Hölder inequality for functions integrable with respect to bilinear maps

Abstract  Let $(\Omega, \Sigma, \mu)$ be a finite measure space, $1 \leq p < \infty$, $X$ be a Banach space $X$ and $B : X \times Y \to Z$ be a bounded bilinear map. We say that an $X$-valued function $f$ is $p$-integrable with respect to $B$ whenever $\sup_{\|y\|=1} \int_{\Omega} \|B(f(w), y)\|^p d\mu < \infty$. We get an analogue to Hölder’s inequality in this setting.

Keywords: Vector-valued functions; Pettis and Bochner integrals; bilinear maps.

AMS 2000 Mathematics subject classification: Primary 42B30, 42B35
Secondary 47B35
HÖLDER INEQUALITY FOR FUNCTIONS INTEGRABLE WITH RESPECT TO BILINEAR MAPS

OSCAR BLASCO\textsuperscript{1}, JOSÉ M. CALABUIG\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Universitat de Valencia, Burjassot 46100 (Valencia) Spain
\textsuperscript{2} Department of Applied Math., Universitat Politècnica de Valencia, 46022 (Valencia) Spain

(Received )

1. Introduction

Throughout the paper 1 \( \leq p < \infty \), \((\Omega, \Sigma, \mu)\) will be a finite complete measure space, \(X, Y\) and \(Z\) will stand for Banach spaces over \(\mathbb{K}\) (\(\mathbb{R}\) or \(\mathbb{C}\)), and \(B : X \times Y \to Z\) will denote a bounded bilinear map.

We denote by \(L^0(X), L^0_{\text{weak}}(X)\) and \(L^0_{\text{weak}}(X^*)\) the spaces of strongly, weakly measurable and weak* measurable functions and write \(L^p(X), L^p_{\text{weak}}(X)\) and \(L^p_{\text{weak}}(X^*)\) for the space of functions in \(L^0(X), L^0_{\text{weak}}(X)\) and \(L^0_{\text{weak}}(X^*)\) such that \(\|f\| \in L^p(\mu)\), \((f,x^*) \in L^p(\mu)\) for \(x^* \in X^*\) and \(\langle x,f \rangle \in L^p(\mu)\) for \(x \in X\) respectively. Finally we use the notation \(P^0(X)\) for the space of Pettis \(p\)-integrable functions \(P^p(X) = L^0_{\text{weak}}(X) \cap L^0(X)\).

In this paper we shall consider spaces of \(X\)-valued functions which are \(p\)-integrable with respect to a bounded bilinear map \(B : X \times Y \to Z\), that is to say functions \(f\) satisfying the condition \(B(f,y) \in L^p(Z)\) for all \(y \in Y\).

Although theses classes have been around for a long time in particular cases such us

\[
B_X = B : X \times \mathbb{K} \to X, \quad B(x,\lambda) = \lambda x, \tag{1.1}
\]

\[
D_X = D : X \times X^* \to \mathbb{K}, \quad D(x, x^*) = \langle x, x^* \rangle, \quad \tag{1.2}
\]

\[
D_{1,X} = D_1 : X^* \times X \to \mathbb{K}, \quad D_1(x^*, x) = \langle x, x^* \rangle, \tag{1.3}
\]

or

\[
\pi_Y : X \times Y \to X \otimes Y, \quad \pi_Y(x, y) = x \otimes y, \tag{1.4}
\]

\[
\hat{\mathcal{O}}_Y : Y \times \mathcal{L}(Y, X) \to Y, \quad \hat{\mathcal{O}}_Y(x, T) = T(x), \tag{1.5}
\]

\[
\mathcal{O}_{Y,Z} : \mathcal{L}(Y, Z) \times Y \to Z, \quad \mathcal{O}_{Y,Z}(T, y) = T(y), \tag{1.6}
\]

a systematic study for general bilinear maps has been iniciated in \([6]\). This approach has been used to extend the results on boundedness from \(L^p(Y)\) to \(L^p(Z)\) of operator-valued kernels by M. Girardi and L. Weiss \([10]\) to the case where \(K : \Omega \times \Omega \to X\) is measurable and the integral operators are defined by

\[
T_K(f)(w) = \int_{\Omega'} B(K(w,w'), f(w')) d\mu(w').
\]

The reader is referred to \([7]\) for the introduction of Fourier Analysis in the bilinear context. This allows to extend the results in \([2, 4, 5]\) regarding convolution by means of bilinear maps and Fourier coefficients for functions in these wider classes.

Let us mention some notions that were relevant for developing the general theory (see \([6]\)). Given \(x \in X\) and \(y \in Y\) we shall be denoting by \(B_x \in \mathcal{L}(Y, Z)\) and \(B^y \in \mathcal{L}(X, Z)\) the corresponding linear operators

\[
B_x(y) = B(x, y) \text{ and } B^y(x) = B(x, y).
\]

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Hölder inequality for functions integrable with respect to bilinear maps

The triple \((Y,Z,\mathcal{B})\) is admissible for \(X\) if the map \(x \to \mathcal{B}_x\) is injective from \(X \to \mathcal{L}(Y,Z)\) and \(X\) is said to be \((Y,Z,\mathcal{B})\)-normed (or normed by \(\mathcal{B}\)) if there exists \(C > 0\) such that for all \(x \in X\)
\[
\|x\| \leq C\|\mathcal{B}_x\|.
\]

Given a bounded bilinear map \(\mathcal{B} : X \times Y \to Z\), we can define the "adjoint" \(\mathcal{B}^* : X \times Z^* \to Y^*\) by the formula
\[
(y, \mathcal{B}^*(x,z^*)) = (\mathcal{B}(x,y), z^*).
\]

Note that \(\mathcal{B}^* = \mathcal{B} \circ (\pi_X)^*\) and \((\mathcal{O}_X,\mathcal{O}_Y,\mathcal{O}_Z)^* = \mathcal{O}_{Z^*} \circ (\mathcal{O}_X \times \mathcal{O}_Y)^*\). Let us start with the following definitions:

**Definition 1.1.** (see [6]) We say that \(f : \Omega \to X\) belongs to \(L^p(X)\) if \(\mathcal{B}(f,y) \in L^p(Z)\) for any \(y \in Y\).

We write \(L^p_b(X)\) for the space of functions \(f\) in \(L^p_b(X)\) such that
\[
\|f\|_{L^p_b(X)} = \sup\{\|\mathcal{B}(f,y)\|_{L^p(Z)} : \|y\| = 1\} < \infty.
\]

A function \(f \in L^p_b(X)\) is said to belong to \(L^p_b(X)\) if there exists a sequence of simple functions \((s_n)_n \in \mathcal{S}(X)\) such that \(s_n \to f\) a.e. and \(\|s_n - f\|_{L^p_b(X)} \to 0\).

For \(f \in L^p_b(X)\) we write \(\|f\|_{L^p_b(X)}\) instead of \(\|f\|_{L^p_b(X)}\).

In particular
\[
L^p_b(X) = L^0(X), L^0_b(X) = L^0_{\text{weak}}(X) \text{ and } L^0_{\mathcal{D}_1}(X^*) = L^0_{\text{weak}}(X^*).
\]

Observe that \(L^p_b(X) \subset L^p_b(X)\) for any \(\mathcal{B}\) and, in general, \(L^p_b(X) \not\subset L^p_b(X)\) (see [8] page 53, for the case \(\mathcal{B} = \mathcal{D}\)). It was shown in [6] that \(L^p_b(X) \subset L^p_{\text{weak}}(X)\) if and only if \(X\) is \(\mathcal{B}\)-normed.

The answer is negative for any infinite dimensional Banach space \(X\). Indeed, take \(p_1 = p_2 = 2\) and \(p_2 = 1\), let \(X\) be an infinite dimensional Banach space, \(Y = X^*\) and \(Z = K\) and \(\mathcal{B} = \mathcal{D}\). Take \((x_n) \in L^p_b(X) \setminus \ell_2(X)\). This allows to find \((x_n^*) \in \ell_2(X^*)\) such that \(\sum_n |\langle x_n, x_n^* \rangle| = \infty\).

One might think that the difficulty comes from allowing the functions to belong to \(L^p_b(X)\) instead of \(L^p_b(X)\). Let us then modify the question: Does \(\mathcal{B}(f,g)\) belong to \(L^p_b(X)\) for any \(f \in L^p_b(X)\) and \(g \in L^p_b(Y)\) if \(f \in L^p_b(X)\) and \(g \in L^p_b(Y)\)?

The answer is again negative. If the result hold true we would have that there exists \(M > 0\) such that
\[
\|\mathcal{B}(s,f)\|_{L^1(Z)} \leq M\|s\|_{L^1(X)}\|f\|_{L^1(Y)}\text{ for any } s \in \mathcal{S}(X)\text{ and } f \in \mathcal{S}(Y).
\]

We now consider \(X = Y = \ell_2, Z = \ell_1\) and \(\mathcal{B}_1 : \ell_2 \times \ell_2 \to \ell_1\) given by \(\mathcal{B}(\lambda_n,\mu_n) = (\lambda_n,\mu_n)\). Let us now consider \(s_N = t_N = \sum_{k=1}^N 2^{-k} e_k 1_{I_k}\) where \(e_k\) is the canonical basis and \(I_k\) are chosen as above. Hence \(\mathcal{B}(s_N,y) = \sum_{k=1}^N 2^{-k} \beta_k e_k 1_{I_k}\) for \(y = (\beta_n)_{n \in \ell_2}\). Therefore \(\|s_N\|_{L^1(\ell_2)} \leq 1\). On the other hand \(\|s_N\|_{L^2(\ell_2)} = \sqrt{N}\). Finally observe that \(\mathcal{B}(s_N,s_N) = \sum_{k=1}^N 2^{-k} e_k 1_{I_k}\) and \(\|\mathcal{B}(s_N,s_N)\|_{L^1(\ell_1)} = N\). This contradicts (1).

Modifying the previous argument with \(Z = K\) and \(\mathcal{B} = \mathcal{D}\) one can even show that there exist \(f \in L^p_b(X)\) and \(g \in L^p_b(Y)\) such that \(\mathcal{B}(f,g) \not\in L^p_b(X)\).

The objective of this paper is to present an analogue to Hölder inequality in the setting of vector-valued functions integrable with respect to bilinear maps. We shall then study the following general problem:
2. A bilinear version of Hölder’s Inequality.

It is well known and easy to see the following analogues of Hölder’s inequality in the vector-valued setting: Let \( 1 \leq p_1, p_2, p_3 \leq \infty \) and \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} \), and let \( B : X \times Y \rightarrow Z \) be a bounded bilinear map. If \( B_1 : X \times X_1 \rightarrow X_2 \) and \( B_2 : Y \times Y_1 \rightarrow Y_2 \) are bounded bilinear maps, find \( B_3 : Z \times Z_1 \rightarrow Z_2 \) such that for any \( f \in L_{p_1}^p(X) \) and \( g \in L_{p_2}^p(Y) \) one has \( B(f, g) \in L_{p_3}^p(Z) \).

**Problem:** Let \( 1 \leq p_1, p_2, p_3 \leq \infty \) and \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} \), and let \( B : X \times Y \rightarrow Z \) be a bounded bilinear map. If \( B_1 : X \times X_1 \rightarrow X_2 \) and \( B_2 : Y \times Y_1 \rightarrow Y_2 \) are bounded bilinear maps, find \( B_3 : Z \times Z_1 \rightarrow Z_2 \) such that for any \( f \in L_{p_1}^p(X) \) and \( g \in L_{p_2}^p(Y) \) one has \( B(f, g) \in L_{p_3}^p(Z) \).

**Example 2.4.** Let \( B_x : X \rightarrow X' \) and \( B_y : Y \rightarrow Y' \) be bounded linear maps. If \( B_1 = B_x \) and \( B_2 = B_y \) are compatible triples, then \( B_3 = B_x \otimes B_y \) is a compatible triple.

**Example 2.5.** \( (\pi_Y, B_X, O_X^*) \) is a compatible triple.

**Example 2.6.** \( (\pi_Y, B_X, O_X^*) \) is a compatible triple.

**Example 2.7.** \( (\pi_Y, B_X, O_X^*) \) is a compatible triple.
Indeed, if $\mathcal{B} = \pi_Y : X \times Y \to X \otimes Y$, $\mathcal{B}_1 = \mathcal{B}_1 : X \to X$ and $\mathcal{B}_2 = \mathcal{B}_2 : Y \to Y$ then we can take $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D}_X : X \times X^* \to \mathbb{K}$ and $\mathcal{P} = \mathcal{D}_Y : Y \times Y^* \to \mathbb{K}$. The compatibility now follows from

$$\overline{\mathcal{P}}(\mathcal{B}(x,y), \lambda) = (x \otimes y, \lambda T) = (x, Ty) = \mathcal{P}(\mathcal{B}_1(x, \lambda), \mathcal{B}_2(y, T)).$$

\[\square\]

**Example 2.6.** Let $\mathcal{B} : \mathcal{L}(X, Z) \times \mathcal{L}(Y, Z^*) \to \mathcal{L}(X, Y^*)$ be given by $(T, S) \to T^* S$. Then $(\mathcal{B}, \mathcal{O}_X, \mathcal{O}_Y, \mathcal{O}_Z)$ is a compatible triple.

Indeed, if $\mathcal{B}_1 = \mathcal{O}_X, \mathcal{B}_2 = \mathcal{O}_Y, \mathcal{B}_3 = \mathcal{O}_Z$ then we can take $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D}_Z : Z \times Z^* \to \mathbb{K}$ and $\mathcal{P} = (\mathcal{D}_1)_{X \otimes Y} : \mathcal{L}(X, Y^*) \times \mathcal{L}(Y, Y^*) \to \mathbb{K}$ given by $\overline{\mathcal{P}}(T, x \otimes y) = (x, Ty)$.

Observe that the compatibility follows from the formula

$$\overline{\mathcal{P}}(\mathcal{B}(T, S), x \otimes y) = (x, T^* S y) = (T x, S y) = \mathcal{P}(\mathcal{B}_1(T, x), \mathcal{B}_2(S, y)).$$

\[\square\]

**Theorem 2.7.** (Hölder’s inequality I) Let $1 \leq p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$. Assume that $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ is a compatible triple for some $F$, $\mathcal{P}$ and $\mathcal{P}$.

1. If $f \in L_{p_1}^0(\Omega)$ and $g \in L_{p_2}^0(\Omega)$ then $\mathcal{B}(f, g) \in L_{p_3}^0(\Omega)$.

2. If $f \in L_{p_1}^0(\Omega)$ and $g \in L_{p_2}^0(\Omega)$ then $\mathcal{B}(f, g) \in L_{p_3}^0(\Omega)$.

Moreover $\|\mathcal{B}(f, g)\|_{L_{p_3}^0(\Omega)} \leq \|\mathcal{P}\|_{L_{p_1}^{p_3}(\Omega)} \|g\|_{L_{p_2}^{p_3}(\Omega)}$.

**Proof.** (1) Let us first show that if $f \in L_{p_1}^0(\Omega)$ and $g \in L_{p_2}^0(\Omega)$ then $h = \mathcal{B}(f, g) \in L_{p_3}^0(\Omega)$.

Indeed, if $x_k \in X_1$ and $y_l \in Y_1$ then $\overline{\mathcal{P}}(h, x_k \otimes y_l) = \mathcal{P}(\mathcal{B}_1(f, x_k), \mathcal{B}_2(g, y_l))$. Now since $\mathcal{B}_1(f, x_k) \in L^p(X_1)$, $\mathcal{B}_2(g, y_l) \in L^p(Y_1)$ and $\mathcal{P}$ is continuous then $\mathcal{P}(h, x_k \otimes y_l) \in L^p(F)$. For general $\varphi = \sum_n x_k^* \otimes y_l^*$ with $\sum_n \|x_k^*\| \|y_l^*\| < \infty$. Then, using the continuity of $\mathcal{P}$ and $\mathcal{P}$, one has

$$\mathcal{P}(h, \varphi) = \lim_{N \to \infty} \sum_{n=1}^N \mathcal{P}(\mathcal{B}_1(f, x_k^*), \mathcal{B}_2(g, y_l^*)) \in L^p(F).$$

Assume $f \in L_{p_1}^0(\Omega)$ and $g \in L_{p_2}^0(\Omega)$. Let us show that $h \in L_{p_3}^0(\Omega)$.

If $x_k \in X_1$ and $y_l \in Y_1$ then

$$\left( \int_\Omega \|\overline{\mathcal{P}}(h, x_k \otimes y_l)\|_{L_{p_3}^0(\mu)} \right)^{\frac{1}{p_3}} = \left( \int_\Omega \|\mathcal{P}(\mathcal{B}_1(f, x_k), \mathcal{B}_2(g, y_l))\|_{L_{p_3}^0(\mu)} \right)^{\frac{1}{p_3}} \leq \|\mathcal{P}\| \left( \int_\Omega \|\mathcal{B}_1(f, x_k)\|_{L_{p_3}^0(\mu)} \right)^{\frac{1}{p_3}} \leq ||\|f\|_{L_{p_1}^{p_3}(\Omega)} \|g\|_{L_{p_2}^{p_3}(\Omega)} \|x_k\| \|y_l\|.$$

In general, for each $\varphi = \sum_n x_k^* \otimes y_l^*$ in $X_1 \otimes Y_1$, one has $\mathcal{P}(h, \sum_n x_k^* \otimes y_l^*) = \sum_n \overline{\mathcal{P}}(h, x_k^* \otimes y_l^*)$.

Therefore $\left( \int_\Omega \|\overline{\mathcal{P}}(h, \sum_n x_k^* \otimes y_l^*)\|_{L_{p_3}^0(\mu)} \right)^{\frac{1}{p_3}} \leq \sum_n \left( \int_\Omega \|\mathcal{P}(\mathcal{B}_1(f, x_k^*), \mathcal{B}_2(g, y_l^*))\|_{L_{p_3}^0(\mu)} \right)^{\frac{1}{p_3}} \leq \|\mathcal{P}\| \left( \sum_n \|x_k^*\| \|y_l^*\| \right) \left( \int_\Omega \|f\|_{L_{p_1}^{p_3}(\Omega)} \|g\|_{L_{p_2}^{p_3}(\Omega)} \right)^{\frac{1}{p_3}}.$

This gives $\|h\|_{L_{p_3}^0(\Omega)} \leq \|\mathcal{P}\|_{L_{p_1}^{p_3}(\Omega)} \|g\|_{L_{p_2}^{p_3}(\Omega)}$.

(2) Assume that $f$ and $g$ are simple functions. If $f = \sum_k x_k 1_{E_k} \in S(X)$ and $g = \sum_p y_p 1_{F_p} \in S(Y)$ then
2.2. Let us recall the following fact that will be used in the proof.

Let \( B \) be a Banach space, \( 1 \leq p < \infty \) and \( (x_n^*)_n \subseteq X^* \). Then

\[
\sup \{ (\sum_n |(x^*_n, x^*)|^p)^{1/p} : \|x^*\| = 1 \} = \sup \{ (\sum_n |(x, x^*)|^p)^{1/p} : \|x\| = 1 \}
\]

Let \( X, X_1, Y_1, Y \) be Banach spaces and \( 1 \leq p_1, p_2, p_3 < \infty \) such that \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} \). Let \( B_1 : X \times X_1 \to U, B_2 : Y \times Y_1 \to U^* \) be bounded bilinear maps and let \( B(B_1, B_2) = \mathcal{B} : X \times Y \to \mathcal{L}(X_1, Y_1^*) \) be defined by the formula

\[
\langle B(x, y)(x_1, y_1) \rangle = \langle B_1(x, x_1), B_2(y, y_1) \rangle.
\]

If \( f \in L^p_{B_1}(X) \) and \( g \in L^p_{B_2}(Y) \) then \( B(f, g) \in P^{p_3}(\mathcal{L}(X_1, Y_1^*)) \).

Moreover \( \|B(f, g)\|_{L^{p_3}_{\text{weak}}(\mathcal{L}(X_1, Y_1^*))} \leq \|f\|_{L^{p_1}_X(X)} \|g\|_{L^{p_2}_Y(Y)} \).

Proof. Assume first that \( f \) and \( g \) are simple functions. If \( f = \sum_k x_k 1_{E_k} \in S(X) \) and \( g = \sum_k y_k 1_{F_k} \in S(Y) \) then \( h = B(f, g) = \sum_{k,p} B(x_k, y_p) 1_{E_k \cap F_p} \in S(\mathcal{L}(X_1, Y_1^*)) \). Note that \( \mathcal{L}(X_1, Y_1^*)(X_1 \otimes Y_1)^* \). Hence from Lemma 2.8

\[
\|h\|_{L^{p_3}_{\text{weak}}((X_1 \otimes Y_1)^*)} = \sup \{ (\sum_{k,p} |\langle B(x_k, y_p), \psi \rangle|^{p_3} \mu(E_k \cap F_p))^{1/p_3} : \|\psi\|_{(X_1 \otimes Y_1)^*} = 1 \}
\]

\[
= \sup \{ (\sum_{k,p} |\langle \psi, B(x_k, y_p) \rangle|^{p_3} \mu(E_k \cap F_p))^{1/p_3} : \|\psi\|_{X_1 \otimes Y_1} = 1 \}
\]

\[
= \|h\|_{L^{p_3}_{\text{weak}}((X_1 \otimes Y_1)^*)}.
\]

We conclude, using Theorem 2.7, that

\[
\|h\|_{L^{p_3}_{\text{weak}}(\mathcal{L}(X_1, Y_1^*))} \leq \|f\|_{L^{p_1}_X(X)} \|g\|_{L^{p_2}_Y(Y)}.
\]

Now, if we take \( f \in L^p_{B_1}(X) \) and \( g \in L^p_{B_2}(Y) \) then there exists \( (f_n)_n \subseteq S(X) \) and \( (g_n)_n \subseteq S(Y) \) such that \( f_n \to f \) a.e., \( g_n \to g \) a.e., \( \|f_n - f\|_{L^{p_1}_X(X)} \to 0 \) and \( \|g_n - g\|_{L^{p_2}_Y(Y)} \to 0 \). Clearly \( B(f_n, g_n) \to B(f, g) \) a.e. and therefore \( \mathcal{B}(f, g) \) is strongly measurable and

\[
|\langle B(f_n, g_n), \psi \rangle|^{p_3} \to |\langle B(f, g), \psi \rangle|^{p_3} \text{ a.e.}
\]

for all \( \psi \in (X_1 \otimes Y_1)^* \).

To see that \( \mathcal{B}(f, g) \in P^{p_3}(\mathcal{L}(X_1, Y_1^*)) \) it suffices to show that \( \mathcal{B}(f, g) \in L^{p_3}_{\text{weak}}(\mathcal{L}(X_1, Y_1^*)) \).
Then using Fatou's Lemma and the inequality for simple functions we have that

\[ \|B(f, g)\|_{L^p_{weak}((X_1 \otimes Y_1)^*)}^p = \sup \left\{ \int_{\Omega} |(B(f, g), \psi)|^p d\mu : \|\psi\|_{(X_1 \otimes Y_1)^*} = 1 \right\} \]

\[ = \sup \left\{ \int_{\Omega} \lim_{n} \|(B(f_n, g_n), \psi)|^p d\mu : \|\psi\|_{(X_1 \otimes Y_1)^*} = 1 \right\} \]

\[ \leq \sup \left\{ \liminf_{n} \int_{\Omega} |(B(f_n, g_n), \psi)|^p d\mu : \|\psi\|_{(X_1 \otimes Y_1)^*} = 1 \right\} \]

\[ \leq \liminf_{n} \|B(f_n, g_n)\|_{L^p_{weak}((X_1 \otimes Y_1)^*)}^p \]

\[ = \|f\|_{L^p_{weak}^{p_3}(X)}^p \|g\|_{L^p_{weak}^{p_3}(Y)}^p \]

Applying Theorem 2.7 to the examples given above one obtains the following applications.

**Corollary 2.10.** Let 1 ≤ p_1, p_2, p_3 < ∞ such that \( \frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} \).

Let \( B : X \times Y \to Z \) be a bounded bilinear map.

1. If \( f \in L^{p_1}(X) \) and \( g \in L^{p_2}(X^*) \) then \( (f, g) \in L^{p_3} \).
2. If \( f \in L^{p_1}(X) \) and \( g \in L^{p_2}(Y) \) then \( B(f, g) \in L^{p_3}_{weak}(Z) \), where

   \[ B : (x, y) \mapsto (x, y, z^*) \] \( \in \) \( \mathcal{B} \) 

3. If \( f \in L^{p_1}_{\mathcal{B}_1}(X) \) and \( g \in L^{p_2}(Z^*) \) then \( (f, g) \in L^{p_3}_{weak}(Y) \) \( \mathcal{B} \) 

4. If \( f \in L^{p_1}_{\mathcal{B}_1}(X) \) and \( g \in L^{p_2}(Y) \) then \( f \otimes g \in L^{p_3}_{weak}(X \otimes Y) \).

5. If \( f \in L^{p_1}_{\mathcal{B}_1}(\mathcal{L}(X, Z)) \) and \( g \in L^{p_2}(\mathcal{L}(Y, Z^*)) \) and if we put \( f^*(t) = f(t)^* \in \mathcal{L}(Z^*, X^*) \) then \( f^* \otimes g \in L^{p_3}_{weak}(\mathcal{L}(Y, X^*)) \).

**Acknowledgements.** The authors gratefully acknowledges support by Proyecto BMF2002-04013 and MTN2004-21420-E.

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