ON THE AREA FUNCTION FOR $H^(\sigma_p), 1 \leq p \leq 2$.

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SUMMARY: It is shown that the inequality

$$\int_0^{2\pi} (\int_{\Omega(\theta)} |f'(z)|^2 dA(z))^{\frac{1}{2}} d\theta \leq C||f||_1$$

holds for Hardy spaces of function taking values in the Schatten classes $\sigma_p, 1 \leq p \leq 2$.

1. INTRODUCTION. It is a well known result that the norm in the Hardy space H^1 is equivalent to the L^1 norm of the Lusin area function (see [20, 5]), in particular,

$$\int_0^{2\pi} S(f,\theta) d\theta \leq C||f||_1 \tag{1.1}$$

where $S(f,\theta) = \left(\int_{\Omega(\theta)} |f'(z)|^2 dA(z)\right)^{\frac{1}{2}}$ and $\Omega(\theta)$ stands for the Stolz domain given by $\Omega(\theta) = \{z = re^{2\pi it} : |t-\theta| \leq 1-r\}$ and $dA(z)$ is the area measure on the unit disc D.

As usual the vector-valued consideration of classical inequalities leads to properties on the Banach spaces theory. This has been the case of lots of properties that have been depply studied. The aim of this note is the consideration of the previous inequalitiy (1.1) in the setting of functions taking values in the Schatten classes σ_p.

Throughout the paper X stand for a complex Banach space, $1 \leq p \leq 2$ and we shall denote by $H^p(X)$ the space of X-valued Bochner p-integrable functions on the circle $\mathbf{T}$ whose negative Fourier coefficients vanish, i.e. $f \in L^p(\mathbf{T}, X)$ such that $\hat{f}(n) = 0$ for $n \leq 0$.

Given $f \in H^1(X)$ we keep the notation f for the analytic function in the disc D whose Taylor coefficients are the Fourier coefficients of f and we shall write $||f||_{p,X} = (\int_0^{2\pi} ||f(e^{it})||^p \frac{dt}{2\pi})^{\frac{1}{p}}$ and $M_{p,X}(f,r) = ||f_r||_{p,X} = (\int_0^{2\pi} ||f(re^{it})||^p \frac{dt}{2\pi})^{\frac{1}{p}}$.

We shall denote by σ_p the Banach space of compact operators $x : l^2 \to l^2$ such that $||x||_p = \left(tr(x^*x)^{\frac{p}{2}}\right)^{\frac{1}{p}} < \infty$. It is well known that σ_1 coincides with the space of nuclear operators on l^2 and σ_2 with the space of Hilbert-Schmidt operators on l^2. The reader is referred to [6] for general properties on σ_p.

Before stating the main theorem of this note, let us recall several previous inequalities which hold in the setting of Schatten classes.

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It was proved by N. Tomczac-Jaegerman ([18]) the cotype 2 property for $\sigma_p, 1 \leq p \leq 2$, or equivalently that there exists a constant $C_p > 0$ such that

$$(1.2) \qquad \left(\sum_{k=0}^{\infty} ||x_k||^2_{\sigma_p}\right)^{\frac{1}{2}} \leq C_p || \sum_{k=0}^{\infty} x_k e^{i2^k t} ||_{1,\sigma_p}$$

Several extensions of this notion were shown to be true for these classes. The reader is referred to [4] for the notion of PL-uniformly convexity, to [10] for the analogue to the version of Kintchine-inequalities for Banach lattices in the context of Schatten classes and to ([19], [7]) for the notion of H^1-convexity and related properties.

Another improvement of the inequality (1.2) is the vector-valued formulation of Paley inequality (see [13]) that was proved by A. Pelzcinsky and the author in [3], this is for $1 \leq p \leq 2$ there exists a constant $C_p > 0$ such that

$$(1.3) \qquad \left(\sum_{k=0}^{\infty} ||x_{2^k}||^2_{\sigma_p}\right)^{\frac{1}{2}} \leq C_p || \sum_{n=0}^{\infty} x_n e^{int} ||_{1,\sigma_p}$$

The reader is referred to [11] for an interesting extension of inequality (1.3).

The following improvement of (1.3) is due to G. Pisier (see [15]) who showed that for any sequence $0 \leq r_0 < r_1 < ... < r_n < ... < 1$ there exist constants $\delta_p, C_p > 0$ such that

$$(1.4) \qquad \left(||f_{r_0}||^2_{1,\sigma_p} + \delta_p \sum_{n=1}^{\infty} ||f_{r_n} - f_{r_{n-1}}||^2_{1,\sigma_p}\right)^{\frac{1}{2}} \leq C_p ||f||_{1,\sigma_p}$$

Recently it has been shown by the author in [2] that still another inequality due to Hardy and Littlewood (see [8]) holds for Schatten classes, that is there exists a constant $C_p > 0$ such that

$$(1.5) \qquad \left(\int_0^1 (1-r) M^2_{1,\sigma_p}(f', r) dr\right)^{\frac{1}{2}} \leq C ||f||_{1,\sigma_p}$$

Let us now formulate the main theorem proved in this note, which is the extension of (1.1) to the setting of Hardy spaces with values in Schatten classes, and which improves all the previous estimates given above.

Theorem. Let $1 \leq p \leq 2$. There exists a constant $C_p > 0$ such that

$$\int_0^{2\pi} \Big(\int_{\Omega(\theta)} ||f'(z)||^2_{\sigma_p} dA(z) \Big)^{\frac{1}{2}} \frac{d\theta}{2\pi} \leq C_p ||f||_{1,\sigma_p}$$

The proof follows similar ideas than those used in the scalar valued case. The main tools are the use of the non-conmutative version of a result on factorization of analytic functions with values on theses classes together with some interpolation arguments. The reader is referred to [2, 3, 7,15, 19] for the use similar arguments in related questions.

2. Related properties in geometry of Banach spaces. Let us recall all the notions apperared in the previous section and their connections.

Although the notions of cotype and type are defined in terms of the Rademacher functions we shall replace them by lacunary sequences $e^{i2^n t}$, which gives an equivalent definition.

A Banach space has *cotype* 2 (see [12, 14]) if there exists a constant $C > 0$ such that

$$\left(\sum_{k=0}^{\infty} ||x_k||^2\right)^{\frac{1}{2}} \leq ||f||_{1,X} \tag{2.1}$$

for any $f(z) = \sum_{k=0}^{\infty} x_k z^{2^k}$.

A complex Banach space is said to be a *Paley space* (see [3]) if

$$\left(\sum_{k=0}^{\infty} ||x_{2^k}||_X^2\right)^{\frac{1}{2}} \leq C||f||_{1,X} \tag{2.2}$$

for any $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$.

Definition 2.1 A complex Banach space is said to satisfy *lacunary radial 2-lower estimate* (see [15] for the corresponding definition for general increasing sequences r_n) if there exist constants $\delta, C > 0$ such that if $r_k = 1 - 2^{-k}$

$$\left(||f(0)||_X^2 + \delta \sum_{n=1}^{\infty} ||f_{r_n} - f_{r_{n-1}}||_{1,X}^2\right)^{\frac{1}{2}} \leq C||f||_{1,X} \tag{2.3}$$

for any $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$.

A complex Banach space X is said to have *property (HL)*, in short $X \in (HL)$, (see [2]) if there exists a constant $C > 0$ such that

$$\left(\int_0^1 (1-r) M_{1,X}^2(f', r) dr\right)^{\frac{1}{2}} \leq C||f||_{1,X} \tag{2.4}$$

for any $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$.

Definition 2.2 A complex Banach space X is said to have *property (LP)*, in short $X \in (LP)$, if there exists a constant $C > 0$ such that

$$\int_0^{2\pi} \left(\int_{\Omega(\theta)} ||f'(z)||_X^2 dA(z)\right)^{\frac{1}{2}} d\theta \leq C||f||_{1,X} \tag{2.5}$$

for any $f(z)=\sum_{n=0}^{\infty} x_n z^n \in H^1(X)$.

Proposition 2.1. *If $X \in (LP)$ then $X \in (HL)$.*

Proof. Let us consider the function $h(r,\theta)=(1-r)||f'(re^{i\theta})||$ and apply vector-valued Minkowski's inequality to get

$$||\int_0^{2\pi} h(r,\theta)d\theta||_{L^2(\frac{dr}{1-r})} \leq \int_0^{2\pi} ||h(r,\theta)||_{L^2(\frac{dr}{1-r})} d\theta,$$

that is

$$\left(\int_0^1 (1-r)M_{1,X}^2(f',r)dr\right)^{\frac{1}{2}} \leq \int_0^{2\pi} (\int_0^1 (1-r)||f'(re^{i\theta})||^2 dr)^{\frac{1}{2}} d\theta$$

On the other hand, if we write $g(f,\theta)=(\int_0^1 (1-r)||f'(re^{i\theta})||^2 dr)^{\frac{1}{2}}$ for the g-function defined by Littlewood and Paley (see [10]), then same proof as in the scalar case (see [20, page 210]) shows that

$$g(f,\theta) \leq CS(f,\theta).$$

Combining both estimates we have

$$\left(\int_0^1 (1-r)M_{1,X}^2(f',r)dr\right)^{\frac{1}{2}} \leq C\int_0^{2\pi} \left(\int_{\Omega(\theta)} ||f'(z)||_X^2 dA(z)\frac{d\theta}{2\pi}\right)^{\frac{1}{2}},$$

and the proof is finished.

Proposition 2.2. *If $X \in (HL)$ then X satisfies a lacunary radial 2-lower estimate.*

Proof. Write $f_{r_n}(e^{i\theta})-f_{r_{n-1}}(e^{i\theta})=\int_{r_{n-1}}^{r_n} f'(se^{i\theta})ds$. Therefore

$$||f_{r_n}-f_{r_{n-1}}||_{1,X} \leq \int_{r_{n-1}}^{r_n} M_{1,X}(f',s)ds \leq (r_n-r_{n-1})^{\frac{1}{2}} \left(\int_{r_{n-1}}^{r_n} M_{1,X}^2(f',s)ds\right)^{\frac{1}{2}}.$$

Using that $r_n-r_{n-1}=1-r_n$ we have

$$\begin{aligned}\sum_{n=1}^{\infty} ||f_{r_n}-f_{r_{n-1}}||_{1,X}^2 &\leq \sum_{n=1}^{\infty}(r_n-r_{n-1})\int_{r_{n-1}}^{r_n} M_{1,X}^2(f',s)ds\\ &\leq \sum_{n=1}^{\infty}\int_{r_{n-1}}^{r_n}(1-s)M_{1,X}^2(f',s)ds\\ &= \int_0^1 (1-s)M_{1,X}^2(f',s)ds.\end{aligned}$$

This estimate gives the desired result.

It is rather elementary to show that actually lacunary radial 2-lower estimate implies Paley space and Paley implies cotype 2 (see [15], [3] respectively).

3. Proof of the main theorem. We need certain lemmas to prepare the proof.

Lemma 3.1. (Non commutative Factorization, see [17]) *Let $f \in H^1(\sigma_1)$. Then there exist two functions $h_1, h_2 \in H^2(\sigma_2)$ such that*

$$f(e^{it}) = h_1(e^{it})h_2(e^{it}), \text{ and } ||f||_{1,\sigma_2} = ||h_1||^2_{1,\sigma_2} = ||h_2||^2_{1,\sigma_2}.$$

Using Plancherel's one easily gets the following fact.

Lemma 3.2.*Let X be a Hilbert space and $f \in H^2(X)$. Then*

$$\left(||f(0)||^2 + \int_0^{2\pi} \int_{\Omega(\theta)} ||f'(z)||^2 dA(z) d\theta \right)^{\frac{1}{2}} \approx ||f||^2_{2,X}.$$

Proposition 3.1. *Hilbert spaces have (LP) property.*

Proof. Assume without lost of generality that $X = l^2$.

Given $f = (f_n)_{n \in \mathbf{N}} \in H^1(l^2)$ we have that

$$\int_0^{2\pi} \left(\int_{\Omega(\theta)} ||f'(z)||^2_{l^2} dA(z) \right)^{\frac{1}{2}} d\theta = \int_0^{2\pi} \left(\int_{\Omega(\theta)} \sum_{n \in \mathbf{N}} |f'_n(z)|^2 dA(z) \right)^{\frac{1}{2}} d\theta.$$

Applying Kintchine's and vector-valued Minkowsky's inequality and then the scalar-valued case together with Kintchine's again we obtain

$$\begin{aligned} \int_0^{2\pi} (\int_{\Omega(\theta)} \sum_{n \in \mathbf{N}} |f'_n(z)|^2 dA(z))^{\frac{1}{2}} d\theta &= \int_0^{2\pi} (\int_{\Omega(\theta)} (\int_0^1 |\sum_{n \in \mathbf{N}} f'_n(z) r_n(t)| dt)^2 dA(z))^{\frac{1}{2}} d\theta \\ &\le \int_0^{2\pi} (\int_0^1 (\int_{\Omega(\theta)} |\sum_{n \in \mathbf{N}} f'_n(z) r_n(t)|^2 dA(z))^{\frac{1}{2}} dt) d\theta \\ &\le C \int_0^1 \int_0^{2\pi} |\sum_{n \in \mathbf{N}} f_n(e^{i\theta}) r_n(t)| d\theta dt \\ &= C \int_0^{2\pi} (\sum_{n \in \mathbf{N}} |f_n(e^{i\theta})|^2)^{\frac{1}{2}} d\theta. \end{aligned}$$

σ_1 has (LP)-property. Given $f \in H^1(\sigma_1)$ take $h_1, h_2 \in H^2(\sigma_2)$ such that

$$f(e^{it}) = h_1(e^{it})h_2(e^{it}), \qquad ||h_1||^2_{2,\sigma_2} = ||h_2||^2_{2,\sigma_2} = ||f||_{1,\sigma_1}.$$

Note that for $i,j \in \{1,2\}, i \neq j$

$$\int_{\Omega(\theta)} ||h_i'(z)h_j(z)||_{\sigma_1}^2 dA(z) \leq \int_{\Omega(\theta)} ||h_i'(z)||_{\sigma_2}^2 ||h_j(z)||_{\sigma_2}^2 dA(z)$$
$$\leq \sup_{z\in\Omega(\theta)} ||h_j(z)||_{\sigma_2}^2 \int_{\Omega(\theta)} ||h_i'(z)||_{\sigma_2}^2 dA(z).$$

This gives

$$\int_0^{2\pi} \left(\int_{\Omega(\theta)} ||f'(z)||_{\sigma_1}^2 dA(z) \right)^{\frac{1}{2}} d\theta \leq \int_0^{2\pi} \sup_{z\in\Omega(\theta)} ||h_1(z)||_{\sigma_2} \left(\int_{\Omega(\theta)} ||h_2'(z)||_{\sigma_2}^2 dA(z) \right)^{\frac{1}{2}} d\theta$$
$$+ \int_0^{2\pi} \sup_{z\in\Omega(\theta)} ||h_2(z)||_{\sigma_2} \left(\int_{\Omega(\theta)} ||h_1'(z)||_{\sigma_2}^2 dA(z) \right)^{\frac{1}{2}} d\theta.$$

Therefore, denoting by $g^*(e^{i\theta}) = \sup_{z\in\Omega(\theta)} ||g(z)||_X$ the non tangential maximal function of a function $g \in H^1(X)$ we have

$$\int_0^{2\pi} \left(\int_{\Omega(\theta)} ||f'(z)||_{\sigma_1}^2 dA(z) \right)^{\frac{1}{2}} d\theta$$
$$\leq \left(\int_0^{2\pi} |h_1^*(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} \int_{\Omega(\theta)} ||h_2'(z)||_{\sigma_2}^2 dA(z) d\theta \right)^{\frac{1}{2}}$$
$$+ \left(\int_0^{2\pi} |h_2^*(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} \int_{\Omega(\theta)} ||h_1'(z)||_{\sigma_2}^2 dA(z) d\theta \right)^{\frac{1}{2}}.$$

Using now Lemma 3.1 and the well known result about the boundedness of the maximal operator one has

$$\int_0^{2\pi} \left(\int_{\Omega(\theta)} ||f'(z)||_{\sigma_1}^2 dA(z) \right)^{\frac{1}{2}} d\theta \leq C||h_1||_{2,\sigma_2} ||h_2||_{2,\sigma_2} = C||f||_{2,\sigma_1}$$

The case $1 < p < 2$. Observe that $X \in (LP)$ means that the operator

$$f \to f'(z)\chi_{\Omega(\theta)}$$

is bounded from $H^1(X)$ into $L^1\big(d\theta, L^2(dA(z), X)\big)$.

Therefore the proposition 3.1 and the previous case give the boundedness of T considered as operator $H^1(\sigma_2)$ into $L^1\big(d\theta, L^2(dA(z), \sigma_2)\big)$ and $H^1(\sigma_1)$ into $L^1\big(d\theta, L^2(dA(z), \sigma_1)\big)$.

Let us choose $0 < \theta < 1$ so that $\frac{1}{p} = 1 - \frac{\theta}{2}$. Using the well known results of interpolation (see [1])

$$\big(L^1\big(d\theta, L^2(dA(z), X_1)\big), L^1\big(d\theta, L^2(dA(z), X_2)\big)\big)_\theta = L^1\big(d\theta, L^2(dA(z), (X1, X_2)_\theta)\big)$$

for any couple af Banach spaces X_1, X_2, the fact $(\sigma_1, \sigma_2)_\theta = \sigma_p$ and the recent results on interpolation for vector valued Hardy spaces due to Pisier and Xu (see [19, 16]),

$$\big(H^1(\sigma_1), H^1(\sigma_2)\big)_\theta = H^1(\sigma_p).$$

one gets that T is also bounded from $H^1\big(\sigma_p\big)$ into $L^1\big(d\theta, L^2(dA(z), \sigma_p\big)$ what gives that σ_p has (LP)-property.

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