Norm continuity and related notions for semigroups on Banach spaces*

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Abstract

We find some conditions on a c_0-semigroup on a Banach space and its resolvent connected with the norm continuity of the semigroup. We use them to get characterizations of norm continuous, eventually norm continuous and eventually compact semigroups on Hilbert spaces in terms of the growth of the resolvent of their generator.

1 Introduction.

Quite recently a characterization of norm continuous semigroups on Hilbert spaces in terms of the convergence to zero of the resolvent on vertical lines was achieved by Y. Puhong in [8]. Later O. ElMennaoui and K-J. Engel gave a simpler approach to the same result in [1]. It is clear that from this one can easily get also complete characterizations for compact semigroups (see [8]) and also for eventually norm continuous ones (see [1]) in a similar fashion.

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The problem of characterizing the norm continuity of exponentially stable semigroups $(T(t))_{t\geq 0}$ for $t > 0$ on arbitrary Banach spaces seems to be much harder. A step to the solution of the problem was given in [2].

The aim of this paper is give some more information on the problem for general Banach spaces by means of the condition of relative compactness of the sets

$$\{(T(\cdot)x, \phi) : ||x||_B \leq 1, ||\phi||_{B^*} \leq 1\}$$

or,

$$\{(R(i\cdot, A)x, \phi) : ||x||_B \leq 1, ||\phi||_{B^*} \leq 1\}$$

regarded as subsets in $L^p(R^+)$ and $L^p(R)$, for certain values of p, where $R(\lambda, A)$ stands for the resolvent of the semigroup. We apply such results to find other characterizations of norm continuous, eventually norm continuous and eventually compact semigroups on Hilbert spaces in terms of the growth of the resolvent of their generator.

The paper is divided into three sections. In the first one we shall give the connection of the norm continuity of the semigroup, the fact $R(i\cdot, A) \in C_0(R)$ and the notion of relative compactness in certain L^p spaces of sets as the ones considered above. This relationship holds for general Banach spaces and gives a chain of implications presented in Theorem 2 below. All of them turn out to be equivalent for Hilbert spaces (which, in particular, allows us to get still another proof of Puhong's result).

The second and third section will be devoted to characterize eventually norm continuous and eventually compact semigroups respectively. Our study of eventual norm continuity is not based on the results of the previous section and depends on the asymptotic behaviour of the powers of the resolvent. Finally we characterize eventually compact semigroups by means of growth estimates of the essential norm of powers of the resolvent together with the eventual norm continuity of the semigroup.

Throughout the paper $\{T(t) : t \geq 0\}$ will stand for a strongly continuous semigroup such that $||T(t)|| \leq Me^{-t}$ (there is no lost of generality for such an assumption which will be more suitable for our formulations) and A will denote its infinitesimal generator. We shall use the notation $L^2_B(R)$ for the space of measurable functions with values in a Banach space $(B, ||.||_B)$ such that $||f||_{L^2_B(R)} = (\int_R ||f(x)||^2_B dx)^{\frac{1}{2}} < \infty$. As usual C will stand for a positive constant which may vary from line to line.

2 Norm continuous semigroups.

Since the notion of relative compactness in $L^p(R)$ will be used in the sequel, let us start by recalling the following result.

Theorem 1 ((see [3], Theorem IV.8.20)) *Let $1 \leq p < \infty$. A subset $\mathcal{K}$ of $L^p(R)$ is relatively compact if and only if $\mathcal{K}$ is bounded and*

$$(2.1) \qquad \lim_{h\to 0}\int_R |f(x)-f(x+h)|^p dx = 0 \qquad uniformly \;\; for \;\; f\in\mathcal{K}.$$

$$(2.2) \qquad \lim_{M\to\infty}\int_{|x|\geq M} |f(x)|^p dx = 0 \qquad uniformly \;\; for \;\; f\in\mathcal{K}.$$

Let us present first a chain of implications which hold in any Banach space.

Theorem 2 *Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup on a Banach space B with $||T(t)|| \leq Me^{-t}$. Consider the following statements.*

(i) $\{(R^3(i\cdot, A)x, \phi) : \|x\|_B \leq 1, \|\phi\|_{B^*} \leq 1\}$ *is a relatively compact subset of $L^1(R)$.*

(ii) $\{T(t) : t \geq 0\}$ *is norm continuous for $t > 0$.*

(iii) $\{(T(\cdot)x, \phi) : \|x\|_B \leq 1, \|\phi\|_{B^*} \leq 1\}$ *is a relatively compact subset of $L^1(R^+)$.*

(iv) $\lim_{|r|\to\infty} \|R(ir, A)\| = 0$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof.- $(i) \Rightarrow (ii)$. Let us denote $T_1(t) = t^2T(t)$. It suffices to show that T_1 is a continuous map from $(0, \infty)$ to $\mathcal{L}(B)$, or, in other words the family

$$\{(T_1(\cdot)x, \phi) : \|x\|_B \leq 1, \|\phi\|_{B^*} \leq 1\}$$

is equicontinuous. As

$$\lim_{t\to\infty} \|t^2T(t)\| = 0,$$

Arzela-Ascoli's theorem asserts that it is enough to show that

$$\{(T_1(\cdot)x, \phi) : \|x\|_B \leq 1, \|\phi\|_{B^*} \leq 1\}$$

is relatively compact in $C((0,\infty))$.

Given any $x \in B, \phi \in B^*$ we have that $(2R^3(i\cdot, A)x, \phi)$ is the Fourier transform of $(T_1(\cdot)x, \phi)$.

Now observe that

$$\mathcal{U} = \{(2R^3(i\cdot, A)x, \phi) : \|x\|_B \leq 1, \|\phi\|_{B^*} \leq 1\} \subseteq L^1(R)$$

is mapped continuously by the Fourier antitransform $\mathcal{F} : L^1(R) \longrightarrow C(R)$ onto:

$$\mathcal{V} = \{(T_1(\cdot)x, \phi) : \|x\|_B \leq 1, \|\phi\|_{B^*} \leq 1\} \subseteq C((0,\infty)),$$

what gives the desired conclusion.

(ii) $\Rightarrow$ *(iii)*. Note that from the assumption $||T(t)|| \leq Me^{-t}$ we only need to show (2.1) in Theorem 1. Given any $m \in N$ we split the integral as follows:

$$\int_0^\infty |(T(t)x,\phi) - (T(t+h)x,\phi)|^p dt =$$

$$= \int_0^{\frac{1}{m}} + \int_{\frac{1}{m}}^m + \int_m^\infty |(T(t)x,\phi) - (T(t+h)x,\phi)|^p dt.$$

Therefore

$$\int_0^\infty |(T(t)x,\phi) - (T(t+h)x,\phi)|^p dt \leq$$

$$\leq \tfrac{2^p M^p}{m} + \textstyle\int_{\frac{1}{m}}^m \|T(t) - T(t+h)\|^p dt + \tfrac{2^p M^p}{p} e^{-mp}.$$

Now given $\varepsilon > 0$ we first take $m > 1$ such that $max\{\frac{2^pM^p}{m}, \frac{2^pM^p}{p}e^{-mp}\} < \frac{\varepsilon}{3}$ and then, using the uniform continuity on $[\frac{1}{m}, m]$ find $\delta > 0$ such that if $0 < h < \delta$ then

$$||T(t) - T(t+h)||^p < \frac{\varepsilon}{3(m - \frac{1}{m})} \qquad t \in [\frac{1}{m}, m].$$

Hence for $0 < h < \delta$ we get

$$\int_0^\infty |(T(t)x,\phi) - (T(t+h)x,\phi)|^p dt < \varepsilon \text{ uniformly in } ||x||_B \leq 1, ||\phi||_{B^*} \leq 1.$$

$(iii) \Rightarrow (iv)$. Given $x \in B, \phi \in B^*$ with $||x||_B \leq 1, ||\phi||_{B^*} \leq 1$, we have for $s > 0$ (and similarly for $s < 0$)

$$(R(is, A)x, \phi) = \int_0^\infty e^{-ist}(T(t)x, \phi)dt.$$

We rewrite this as follows

$$\begin{aligned}(R(is, A)x, \phi) \quad &= \tfrac{1}{2}\int_0^\infty (e^{-ist} - e^{-i(t-\frac{\pi}{s})s})(T(t)x, \phi)dt \\ &= -\tfrac{1}{2}\int_0^{\frac{\pi}{s}} e^{-is(t-\frac{\pi}{s})}(T(t)x, \phi)dt \\ &+\tfrac{1}{2}\int_0^\infty e^{-ist}(T(t) - T(t + \frac{\pi}{s})x, \phi)dt.\end{aligned}$$

Hence

$$|(R(is, A)x, \phi)| \leq \frac{1}{2}\int_0^{\frac{\pi}{s}} |(T(t)x, \phi)|dt + \frac{1}{2}\int_0^\infty |(T(t) - T(t + \frac{\pi}{s})x, \phi)|dt.$$

Now a simple application of Theorem 1 gives (iv).

Let us present some equivalent formulations of condition (iii) in Theorem 2.

Proposition 1 *Let A be the generator of a strongly continuous semigroup on a Banach space B with $||T(t)|| \leq Me^{-t}$. Then the following are equivalent:*

(iii a) $\{(T(\cdot)x, \phi) : \|x\|_B \leq 1, \|\phi\|_{B^*} \leq 1\}$ *is a relatively compact subset of $L^p((0, \infty))$ for some $1 \leq p < \infty$.*

(iii b) $\{(T(\cdot)x, \phi) : \|x\|_B \leq 1, \|\phi\|_{B^*} \leq 1\}$ *is a relatively compact subset of $L^p((0, \infty))$ for all $1 \leq p < \infty$.*

(iii c) $\{(R(i\cdot, A)x, \phi) : \|x\|_B, \|\phi\|_{B^*} \leq 1\}$ *is a relatively compact subset of $L^2(R)$.*

(iii d) $\lim_{M\to\infty} \int_M^\infty |(R(is, A)x, \phi)|^2 ds = 0$ *uniformly in* $\|x\|_B \leq 1, \|\phi\|_{B^*} \leq 1$.

Proof.- $(iiia) \Leftrightarrow (iiib)$. Assume the condition on p_0 and take any $1 \leq p < \infty$. Given a pair of sequences $(x_n, \phi_n)_{n\in N} \in B \times B^*$ we have to find a subsequence n_k such that

$$f_k(t) = (T(t)x_{n_k}, \phi_{n_k}) \text{ converges in } L^p((0,\infty)).$$

From the assumption there exist a function $f \in L^{p_0}$ and a subsequence n_k such that

$$g_k(t) = (T(t)x_{n_k}, \phi_{n_k}) - f(t) \to 0 \qquad a.e.$$

Using the fact $|g_k(t)|^p \leq 2^p M^p e^{-pt}$ and the Lebesgue dominated convergence theorem the desired result is achieved.

$(iiib) \Leftrightarrow (iiic)$. It follows from the fact that the Fourier transform is an isomorphism in $L^2(R)$.

$(iiic) \Leftrightarrow (iiid)$. It follows from Theorem 1 since (2.1) is always true due to the resolvent identity.

We shall now prove that all the conditions appearing in Theorem 2 are equivalent in the case of Hilbert spaces.

Theorem 3 *Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup on a Hilbert space E with $||T(t)|| \leq Me^{-t}$ such that* $\lim_{|r|\to\infty} \|R(ir, A)\| = 0$.

Then $\{(R^3(i\cdot, A)x, y) : \|x\|_E \leq 1, \|y\|_E \leq 1\}$ is a relatively compact subset of $L^1(R)$.

Proof.- It is well known that for Hilbert spaces E we still have that the Fourier transform is an isomorphism from $L^2_E(R)$ onto $L^2_E(R)$. Therefore given any $x, y \in E$ we have that

$$\sup_{||x||\leq 1, ||y||\leq 1} \{\|R(i\cdot, A)x\|_{L^2_E(R)}, \|R(i\cdot, A')y\|_{L^2_E(R)}\} < \infty \tag{2.3}$$

Note that

$$(R^3(is, A)x, y) = (R^2(is, A)x, R(is, A')y).$$

Therefore

$$|(R^3(is, A)x, y)| \leq \|R(is, A)\| \; \|R(is, A)x\|_E \; \|R(is, A')y\|_E \tag{2.4}$$

Using (2.3) and (2.4) we have that

$$\mathcal{K} = \{(3R^3(i\cdot, A)x, y) : \|x\| \leq 1, \|y\| \leq 1\}$$

is a bounded subset of $L^1(R)$.

To show the relative compactness of $\mathcal{K}$ we shall use Theorem 1.

Observe that

$$(2R^3(is,A)x,y)-(2R^3(i(s+h),A)x,y)=\int_0^\infty(e^{-ist}-e^{-i(s+h)t})t^2(T(t)x,y)dt.$$

This implies

$$\begin{aligned}(2R^3(is,A)x,y)-(2R^3(i(s+h),A)x,y) &= -i\int_0^\infty(\int_s^{s+h}e^{-iut}du)t^3(T(t)x,y)dt\\ &= -i\int_s^{s+h}(\int_0^\infty e^{-iut}t^3(T(t)x,y)dt)du\\ &= -6i\int_s^{s+h}(R^4(iu,A)x,y)du.\end{aligned}$$

(Another way to see this is using that $(3iR^4(is,A)x,y)$ is the derivative of $(R^3(is,A)x,y)$ and the Fundamental Theorem of Calculus.)

Now using Fubini theorem and Cauchy-Schwarz inequality we can write

$$\begin{aligned}\int_R|(2R^3(is,A)x,y) & -(2R^3(i(s+h),A)x,y)|ds\leq 6\int_R\int_s^{s+h}|(R^4(iu,A)x,y)|duds\\ &\leq 6h\int_R|(R^4(iu,A)x,y)|du\\ &\leq 6h\sup_{s\in R}||R(is,A)||^2.\\ &.(\int_R||R(is,A)x||_E^2ds)^{\frac{1}{2}}(\int_R||R(is,A')y||_E^2ds)^{\frac{1}{2}}\\ &\leq Ch.\end{aligned}$$

On the other hand, using (2.4), we have

$$\begin{aligned}\int_{|s|>M}|(2R^3(is,A)x,y)|ds &\leq 2\sup_{|s|>M}||(R(is,A)||\int_R||(R(is,A)x||_E\ ||(R(is,A')y||_E ds\\ &\leq C\sup_{|s|>M}||(R(is,A)||.\end{aligned}$$

Therefore the assumption gives (2.2) in Theorem 1 and the proof is finished.

3 Eventually norm continuous semigroups in Hilbert spaces.

In this section we will be concerned with a characterization of eventually norm continuous semigroups on Hilbert spaces in terms of the growth of the resolvent of their generator.

Let us mention the particular case of nilpotent semigroups which are characterized as follows.

Theorem 4 (see [6], Theorem 6.11) *Let* $\{T(t) : t \geq 0\}$ *be strongly continuous semigroup on a Banach space* B *with* $||T(t)|| \leq Me^{-t}$. $\{T(t) : t \geq 0\}$ *is a nilpotent semigroup if and only if*

$$\sup_{n \in N}\{(n!||R^n(0, A)||)^{\frac{1}{n}}\} < \infty. \tag{3.1}$$

We shall prove that an asymptotic version of (3.1) characterizes the eventual norm continuity. Before seeing this, let us introduce some constants associated to a semigroup which will be needed in the sequel.

Definition 1 *Let* A *be the generator of a strongly continuous semigroup* $\{T(t) : t \geq 0\}$ *on a Hilbert space* E. *Given* $n \in N$ *we define*

$$\rho_n = \lim_{k \to \infty} \sup_{||x|| \leq 1, ||y|| \leq 1} \int_{|s|>k} (n+1)! \ |(R^{n+2}(is, A)x, y)| ds.$$

It is not hard to see that $\rho_n \in R^+$.

Theorem 5 *Let* A *be the generator of a strongly continuous semigroup* $\{T(t) : t \geq 0\}$ *with* $||T(t)|| \leq Me^{-t}$ *on a Hilbert space* E. *Then the following assertions are equivalent*

(i) $\{T(t) : t \geq 0\}$ *is eventually norm continuous, i.e., there exists* $t' > 0$ *such that* $t \longrightarrow T(t)$ *from* (t', ∞) *into* $\mathcal{L}(E)$ *is norm continuous.*

(ii) There exists $C > 0$ *such that*

$$\limsup_{|s|\to\infty} \|n!R^n(is,A)\|^{\frac{1}{n}} \leq C \qquad \forall n \in N.$$

(iii) There exist $t_0 > 0$ *such that*

$$\lim_{n\to\infty} \frac{\rho_n}{t_0^n} = 0.$$

Proof.- $(i) \Rightarrow (ii)$ is a refinement of the proof of Theorem 1.20. in [5]. Let now t' be such that $t \longrightarrow T(t)$ is norm continuous on $[t', \infty)$. Since Plancherel's holds in the setting of Hilbert spaces, we have the following identification

$$n!R^{n+1}(is,A)x = \int_0^\infty e^{-ist} t^n T(t) x dt.$$

then we get

$$\|n!R(is,A)^{n+1}\| \leq \int_0^{t'} t^n M e^{-t} dt + \| \int_{t'}^\infty e^{-ist} t^n T(t) dt \|$$

$$\leq M \tfrac{(t')^{n+1}}{n+1} + \| \textstyle\int_{t'}^\infty e^{-ist} t^n T(t) dt \|.$$

It follows now from the Riemann-Lebesgue lemma:

$$\limsup_{|s|\to\infty} \|n!R^{n+1}(is,A)\| \leq M \frac{(t')^{n+1}}{n+1}$$

which gives the desired estimate.

$(ii) \Rightarrow (iii)$

$$\rho_n = \lim_{k\to\infty} \sup_{||x||\leq 1, ||y||\leq 1} \int_{|s|>k} |((n+1)!(R^{n+2}(is,A)x, y)| ds$$

$$\leq \lim_{k\to\infty} \sup_{|s|>k} \{(n+1)! ||R^n(is,A)||\}.$$

$$\sup_{||x||\leq 1, ||y||\leq 1} \textstyle\int_R ||R(is,A)x||_E \; ||R(is,A')y||_E ds$$

$$\leq \tfrac{M^2}{2} \limsup_{|s|\to\infty} \{(n+1)! ||R^n(is,A)||\}.$$

Therefore $\rho_n \leq \frac{M^2}{2}(n+1)C^n$. Hence it suffices to take $t_0 > C$ to get (iii).

$(iii) \Rightarrow (i)$ Given any $x \in E$, the vector-valued function $n!R^{n+1}(i\cdot, A)x$ is the Fourier transform of the function $T_n(\cdot)x$ where $T_n(t)x = t^nT(t)x$. As the Fourier transform is a isomorphism from $L^2_E(R)$ onto $L^2_E(R)$ it may be concluded that for any $x, y \in E$

$$R(i\cdot, A')y, \; R^n(i\cdot, A)x \in L^2_E(R).$$

Given $x, y \in E$, observe that

$$|(R^{n+1}(is, A)x, y)| \leq \|R^n(is, A)x\| \|R(is, A')y\|_E$$

what shows that

$$|(R^{n+1}(i\cdot, A)x, y)| \in L^1(R)$$

what allows us to write

$$t^{n+1}(T(t)x, y) = \frac{1}{2\pi}\int_R (n+1)! e^{ist}(R^{n+2}(is, A)x, y)ds. \tag{3.2}$$

Let us first note that

$$\lim_{h\to 0} t^{n+1}\|T(t) - T(t+h)\| \leq \lim_{h\to 0} \|t^{n+1}T(t) - (t+h)^{n+1}T(t+h)||.$$

Let us now fix $k, n \in N$ and $t, h > 0$. Then from (3.2)

$$\begin{aligned} \|t^{n+1}T(t) \quad & -(t+h)^{n+1}T(t+h)|| \leq \\ & \leq \tfrac{1}{2\pi}\sup_{||x||\leq 1, ||y||\leq 1}\textstyle\int_R |(1-e^{ihs})| \; |(n+1)!(R^{n+2}(is, A)x, y)|ds \\ & \leq \tfrac{1}{2\pi}\sup_{||x||\leq 1, ||y||\leq 1}\textstyle\int_{|s|\leq k} |(1-e^{ihs})| \; |(n+1)!(R^{n+2}(is, A)x, y)|ds \\ & + \tfrac{1}{\pi}\sup_{||x||\leq 1, ||y||\leq 1}\textstyle\int_{|s|>k} |(n+1)!(R^{n+2}(is, A)x, y)|ds. \end{aligned}$$

Taking the limit as $h \to 0$ we get

$$\lim_{h\to 0} t^{n+1}\|T(t) - T(t+h)\| \leq \frac{1}{\pi}\sup_{||x||\leq 1, ||y||\leq 1}\int_{|s|>k} |(n+1)!(R^{n+2}(is, A)x, y)|ds.$$

Taking now the limit as $k \to \infty$ we get

$$\lim_{h\to 0} t^{n+1}\|T(t) - T(t+h)\| \leq \frac{\rho_n}{\pi}.$$

This clearly forces the semigroup to be norm continuous for $t \geq t_0$.

4 Eventually compact semigroups

Let us recall that compact semigroups, i.e., those that $T(t)$ are compact operators for $t > 0$ are known to be the norm continuous semigroups such that $R(\lambda, A)$ is compact for any $\lambda \in \rho(A)$. (see [7] Theorem 3.3). The aim of this section is to find the corresponding result for eventually compact semigroups, i.e., where $T(t)$ are compact operators for $t > t_0$ for some value $t_0 \geq 0$.

To find such a characterization we shall need the notion of essential norm. Let us recall that given a Banach space B and an operator $T \in \mathcal{L}(B)$ we denote by

$$||T||_{ess} = inf\{||T - K|| : K \in \mathcal{K}(B)\}.$$

As usual $\mathcal{K}(B)$ stands for the space of compact operators on B.

Theorem 6 *Let A be the generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ on a Banach space B with $||T(t)|| \leq Me^{-t}$. Then the following assertions are equivalent*

(i) $\{T(t) : t \geq 0\}$ is eventually compact, i. e., there exists $t_0 > 0$ such that $T(t_0)$ is a compact operator.

(ii) $\{T(t) : t \geq 0\}$ is eventually norm continuous and

$$\sup_{n \in N}\{\|n!R^n(0, A)\|_{ess}^{\frac{1}{n}}\} < \infty. \tag{4.1}$$

Proof.- $(i) \Rightarrow (ii)$. It is well known that eventual compactness implies eventual norm continuity (see [7], Theorem 3.2). Let us now observe that

$$n!R^{n+1}(0, A)x = \int_0^{t_0} t^n T(t)xdt + \int_{t_0}^{\infty} t^n T(t)xdt.$$

Since $K_0 = \int_{t_0}^{\infty} t^n T(t)dt$ is a compact operator on $\mathcal{L}(B)$ then

$$||n!R^{n+1}(0, A) - K_0|| \leq M\frac{{t_0}^{n+1}}{n+1}.$$

This clearly implies (4.1).

$(ii) \Rightarrow (i)$. Let us first define the spaces:

$$l^{\infty}(B) = \{(y_n) \subset B : \{y_n : n \in N\} \text{ is bounded }\},$$

$$l_T^\infty(B) = \{(y_n) \subset l^\infty(B) : \lim_{t\to 0} \|T(t)y_n - y_n\|_B = 0 \text{ uniformly for } n \in N\},$$

$$m(B) = \{(x_n) \in l^\infty(B) : \{x_n : n \in N\} \text{ is relatively compact } \} \subset l_T^\infty(B).$$

On the quotient space $\hat{B} = l_T^\infty(B)/m(B)$, we define:

$$\widehat{T(t)}(y_n + m(B)) := (T(t)y_n) + m(B) \qquad \forall (y_n) \in l_T^\infty(B).$$

It is known that $\widehat{T(t)}$ is also a strongly continuous semigroup (see [5]). Let us denote by $\hat{A}$ its generator.

Moreover, since $(Ky_n) \in m(B)$ for any $(y_n) \in l^\infty(B)$ and any compact operator K then it is easy to show

$$\|R^n(0, \hat{A})\| \leq \|R^n(0, A)\|_{ess}. \tag{4.2}$$

Hence assumption (4.1) gives

$$(n!\|R^n(0, \hat{A})\|)^{\frac{1}{n}} \leq M \qquad \forall n \in N$$

Therefore, from Theorem 4, $\hat{T}$ is a nilpotent semigroup.

Now, by the norm continuity of $T(t)$ we have that there exists $t' > 0$ such that for $t \geq t'$

$$(T(t)x_n) \in l_T^\infty(B) \qquad \forall (x_n) \in l^\infty(E).$$

Since $\hat{T}$ is nilpotent, for some $t_0 > 0$,

$$(T(t_0)x_n) \in m(B) \qquad \forall (x_n) \in l^\infty(B)$$

which shows that $T(t_0)$ is compact.

Corollary 1 *Let A be the generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ on a Hilbert space E with $||T(t)|| \leq Me^{-t}$. Then $\{T(t) : t \geq 0\}$ is eventually compact if and only if*

$$\sup_{n\in N} \limsup_{|s|\to\infty} \{\|n!R^n(is, A)\|^{\frac{1}{n}}\} < \infty \text{ and} \tag{4.3}$$

$$\sup_{n\in N} \{\|n!R^n(0, A)\|_{ess}^{\frac{1}{n}}\} < \infty. \tag{4.4}$$

References

[1] O. ElMennaoui, K.-J. Engel, On the characterization of eventually norm continuous semigroups in Hilbert spaces, Archiv Math, to appear.

[2] O. ElMennaoui, K.-J. Engel, Towards a characterization of eventually norm continuous semigroups on Banach spaces, preprint.

[3] N. Dunford, J.T. Schwarz, Linear operators. Part I, John Wiley and sons., New York 1958.

[4] E. Hille and R.S. Phillips, Functional Analysis and Semigroups, Amer. Math. Soc. Colloquium Publ., 31, Revised 1981.

[5] R. Nagel (ed), One-parameter Semigroups of Positive Operators, Lecture Notes in Math. 1184, Springer-Verlag 1986.

[6] A. M. Sinclair, Continuous Semigroups in Banach Algebras, London Math. Soc. Lecture Notes 63, Cambridge Univ. Press, Cambridge , 1982.

[7] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci. 44, Springer-Verlag 1983.

[8] Y. Puhong, Characteristic conditions for a c_0 -semigroup with continuity in the uniform topology for $t > 0$ in Hilbert space, Proc. Amer. Math. Soc. 116 , 1992, 991-997.