|} |\langle z_{[\frac{1}{t_k}]} - z_{[\frac{1}{t_{k-1}}]}, z^* \rangle|$$

$$\leq 2 \sum_{k=1}^{\infty} |\langle z_k, z^* \rangle| < \infty.$$

Then semivar$(f) < \infty$ what leads to a contradiction. $\square$

Corollary 4. *Let Z be a Banach space. The following assertions are equivalent:*

(1) Z does not contain copy of c_0.

(2) For every $\mathcal{B} = (B; X, Y, Z)$ where X and Y are Banach spaces and $B : X \times Y \to Z$ is a bounded bilinear map , we have that every function $f : [0,1] \to X$ with finite $(\mathcal{B})$-variation is $(\mathcal{B})$-regulated.

Proof. (I) Given $\mathcal{B} = (B; X, Y, Z)$ and $f : [0,1] \to X$ with $(\mathcal{B})\mathrm{var}(f) < \infty$, since semivar$(f_{B,y}) < \infty$ for every $y \in Y$ then the previous theorem gives that function $f_{B,y}$ is regulated and therefore f is $(\mathcal{B})$-regulated.

(II) If Z has a copy of c_0, we choose $(z_n)_n \subset Z$ as it is described in (1). For the situation $\mathcal{B} = (B; Z, \mathbb{K}, Z)$ we can use now the function defined in the theorem to reach the contradiction. $\square$

Finally, we set the application of this result to the concept of the Henstock-Kurzweil integral where, for a given Banach space X, it is of interest to have functions $f : [0,1] \to L(X)$ of finite $(\mathcal{B})$-variation and $(\mathcal{B})$-regulated for the particular situation $\mathcal{B} = (B; L(X), X, X)$ with $B(T, x) = Tx$.

Theorem 5. *Let X be a Banach space, and let us consider functions $f : [0,1] \to L(X)$ and the situation $\mathcal{B} = (B; L(X), X, X)$. The following assertions are equivalent:*

(1) X does not contain copy of c_0

(2) Every function $f : [0,1] \to L(X)$ with finite $(\mathcal{B})$-variation is $(\mathcal{B})$-regulated.

Proof. (I) It is an immediate corollary of the previous theorem.

(II) Let $f : [0,1] \to L(X)$ be the function $f = 0\chi_{\{0\}} + \sum_{k=1}^{\infty} \widetilde{T}_k \chi_{(\frac{1}{k+1}, \frac{1}{k}]}$. Operators $\widetilde{T}_k$ are asked to map X on X. As long as X has copy of c_0 we can fix a sequence $(z_n)_n$ as described in (1). Let $Y = \overline{\mathrm{span}}\{x_n : n \in \mathbb{N}\}$ be the copy of c_0 lying inside X, and let $T_k : Y \to \mathrm{span}\{x_{2^k+n}\}_{n=1}^{k}$ be defined by $S_k(\sum_{n=1}^{\infty} \alpha_n x_n) = \sum_{n=1}^{k} \alpha_n x_{2^k+n}$. It is easy to check that S_k is a continuous linear map and $\|S_k\| \leq \frac{C_2}{C_1}$ (use (1)). Let $\pi_j : Y \to \mathbb{R}$ be the projection

$\pi_j(\sum_{n=1}^{\infty} \alpha_n x_n) = \alpha_j$. It is easy to see that $\|\pi_j\| \leq \frac{1}{C_1}$ for every $j \in \mathbb{N}$. Applying Hahn-Banach Theorem to the compositions $T_{k,j} = \pi_j \circ S_k$ (which have all norm at most $\frac{C_2}{C_1^2}$) we get extensions $\widetilde{T}_{k,j} : X \to \mathbb{R}$ with the same norm. Finally, let us define $\widetilde{T}_k : X \to X$ such that $\widetilde{T}_k(x) = \sum_{j=1}^{k} \widetilde{T}_{k,2^k+j}(x) x_{2^k+j}$. Then $\widetilde{T}_k \in L(X)$ and $\|\widetilde{T}_k\| \leq \frac{C_2}{C_1^2}$.

Let us show now that f is the desired function. Let us take (for instance) the vector x_1 of the sequence $(x_n)_n$ (normalized if needed) and compute, for $k \in \mathbb{N}$,

$$\begin{aligned}
\|f_{B,x_1}(\tfrac{1}{k}) - f_{B,x_1}(\tfrac{1}{k-1})\|_X &= \|f(\tfrac{1}{k})(x_1) - f(\tfrac{1}{k-1})(x_1)\|_X \\
&= \|\widetilde{T}_k(x_1) - \widetilde{T}_{k-1}(x_1)\|_X \\
&= \|x_{2^k+1} - x_{2^{k-1}+1}\|_X \\
&\geq C_1.
\end{aligned}$$

Then f is not $(\mathcal{B})$-regulated at $t = 0$. Now let $d \in D$ and w_k be norm-one elements of X $(k = 1, \dots, |d|)$, denoting by $n_k = [\frac{1}{t_k}]$ we have

$$\|\sum_{k=1}^{|d|} [f(t_k) - f(t_{k-1})](w_k)\|_X = \|\sum_{k=2}^{|d|} [\widetilde{T}_{n_k} - \widetilde{T}_{n_{k-1}}](w_k)\|_X + \|\widetilde{T}_{n_1}(w_1)\|_X$$

$$= \|\sum_{k=2}^{|d|} (\sum_{j=1}^{n_k} \widetilde{T}_{n_k,2^{n_k}+j}(w_k) x_{2^{n_k}+j} \quad - \sum_{j=1}^{n_{k-1}} \widetilde{T}_{n_{k-1},2^{n_{k-1}}+j}(w_k) x_{2^{n_{k-1}}+j})\|_X$$

$$+ \|\widetilde{T}_{n_1}(w_1)\|_X \leq 2\frac{C_2}{C_1^2}.$$

Hence f has finite $(\mathcal{B})$-variation, and the proof concludes. $\qquad\square$

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