$q$-CONCAVITY AND $q$-ORLICZ PROPERTY ON
SYMMETRIC SEQUENCE SPACES.

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ABSTRACT. We give a general method for constructing symmetric sequence spaces that for $1 < q < 2$ satisfy a lower $q$-estimate but fail to be $q$-concave and, for $2 \leq q < \infty$ have the $q$-Orlicz property but fail to be $q$-concave. In particular this gives examples of spaces with the 2-Orlicz property but without cotype 2.
Let $1 \leq q < \infty$. A Banach lattice $X$ is said to be $q$-concave if there exists a constant $C \geq 0$ such that
\[
\left( \sum_{k=1}^{n} \| x_k \|_X^q \right)^{\frac{1}{q}} \leq C \left( \sum_{k=1}^{n} | x_k |^q \right)^{\frac{1}{q}}
\]
for every choice of elements $x_1, \ldots, x_n$ in $X$.

A Banach lattice $X$ is said to satisfy a lower $q$-estimate if there exists a constant $C \geq 0$ so that, for every choice of elements $x_1, \ldots, x_n$ in $X$, we have
\[
\left( \sum_{k=1}^{n} \| x_k \|_X^q \right)^{\frac{1}{q}} \leq C \left( \sum_{k=1}^{n} | x_k | \right)_X.
\]

Obviously $q$-concavity implies lower $q$-estimate and both notions are the same when $q = 1$. On the other hand, there are Banach lattices that satisfy a lower $q$-estimate but fail to be $q$-concave (see [1, Prop. 3.1], [4, Ex. 1.f.19 and 1.f.20]).

Two related concepts from the theory of Banach spaces are the following:

A Banach space $X$ is said to have cotype $q$, $2 \leq q < \infty$, if there exists a constant $C \geq 0$ so that
\[
\left( \sum_{k=1}^{n} \| x_k \|_X^q \right)^{\frac{1}{q}} \leq C \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) x_k \right\|_X dt
\]
for every choice of elements $x_1, \ldots, x_n$ in $X$, where $r_k$ stands for the Rademacher functions.

$X$ is said to have the $q$-Orlicz property if the identity operator $id : X \to X$ is $(q, 1)$-summing. That is, if there exists a constant $C \geq 0$ such that regardless of the choice of $x_1, \ldots, x_n$ in $X$ we have
\[
\left( \sum_{k=1}^{n} \| x_k \|_X^q \right)^{\frac{1}{q}} \leq C \sup_{\| \epsilon_k \| = 1} \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\|_X.
\]

Let us observe that every Banach space with cotype $q$ has the $q$-Orlicz property, $2 \leq q < \infty$. The converse was an open problem for some time and was solved by Talagrand in [7] and [8]. Indeed, Talagrand in [8] showed that if a Banach space has the $q$-Orlicz property for $2 < q < \infty$ then it also has cotype $q$. Also, he proved in [7] that the situation for
\( q = 2 \) is a bit different. He constructed an example with the 2-Orlicz property but without cotype 2.

There are many connections between all these notions. The reader is referred to [2] or [4] for the following chain of implications.

For \( 2 < q < \infty \) we have that

\[ q\text{-concavity} \Rightarrow \text{cotype } q \iff q\text{-Orlicz property} \iff \text{lower } q\text{-estimate}. \]

The examples mentioned above show that the converse of the first implication fails.

For \( q = 2 \) we have that

\[ 2\text{-concavity} \iff \text{cotype } 2 \Rightarrow 2\text{-Orlicz property} \Rightarrow \text{lower } 2\text{-estimate}. \]

The converse of the two last implications fail. E. M. Semenov and A. M. Shteinberg [6] showed that the Lorentz space \( L_{2,1}((0,1]) \) satisfies a lower 2-estimate but fails to have the 2-Orlicz property. As we said before M. Talagrand in [7] constructed an example with the 2-Orlicz property but without cotype 2. Moreover in [9] he was even able to construct a counterexample in the setting of symmetric sequence spaces.

The aim of this paper is to continue the study of the relationship between all this notions and to give a general method, which is inspired by Talagrand’s techniques in [9], to construct symmetric sequence spaces that satisfy a lower \( q \)-estimate but fail to be \( q \)-concave, \( 1 < q < 2 \), and that have the \( q \)-Orlicz property but fail to be \( q \)-concave for \( 2 \leq q \).

Let us recall that a symmetric sequence space \( (X, \| \cdot \|) \) is a Banach space of sequences such that

1. If \( x \in X \) and \( |y(i)| \leq |x(i)| \) for all \( i \in \mathbb{N} \), then \( y \in X \) and \( \|y\| \leq \|x\| \).
2. If \( x \in X \) and \( \sigma \in \Pi(\mathbb{N}) \), then \( x\sigma \in X \) and \( \|x\sigma\| = \|x\| \).

We shall consider the following method to construct symmetric sequence spaces generated by a family of sequences.

Let \( \mathcal{F} \) be a family sequences in \( \ell_\infty \) with the following properties:

(i) (Solid) If \( f \in \mathcal{F} \) and \( |g(i)| \leq |f(i)| \), for all \( i \in \mathbb{N} \), then \( g \in \mathcal{F} \).
(ii) (Symmetric) If \( f \in \mathcal{F} \) and \( \sigma \in \Pi(\mathbb{N}) \) then \( f\sigma \in \mathcal{F} \).
(iii) **(Bounded)** There exists a constant $C \geq 0$ such that

$$\sup_{f \in F} \|f\|_{\ell_\infty} \leq C.$$ 

In this case, it will be called a generating family.

Given $1 < q < \infty$ we consider $X_q(F)$ the space of sequences such that

$$\|x\|_{X_q(F)} = \sup_{f \in F} \langle |x|, |f|^{\frac{1}{q'}} \rangle < \infty$$

where $\langle x, f \rangle$ means $\sum_{i=1}^{\infty} x(i) f(i)$.

It is easy to see that $X_q(F)$ is a symmetric sequence space and

$$\ell_1 \hookrightarrow X_q(F) \hookrightarrow \ell_\infty$$

with

$$\|x\|_{\ell_\infty} (\sup_{f \in F} \|f\|_{\ell_\infty})^{1/q'} \leq \|x\| \leq \|x\|_{\ell_1} \sup_{f \in F} \|f\|_{\ell_\infty}^{1/q'}.$$ 

Our main theorem can now be stated as follows.

**Theorem 1.1.** Let $1 < q < \infty$. There exists a generating family $F$ such that $X_q(F)$ satisfies a lower $q$-estimate but is not $q$-concave.

As a corollary we have that $X_q(F)$, for $2 < q < \infty$, are examples of spaces of cotype $q$ which are not $q$-concave and $X_2(F)$ satisfies the $2$-Orlicz property but is not of cotype $2$.

2. **Families generated by a function**

In this section we give the main construction for our families.

Let $(k_s)_{s=0}^{\infty}$ be a strictly increasing sequence of natural numbers with $k_0 = k_1 := 1$ and let $(\alpha_s)_{s=0}^{\infty}$ be a sequence in $\mathbb{R}^+$ with $\alpha_0 = \alpha_1$, such that the sequence $(\alpha_s / k_s)_{s=1}^{\infty}$ is decreasing and

$$\lim_{s \to \infty} \frac{\alpha_s}{k_s} = 0.$$  

**Step 1.**

We start with a single function on $\mathbb{N}$

$$h = \sum_{s=2}^{\infty} \frac{\alpha_s}{k_s} \chi_{[k_{s-1}, k_s)}$$
and the set of functions
\[ \mathcal{H} = \{ h\sigma : \sigma \in \Pi(N) \}. \]

By (1) we know that \( h \in c_{o}(N) \) and so \( \mathcal{H} \subseteq c_{o}(N) \). Observe also that \( \mathcal{H} \) is symmetric and bounded by \( \alpha_{2}/k_{2} \).

**Proposition 2.1.** The following properties hold:

1. \( \sum_{i \leq k_{s}} h(i) \leq \sum_{\ell=2}^{s} \alpha_{\ell} \) for \( s \geq 2 \).
2. If \( h' \in \mathcal{H} \) and \( A \subseteq N \) with \( \text{card}(A) \leq k_{s} \), \( s \geq 2 \), then
   \[ \sum_{i \in A} h'(i) \leq \sum_{\ell=2}^{s} \alpha_{\ell}. \]
3. Let \( h' \in \mathcal{H} \) and \( s \geq 0 \). Then, there exists \( A \subseteq N \) such that \( \text{card}(A) = k_{s} \) and \( \| h'\chi_{A'} \|_{\ell_{\infty}} \leq \alpha_{s+1}/k_{s+1} \).
4. Let \( h' \in \mathcal{H} \) and \( s \geq 0 \). Then, there exist \( h'_{1} \) and \( h'_{2} \) functions on \( N \) such that
   \[ h' = h'_{1} + h'_{2} \quad \text{with} \quad \begin{cases} \text{card(supp } h'_{1} \text{)} = k_{s}, \\ \| h'_{2} \|_{\ell_{\infty}} \leq \frac{\alpha_{s+1}}{k_{s+1}}. \end{cases} \]

**Proof.** 1) Let \( s \geq 2 \). Then

\[ \sum_{i \leq k_{s}} h(i) \leq \sum_{\ell=2}^{s} \frac{\alpha_{\ell}}{k_{\ell} k_{\ell-1}} + \frac{\alpha_{s+1}}{k_{s+1}} \leq \sum_{\ell=2}^{s-1} \frac{\alpha_{\ell}}{k_{\ell}} + \frac{\alpha_{s}}{k_{s}} (k_{s} - k_{s-1} + 1) \leq \sum_{\ell=2}^{s} \alpha_{\ell}. \]

3) Suppose that \( h' = h\sigma, \sigma \in \Pi(N) \), and let \( A = \sigma^{-1}([1, k_{s}]) \). If \( i \not\in A \) then \( h'(i) = h(j) \) with \( j > k_{s} \) \((j = \sigma(i))\), hence \( h'(i) = h(j) \leq \alpha_{s+1}/k_{s+1} \).

2) and 4) follows from 1) and 3), respectively. \( \square \)

**Step 2.**

For each \( m \in \mathbb{N} \) we consider the family:

\[ \text{co}_{m}(\mathcal{H}) = \left\{ \sum_{j=1}^{m} \zeta_{j} h_{j} : h_{j} \in \mathcal{H}, \zeta_{j} \in \mathbb{R}^{+}, \sum_{j=1}^{m} \zeta_{j} = 1 \right\}. \]

The family \( \text{co}_{m}(\mathcal{H}) \) is symmetric, bounded by \( \alpha_{2}/k_{2} \).
Let \((m_r)_{r=1}^{\infty}\) be a strictly increasing sequence of natural numbers, \(m_1 \geq 2\). Then, for \(r \in \mathbb{N}\), we define
\[
G_r = \left\{ f : \mathbb{N} \rightarrow \mathbb{R}^+ : f \leq \sum_{\ell=0}^{\infty} 2^{-\ell} f_\ell \text{ with } f_\ell \in c_{m_\ell}(\mathcal{H}) \right\}.
\]
Again \(G_r \subseteq c_0(\mathbb{N})\) and \(\mathcal{H} \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_r \subseteq G_{r+1} \subseteq \ldots\).

**Proposition 2.2.** Let \(r \in \mathbb{N}\), \(f \in G_r\) and \(s \geq 2\). Then,
1. \(\sum_{i \in A} f(i) \leq \sum_{\ell=2}^{s} \alpha_\ell\) for every \(A \subseteq \mathbb{N}\) with \(\text{card}(A) \leq k_s\).
2. There exists \(A \subseteq \mathbb{N}\) such that \(\text{card}(A) = k_s\) and
\[
\|f \chi_A\|_\infty \leq \frac{\sum_{\ell=2}^{s} \alpha_\ell}{k_s}.
\]
3. There exist \(f_1\) and \(f_2\) functions on \(\mathbb{N}\) such that
\[
f = f_1 + f_2 \quad \text{with} \quad \begin{cases} 
\text{card}(\text{supp } f_1) = k_s \\
\|f_2\|_\infty \leq \frac{\sum_{\ell=2}^{s} \alpha_\ell}{k_s}.
\end{cases}
\]

**Proof.** It suffices to show the result for functions in \(c_{m}(\mathcal{H})\) for a fixed \(m \in \mathbb{N}\).

1) is immediate. To prove 2) let \(f \in c_{m}(\mathcal{H}) \subseteq c_0(\mathbb{N})\). Then there exists \(i_1 \in \mathbb{N}\) such that \(f(i_1) \geq f(i)\) for all \(i \in \mathbb{N}\). We consider now \(N_1 = \mathbb{N} \setminus \{i_1\}\). Since \(f \in c_0(N_1)\), then there exists \(i_2 \in N_1\) such that \(f(i_2) \geq f(i)\) for all \(i \in N_1\). Hence we can find \(A = \{i_1, \ldots, i_{k_s}\}\) such that \(f(j) \leq f(i)\) if \(i \in A\) and \(j \not\in A\). Therefore
\[
k_s \sup_{j \not\in A} f(j) \leq \sum_{i \in A} f(i) \leq \sum_{\ell=2}^{s} \alpha_\ell.
\]
3) follows from 2). \(\square\)

The family \(G_r\) is a generating family which is almost convex.

**Lemma 2.3.** Let \(r \in \mathbb{N}\) and let \((f_j)_{j \leq m_r}\) be functions in \(G_r\). Let \(\xi_j \in \mathbb{R}^+, j = 1, \ldots, m_r\), such that \(\sum_{j \leq m_r} \xi_j = 1\). Then
\[
\frac{1}{2} \sum_{j \leq m_r} \xi_j f_j \in G_r.
\]
Proof. Since \( f_j \in G_r \) it holds
\[
f_j \leq \sum_{\ell=0}^{\infty} 2^{-\ell} \sum_{s \leq m_r^\ell} \gamma_{\ell,s,j} h_{\ell,s,j}
\]
with \( h_{\ell,s,j} \in \mathcal{H} \), \( \gamma_{\ell,s,j} \geq 0 \) and \( \sum_{s \leq m_r^\ell} \gamma_{\ell,s,j} = 1 \) for all \( \ell, j \). Hence
\[
\frac{1}{2} \sum_{j \leq m_r} \xi_j f_j \leq \sum_{\ell=0}^{\infty} 2^{-(\ell+1)} \sum_{s \leq m_r^\ell} \xi_j \gamma_{\ell,s,j} h_{\ell,s,j}
\]
and the point is that there are at most \( m_r^{\ell+1} \) terms in the last summation. \( \Box \)

Finally we glue the families \( G_r \) as follows:
\[
G = \left\{ 0 \leq f \leq \sum_{r=1}^{\infty} \gamma_r f_r : f_r \in G_r, \gamma_r \geq 0, \sum_{r=1}^{\infty} \gamma_r = 1 \right\}.
\]
The family \( G \) is again a generating family with the following convexity property.

**Lemma 2.4.** Let \((g_\ell)_{\ell \leq N}\) be a finite collection of functions in \( G \) and let \( \xi_\ell \in \mathbb{R}^+ \), \( \ell = 1, \ldots, N \), such that \( \sum_{\ell \leq N} \xi_\ell = 1 \). Then
\[
\frac{1}{8} \sum_{\ell=1}^{N} \xi_\ell g_\ell \in G.
\]

Proof. Let us write \( g_\ell = \sum_{r=1}^{\infty} \gamma_{\ell,r} f_{\ell,r} \) with \( f_{\ell,r} \in G_r, \gamma_{\ell,r} \in \mathbb{R}^+ \) and \( \sum_{r=1}^{\infty} \gamma_{\ell,r} = 1 \) for all \( \ell \leq N \). We let \( I_N = [1, N] \cap \mathbb{N} \) and for each \( r \geq 1 \) we set
\[
g_r' = \sum_{\ell \in I_N} \xi_\ell \gamma_{\ell,r} f_{\ell,r} \quad \text{and} \quad v_r = \sum_{\ell \in I_N} \xi_\ell \gamma_{\ell,r}.
\]
By Lemma 2.3 we have that \( g_r' \in 2v_r G_r \). On the other hand, if we fix \( r \) and take \( s \leq r \) we can show that
\[
\sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_\ell \gamma_{\ell,s} f_{\ell,s} \in 2w_s G_{r+1}
\]
where \( w_s = \sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_\ell \gamma_{\ell,s} \). Indeed, for all \( s \leq r \), \( f_{\ell,s} \in G_s \) and \( G_s \subseteq G_r \) so that \( f_{\ell,s} \in G_{r+1} \), by Lemma 2.3 we get (2). We take now
\[
g_r'' = \sum_{s \leq r} \sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_\ell \gamma_{\ell,s} f_{\ell,s} \quad \text{and} \quad \delta_r = \sum_{s \leq r} w_s.
\]
Then by Lemma 2.3 we have that $g_r'' \in 4\delta_rG_{r+1}$, since $r \leq m_r$. Now observe that

$$\sum_{r=1}^{\infty} (\nu_r + \delta_r) = \sum_{r=1}^{\infty} \sum_{\ell=1}^{N} \xi_{\ell} \gamma_{\ell,r} = 1$$

because

$$\sum_{r=1}^{\infty} \delta_r = \sum_{r=1}^{\infty} \sum_{s \leq r} \sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_{\ell} \gamma_{\ell,s} = \sum_{r=1}^{\infty} \sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \sum_{s \leq r} \xi_{\ell} \gamma_{\ell,s}$$

$$= \sum_{r=1}^{\infty} \sum_{\ell \in I_N} \xi_{\ell} \gamma_{\ell,r}.$$

Therefore, using Lemma 2.3 one more time we know that the function $g = \sum_{r \geq 1} g'_r + g''_r$ belongs to $8G$. Now we are going to see that $g = \sum_{\ell=1}^{N} \xi_{\ell} g_{\ell},$ so that $\sum_{\ell=1}^{N} \xi_{\ell} g_{\ell} \in 8G$. Indeed,

$$\sum_{\ell=1}^{N} \xi_{\ell} g_{\ell} = \sum_{r=1}^{\infty} \sum_{\ell \in [1, m_r] \cap I_N} \xi_{\ell} \gamma_{\ell,r} f_{\ell,r} = \sum_{r=1}^{\infty} \left( \sum_{\ell \in [1, m_r] \cap I_N} \xi_{\ell} \gamma_{\ell,r} f_{\ell,r} + \sum_{\ell \in (m_r, N] \cap I_N} \xi_{\ell} \gamma_{\ell,r} f_{\ell,r} \right)$$

$$= \sum_{r=1}^{\infty} \left( g'_r + \sum_{\ell \in (m_r, N] \cap I_N} \xi_{\ell} \gamma_{\ell,r} f_{\ell,r} \right).$$

But

$$\sum_{r=1}^{\infty} \sum_{\ell \in (m_r, N] \cap I_N} \xi_{\ell} \gamma_{\ell,r} f_{\ell,r} = \sum_{r=1}^{\infty} \sum_{\ell \in [1, m_{r+1}] \cap I_N} \sum_{s \leq r} \sum_{\ell \in I_N} \xi_{\ell} \gamma_{\ell,s} f_{\ell,s} = \sum_{r=1}^{\infty} g''_r.$$

Therefore,

$$\sum_{\ell=1}^{N} \xi_{\ell} g_{\ell} = \sum_{r=1}^{\infty} \sum_{\ell \in [1, m_r] \cap I_N} \xi_{\ell} \gamma_{\ell,r} f_{\ell,r} = \sum_{r=1}^{\infty} g'_r + g''_r.$$

Our first result about convexity of these spaces is the following.

**Theorem 2.5.** Let $1 < q < \infty$. Then, the space $X_q(G)$ is $q$-concave.

**Proof.** Let $x_1, \ldots, x_N$ be a finite number of elements in $X_q(G)$. We set $S_1^q = \sum_{\ell=1}^{N} \|x_{\ell}\|^q$ and $\xi_{\ell} = \|x_{\ell}\|^q / S_1^q$. Then $\sum_{\ell=1}^{N} \xi_{\ell} = 1$.

For each $\ell$ take $f_{\ell} \in G$ such that $\|x_{\ell}\| \leq 4/3 (\|x_{\ell}\|, \sqrt{\|f_{\ell}\|}).$
Hence,
\[ S^q \leq \frac{4}{3} \sum_{\ell=1}^{N} \|x_\ell\|^{(q-1)} \langle |x_\ell|, \sqrt[3]{|f_\ell|} \rangle = \frac{4}{3} \sum_{\ell=1}^{N} S^{q/q'} \langle \xi_\ell \langle |x_\ell|, \sqrt[3]{|f_\ell|} \rangle \rangle \]
\[ = \frac{4}{3} S^{q-1} \sum_{\ell=1}^{N} \sum_{i=1}^{\infty} |x_\ell(i)| \sqrt[3]{|\xi_\ell f_\ell(i)|}. \]

Using Hölder’s inequality and Lemma 2.4 we have that \( \sum_{\ell \leq N} |\xi_\ell f_\ell| \in 8G \). Now
\[ S^q \leq \frac{4}{3} S^{q-1} \sum_{i=1}^{\infty} \left( \sum_{\ell=1}^{N} |x_\ell(i)|^q \right)^{\frac{1}{q}} \left( \sum_{\ell=1}^{N} |\xi_\ell f_\ell(i)| \right)^{\frac{1}{q}} \leq \frac{1}{6} S^{q-1} \left\| \left( \sum_{\ell=1}^{N} |x_\ell|^q \right)^{\frac{1}{q}} \right\|. \]

This implies
\[ \left( \sum_{\ell=1}^{N} \|x_\ell\|^q \right)^{\frac{1}{q}} \leq \frac{1}{6} \left\| \left( \sum_{\ell=1}^{N} |x_\ell|^q \right)^{\frac{1}{q}} \right\| \]
and the proof is complete. \( \square \)

**Step 3.** For each \( r \geq 1 \) we write
\[ \mathcal{F}_r = \left\{ f \in \mathcal{G}_r : \|f\|_{C_\infty} \leq \frac{\alpha_{r-1}}{k_{r-1}} \right\}. \]

Again \( \mathcal{F}_r \subseteq c_0(\mathbb{N}) \) and \( \mathcal{F}_r \) are generating families with \( \mathcal{F}_1 \subseteq \mathcal{F}_2 \) but now, for \( r \geq 2 \), \( \mathcal{F}_r \not\subseteq \mathcal{F}_{r+1} \).

Finally we define the generating family
\[ \mathfrak{F} = \left\{ 0 \leq f \leq \sum_{r=1}^{\infty} \gamma_r f_r : f_r \in \mathcal{F}_r, \gamma_r \geq 0, \sum_{r=1}^{\infty} \gamma_r = 1 \right\}. \]

We have to observe that the family \( \mathfrak{F} \) depends on the sequences \((k_s)_{s=0}^{\infty}, (\alpha_s)_{s=0}^{\infty} \) and \((m_r)_{r=1}^{\infty} \).

3. **q-Orlicz property and lower q-estimate**

In this section we prove under suitable conditions on \( \mathfrak{F} \) that the space \( X_q(\mathfrak{F}) \) satisfies a lower q-estimate for \( 1 < q < \infty \) and has the q-Orlicz property for \( 2 \leq q < \infty \) (the reader should notice that this is stronger only for \( q = 2 \)).

We begin with some lemmas to be used in the sequel. The first one follows from Lemma 2.3.
Lemma 3.1. Let \( r \in \mathbb{N} \), let \((f_j)_{j \leq m_r}\) functions in \( \mathcal{F}_r \) and let \( \xi_j \in \mathbb{R}^+ \), \( j = 1, \ldots, m_r \), such that \( \sum_{j \leq m_r} \xi_j = 1 \). Then
\[
\frac{1}{2} \sum_{j \leq m_r} \xi_j f_j \in \mathcal{F}_r.
\]

From here on we will assume another property on the sequence \((\alpha_s)_{s=0}^\infty\).

\((\ast)\) There exists a constant \( C \geq 1 \) such that \( \sum_{\ell=2}^{s} \alpha_{\ell} \leq C \alpha_{s} \) for all \( s \geq 2 \).

Lemma 3.2. Let \( s, r \in \mathbb{N} \) with \( s \leq r \), let \((f_j)_{j \leq m_r+1}\) be a collection of functions in \( \mathcal{F}_s \) and let \( \xi_j \in \mathbb{R}^+ \), \( j = 1, \ldots, m_r+1 \), such that \( \sum_{j \leq m_r+1} \xi_j = 1 \). If the sequence \((\alpha_s)_{s=0}^\infty\) satisfies \((\ast)\), then there exists \( A_{s,r} \subseteq \mathbb{N} \) with \( \text{card}(A_{s,r}) = k_r \) such that
\[
\chi_{A_{s,r}} \frac{1}{2C} \sum_{j \leq m_r+1} \xi_j f_j \in \mathcal{F}_{r+1}.
\]

Proof. If \( r = s = 1 \) we only have to notice that \( \mathcal{F}_1 \subseteq \mathcal{F}_2 \). Assume that \( r \geq 2 \). We define \( g = \frac{1}{2} \sum_{j \leq m_r+1} \xi_j f_j \). If we show that \( g \in \mathcal{G}_{r+1} \) and that \( \frac{1}{C} g \chi_{A_{s,r}} \ell_\infty \leq \alpha_c / k_r \) for a set \( A_{s,r} \) of integers then the proof will be finished.

By hypothesis \( f_j \in \mathcal{G}_s \subseteq \mathcal{G}_r \subseteq \mathcal{G}_{r+1} \) for all \( j \leq m_{r+1} \), so by Lemma 2.3 \( g \in \mathcal{G}_{r+1} \). On the other hand, by (2) in Proposition 2.2 and \((\ast)\) we can find \( A_{s,r} \subseteq \mathbb{N} \) with \( \text{card}(A_{s,r}) = k_r \) such that
\[
\left\| \frac{1}{C} g \chi_{A_{s,r}} \right\|_{\ell_\infty} \leq \sum_{\ell=2}^{r} \alpha_{\ell} / \frac{C k_r}{k_r} \leq \frac{\alpha_c}{k_r}.
\]

Our next result shows a convexity property of the family \( \mathcal{F} \).

Theorem 3.3. Let \((g_\ell)_{\ell \leq N} \) a finite collection of functions in \( \mathcal{F} \) given by
\[
g_\ell \leq \sum_{r=1}^{\infty} \gamma_{\ell,r} f_{\ell,r},
\]
where \( f_{\ell,r} \in \mathcal{F}_r \), \( \gamma_{\ell,r} \in \mathbb{R}^+ \) and \( \sum_{r=1}^{\infty} \gamma_{\ell,r} = 1 \) for all \( \ell \leq N \). Let \( \xi_\ell \in \mathbb{R}^+ \) such that \( \sum_{\ell \leq N} \xi_\ell = 1 \) and assume that the sequence \((\alpha_s)_{s=0}^\infty\) satisfies
Then, there exists $B_r \subseteq \mathbb{N}$ with $\text{card}(B_r) \leq rk_r$, $r \geq 1$, such that the functions defined by

$$f'_r = \chi_{B_r^{(\ell)}} \sum_{r=1}^{r(\ell)} \gamma_{t,r} f_{t,r} + \sum_{r=r(\ell)+1}^{\infty} \gamma_{t,r} f_{t,r},$$

satisfy

$$\frac{1}{8C} \sum_{\ell=1}^{N} \xi_{\ell} f'_r \in \mathcal{G},$$

where $r(\ell)$ is chosen so that $m_r(\ell) < \ell \leq m_r(\ell)+1$.

**Proof.** Write $I_N = [1, N] \cap \mathbb{N}$ and set

$$g'_r = \sum_{\ell \in [1,m_r] \cap I_N} \xi_{\ell} f_{t,r} \quad \text{and} \quad \nu_r = \sum_{\ell \in [1,m_r] \cap I_N} \xi_{\ell} \gamma_{t,r}.$$

Then by Lemma 3.1 we have that $g'_r \in 2\nu_r \mathcal{F}_r$.

Fix $r \in \mathbb{N}$ and let $s \leq r$. We consider the functions $(f_{t,s})_{t \in (m_r, m_r+1] \cap I_N} \subseteq \mathcal{F}_s$. Then, by Lemma 3.2 we know that there exists $A_{s,r} \subseteq \mathbb{N}$ with $\text{card}(A_{s,r}) = k_r$ such that

$$\chi_{A_{s,r}} \sum_{\ell \in (m_r, m_r+1] \cap I_N} \xi_{\ell} \gamma_{t,s} f_{t,s} \in 2C w_s \mathcal{F}_{r+1},$$

where $w_s = \sum_{t \in (m_r, m_r+1] \cap I_N} \xi_{\ell} \gamma_{t,s}$. Set $B_r = \bigcup_{s=1}^{r} A_{s,r}$, and note that $\text{card}(B_r) \leq rk_r$. Since $r \leq m_r$, Lemma 3.1 gives that the function

$$g''_r = \chi_{B_r^{(\ell)}} \sum_{s \leq r} \sum_{\ell \in (m_r, m_r+1] \cap I_N} \xi_{\ell} \gamma_{t,s} f_{t,s} \leq \sum_{s \leq r} \chi_{A_{s,r}} \sum_{\ell \in (m_r, m_r+1] \cap I_N} \xi_{\ell} \gamma_{t,s} f_{t,s},$$

belongs to $4C \delta_r \mathcal{F}_{r+1}$, where $\delta_r = \sum_{s \leq r} w_s$. Therefore, applying Lemma 3.1 again we see that the function

$$g = \sum_{r=1}^{\infty} g'_r + g''_r$$

belongs to $8C \mathcal{F}$. Observe also that $\sum_{r=1}^{\infty} \nu_r + \delta_r = 1$.

Now we are going to define functions $f'_r$ such that $\sum_{r \leq N} \xi_{\ell} f'_r = g$. Let us fix $\ell \in \{m_1, \ldots, N\}$, then there exist a unique $r$ such that $m_r < \ell \leq m_{r+1}$. We denote by $r(\ell)$ this unique $r$ and define the function

$$f'_r = \chi_{B_r^{(\ell)}} \sum_{r=1}^{r(\ell)} \gamma_{t,r} f_{t,r} + \sum_{r=r(\ell)+1}^{\infty} \gamma_{t,r} f_{t,r}.$$
For $\ell \in \{1, \ldots, m\}$, we define (corresponding to $r(\ell) = 0$) the function $f'_\ell = \sum_{r=1}^{\infty} \gamma_{\ell,r} f_{\ell,r}$. Thus $f'_\ell$ can also be expressed as

$$f'_\ell = \sum_{r=1}^{\infty} \gamma_{\ell,r} f_{\ell,r} h_{\ell,r}$$

where $h_{\ell,r} = 1$ if $\ell \leq m_r$ and $h_{\ell,r} = \chi_{B^r_{\ell}(t)}$ if $m_r < \ell$. The same proof as in Lemma 2.4 gives that $\sum_{\ell=1}^{N} \xi_{\ell} f'_\ell = g \in SC\mathbb{F}$.

We need also some general lemmas.

**Lemma 3.4.** Let $\mathcal{F}$ be a generating family and let $1 < q < \infty$. Assume that $(x_\ell)_{\ell \leq N}$ is a finite collection of elements in $X_q(\mathcal{F})$ and $B \subseteq \mathbb{N}$. Then,

$$\sum_{\ell=1}^{N} \|x_\ell \chi_B\| \leq \text{card}(B) \sup_{|\epsilon_\ell| = 1} \left\|\sum_{\ell=1}^{N} \epsilon_\ell x_\ell\right\|.$$

**Proof.** Set $c = \sup_{f \in \mathcal{F}} \|f\|_{\ell_\infty}$. Since $c^{1/q'} \|x\|_{\ell_\infty} \leq \|x\| \leq c^{1/q} \|x\|_{\ell_1}$, we have

$$\sum_{\ell=1}^{N} \|x_\ell \chi_B\| \leq \sum_{\ell=1}^{N} \sum_{i \in B} |x_\ell(i)| c^{1/q'} = \sum_{i \in B} \sum_{\ell=1}^{N} |x_\ell(i)| c^{1/q'}$$

$$\leq \text{card}(B) \sup_{|\epsilon_\ell| = 1} \left\|\sum_{\ell=1}^{N} \epsilon_\ell x_\ell\right\|_{\ell_\infty} c^{1/q'} \leq \text{card}(B) \sup_{|\epsilon_\ell| = 1} \left\|\sum_{\ell=1}^{N} \epsilon_\ell x_\ell\right\|$$

which yields the result.

**Lemma 3.5.** Let $\mathcal{F}$ be a generating family, $\xi_\ell \in \mathbb{R}^+$, $\ell = 1, \ldots, N$, and let $(f_\ell)_{\ell \leq N}$ be a finite collection of functions in $\mathcal{F}$ such that $\sum_{\ell \leq N} \xi_{\ell} f_\ell \in \mathcal{F}$.

a) If $1 < q < \infty$, then

$$\sum_{\ell=1}^{N} \langle |x_\ell|, \sqrt{\xi_{\ell} f_\ell} \rangle \leq \left\|\sum_{\ell=1}^{N} |x_\ell|\right\|.$$

b) If $2 \leq q < \infty$, then

$$\sum_{\ell=1}^{N} \langle |x_\ell|, \sqrt{\xi_{\ell} f_\ell} \rangle \leq \sqrt{2} \sup_{|\epsilon_\ell| = 1} \left\|\sum_{\ell=1}^{N} \epsilon_\ell x_\ell\right\|.$$
Proof. Since \( \sum_{\ell \leq N} \xi_{\ell} f_{\ell} \in \mathcal{F} \), by Hölder’s inequality we get

\[
\sum_{\ell=1}^{N} \langle |x_{\ell}|, \sqrt[q]{\xi_{\ell} f_{\ell}} \rangle \leq \left( \sum_{\ell=1}^{N} |x_{\ell}|^q \right)^{\frac{1}{q}}, \sqrt[q]{\left( \sum_{\ell=1}^{N} |\xi_{\ell} f_{\ell}| \right)^{\frac{1}{q}}} \leq \left( \sum_{\ell=1}^{N} |x_{\ell}|^q \right)^{\frac{1}{q}}.
\]

If \( 1 < q < \infty \) then

\[
\left\| \left( \sum_{\ell=1}^{N} |x_{\ell}|^q \right)^{\frac{1}{q}} \right\| \leq \left\| \sum_{\ell=1}^{N} |x_{\ell}| \right\|.
\]

Hence a) is true. If \( q \geq 2 \) by Kintchine’s inequality (see [2, 1.10]) we have that there exists a constant \( B_1 = \sqrt{2} \) such that for all \( i \in \mathbb{N} \)

\[
\left( \sum_{\ell=1}^{N} |x_{\ell}(i)|^q \right)^{\frac{1}{q}} \leq \left( \sum_{\ell=1}^{N} |x_{\ell}(i)|^2 \right)^{\frac{1}{2}} \leq B_1 \int_{0}^{1} \left| \sum_{\ell=1}^{N} r_{\ell}(t)x_{\ell}(i) \right| dt.
\]

Therefore,

\[
\sum_{\ell=1}^{N} \langle |x_{\ell}|, \sqrt[q]{\xi_{\ell} f_{\ell}} \rangle \leq \sqrt{2} \int_{0}^{1} \left\| \sum_{\ell=1}^{N} r_{\ell}(t)x_{\ell}(i) \right\| dt \leq \sqrt{2} \sup_{t \in [0,1]} \left\| \sum_{\ell=1}^{N} r_{\ell}(t)x_{\ell} \right\|.
\]

From this we get b) and the proof is complete. \( \square \)

Lemma 3.6. Let \( \mathcal{F} \) be a generating family and let \( 1 < q < \infty \). Suppose that \( (\eta_r)_{r=1}^{\infty} \) is an increasing sequence of real numbers and that \{\( x_1, \ldots, x_N \)\} is a finite collection of elements in \( X_q(\mathcal{F}) \) such that the sequence \( (\|x_{\ell}\|)_{\ell \leq N} \) is decreasing. Let \( (C_r)_{r \geq 1} \) be subsets of \( \mathbb{N} \). Consider, for \( r \geq 1 \), the subsets of \( \mathbb{N} \)

\[
H_r = \{ \ell : 1 \leq \ell \leq N, \ m_r < \ell \leq m_{r+1} \text{ and } \|x_{\ell}\| \leq \eta_r \|x_{\ell}C_r\| \}
\]

and let \( H = \bigcup_{r \geq 1} H_r \). Then,

\[
\sum_{\ell \in H} \|x_{\ell}\|^q \leq \left( \sum_{\ell=1}^{N} \|x_{\ell}\|^q \right)^{\frac{1}{q}} \sup_{|C_r|=1} \left\| \sum_{\ell=1}^{N} \epsilon_{\ell} x_{\ell} \right\| \left( \sum_{r=1}^{\infty} \frac{\eta_r \text{card}(C_r)}{\sqrt{m_r}} \right).
\]

Proof. We assume that \( \sup_{|C_r|=1} \left\| \sum_{\ell=1}^{N} \epsilon_{\ell} x_{\ell} \right\| = 1 \). By Lemma 3.4 and the definition of \( H_r \) we know that

\[
\sum_{\ell \in H_r} \|x_{\ell}\| \leq \eta_r \sum_{\ell \in H_r} \|x_{\ell}C_r\| \leq \eta_r \text{card}(C_r).
\]

Thus

\[
\sum_{\ell \in H_r} \|x_{\ell}\|^q \leq \left( \max_{\ell \in H_r} \|x_{\ell}\|^{q-1} \right) \left( \sum_{\ell \in H_r} \|x_{\ell}\| \right) \leq \left( \max_{\ell \in H_r} \|x_{\ell}\|^{q-1} \right) \eta_r \text{card}(C_r).
\]
On the other hand, since \( (\|x_\ell\|)_{\ell \leq N} \) is decreasing we get
\[
\|x_\ell\|^q \leq \frac{\sum_{\ell=1}^N \|x_\ell\|^q}{m_r}
\]
if \( \ell \in H_r \) and so \( \|x_\ell\|^{q-1} \leq \left( \frac{\sum_{\ell=1}^N \|x_\ell\|^q}{\sqrt{m_r}} \right)^{\frac{1}{q}} \). Whence we conclude that
\[
\sum_{\ell \in H} \|x_\ell\|^q \leq \left( \sum_{\ell=1}^N \|x_\ell\|^q \right)^{\frac{1}{q}} \left( \sum_{r=1}^\infty \frac{\eta_r \text{card}(C_r)}{\sqrt{m_r}} \right).
\]

We are now ready to study the \( q \)-Orlicz property and a lower \( q \)-estimate of the space \( X_q(\mathcal{F}) \).

**Theorem 3.7.** Let \( (\eta_r)_{r=1}^\infty \) be an increasing sequence of real numbers with \( \eta_r \geq 2 \). Assume that the sequence \( (\alpha_s)_{s=0}^\infty \) satisfies (\( \ast \)) and that the sequences \( (\eta_r)_{r=1}^\infty \), \( (k_r)_{r=1}^\infty \) and \( (m_r)_{r=1}^\infty \) satisfy
\[
\sum_{r=1}^\infty r\eta_r k_r \frac{q}{\sqrt{m_r}} < \infty.
\]

Then if \( 1 < q < \infty \) the space \( X_q(\mathcal{F}) \) satisfies a lower \( q \)-estimate. Furthermore if \( 2 \leq q < \infty \) the space \( X_q(\mathcal{F}) \) has the \( q \)-Orlicz property.

**Proof.** Let \( N \in \mathbb{N} \) and let \( (x_\ell)_{\ell \leq N} \) a collection of elements in \( X_q(\mathcal{F}) \). We assume that the sequence \( (\|x_\ell\|)_{\ell \leq N} \) is decreasing. We set \( S^q = \sum_{\ell=1}^N \|x_\ell\|^q \) and \( \xi_\ell = \frac{\|x_\ell\|^q}{S^q} \). Hence \( \sum_{\ell=1}^N \xi_\ell = 1 \).

By definition of the norm in \( X_q(\mathcal{F}) \), for each \( \ell \) there exists a function \( g_\ell \in \mathcal{F} \) such that
\[
\|x_\ell\| \leq \frac{4}{3} \langle |x_\ell|, g_\ell^{1/q}\rangle.
\]
If we apply Theorem 3.3 to the functions \( g_\ell \) and the numbers \( \xi_\ell = \frac{\|x_\ell\|^q}{S^q} \), then we can find functions \( f'_\ell \) so that \( \sum_{\ell=1}^N \xi_\ell f'_\ell \in 8C\mathcal{F} \) and subsets \( B_r \subseteq \mathcal{F} \) with \( \text{card}(B_r) \leq rk_r \).

In order to estimate \( S^q \) we split it as
\[
S^q = \sum_{\ell=1}^N \|x_\ell\|^q = \sum_{\ell=1}^{m_1} \|x_\ell\|^q + \sum_{\ell \in H} \|x_\ell\|^q + \sum_{\ell \in H \cup \{1, \ldots, m_1\}} \|x_\ell\|^q
\]
where $H = \bigcup_{r \geq 1} H_r$ and

$$H_r = \{ \ell : 1 \leq \ell \leq N, \ m_r < \ell \leq m_{r+1} \ \text{and} \ \| x_\ell \| \leq \eta_r \| x_\ell \chi_{B_r} \| \}.$$

If $\ell \in H$ then by Lemma 3.6 we have

$$\sum_{\ell \in H} \| x_\ell \|^q \leq S^{q/\varrho} \sup_{|r| = 1} \left( \sum_{\ell = 1}^N \epsilon_\ell x_\ell \right) \left( \sum_{r = 1}^\infty \eta_r k_r / \sqrt{m_r} \right) \leq T S^{q-1} \sup_{|r| = 1} \left( \sum_{\ell = 1}^N \epsilon_\ell x_\ell \right),$$

where $T := \sum_{r = 1}^\infty r m_r k_r / \sqrt{m_r}$. On the other hand, if $\ell \in \{1, \ldots, m_1\}$ then $g_\ell \leq f^*_\ell$ and hence

$$\sum_{\ell = 1}^{m_1} \| x_\ell \|^q \leq \frac{4}{3} \sum_{\ell = 1}^{m_1} \| x_\ell \|^q \langle |x_\ell|, \sqrt{g_\ell} \rangle \leq \frac{4}{3} \sum_{\ell = 1}^N \| x_\ell \|^q \langle |x_\ell|, \sqrt{f^*_\ell} \rangle.$$

Finally if we assume that $\ell \notin H \cup \{1, \ldots, m_1\}$, then there exists a number $r(\ell) \geq 1$ such that $m_{r(\ell)} < \ell \leq m_{r(\ell)+1}$ and by the definition of $H_r$ we have for $\eta_r \geq 2$

$$\| x_\ell \chi_{B_{r(\ell)}} \| \leq \| x_\ell \| \leq \frac{\| x_\ell \|}{2}.$$
Thus
\[
\sum_{\ell=1}^{m_1} \|x_\ell\|^q + \sum_{\ell \not\in H \cup \{1, \ldots, m_1\}} \|x_\ell\|^q \leq \left(\frac{4}{3} + 4\right) \sum_{\ell=1}^{N} \|x_\ell\|^q \langle |x_\ell|, \sqrt[3]{f_\ell} \rangle \\
= \frac{16}{3} \sum_{\ell=1}^{N} S_{q-1}^{q-1} \sqrt[3]{\xi_\ell \langle |x_\ell|, \sqrt[3]{f_\ell} \rangle} \\
= \frac{16}{3} S_{q-1}^{q-1} \sum_{\ell=1}^{N} \langle |x_\ell|, \sqrt[3]{\xi_\ell f_\ell} \rangle.
\]

Assume that $1 < q < \infty$. Then, by (a) in Lemma 3.5, we get
\[
S^q \leq \frac{16 \sqrt{8C}}{3} S_{q-1}^{q-1} \left(\sum_{\ell=1}^{N} |x_\ell|\right) + TS_{q-1}^{q-1} \sup_{|x_\ell| = 1} \left(\sum_{\ell=1}^{N} \epsilon_\ell x_\ell\right).
\]

Therefore
\[
\left(\sum_{\ell=1}^{N} |x_\ell|^q\right)^{\frac{1}{q}} \leq \left(\frac{16 \sqrt{8C}}{3} + T\right) \left(\sum_{\ell=1}^{N} |x_\ell|\right)
\]
and the space $X_q(\mathcal{F})$ satisfies a lower $q$-estimate.

If $2 \leq q < \infty$ by (b) in Lemma 3.5 we have
\[
S^q \leq \left(\frac{16 \sqrt{8C}}{3} \sqrt{2} + T\right) S_{q-1}^{q-1} \sup_{|x_\ell| = 1} \left(\sum_{\ell=1}^{N} \epsilon_\ell x_\ell\right)
\]
and hence the space $X_q(\mathfrak{F})$ has the $q$-Orlicz property. \hfill \Box

4. $q$-Concavity

In this section we show that the space $X_q(\mathfrak{F})$ is not $q$-concave if the family $\mathfrak{F}$ satisfies some further conditions. In order to do this we need to introduce another increasing sequence of natural numbers $(n_s)_{s=1}^{\infty}$ with $n_1 = 1$.

Again we need some lemmas.

**Lemma 4.1.** Let $s, r \in \mathbb{N}$ with $r \leq s$. Let $(n_s)_{s=1}^{\infty}$ be an increasing sequence of natural numbers, $n_1 = 1$, such that $n_s \leq k_{s+1}$ for every $s \geq 1$, and assume that the sequence $(\alpha_s)_{s=0}^{\infty}$ satisfies $(\ast)$. Then for every function $f \in \mathcal{F}_r$ there exists a pair of functions $f_1$ and $f_2$ such
that $f = f_1 + f_2$ with

$$\text{card}(\text{supp} f_1) \leq 2m_* k_s \quad \text{and} \quad \sum_{i=1}^{n_s} f_2(i) \leq \alpha_{s+1} \left( \frac{n_s}{k_{s+1}} + \frac{C}{2^s} \right).$$

Proof. Since $f \in \mathcal{G}_r$ we can assume that

$$f = \sum_{\ell=0}^{\infty} 2^{-\ell} \sum_{j \leq m^\ell_r} \zeta_{j,\ell} h_{j,\ell},$$

where $h_{j,\ell} \in \mathcal{H}$, $\zeta_{j,\ell} \in \mathbb{R}^+$ and $\sum_{j \leq m^\ell_r} \zeta_{j,\ell} = 1$ for all $\ell$. We know that for each $h_{j,\ell} \in \mathcal{H}$ we can find $h'_{j,\ell}$ and $h''_{j,\ell}$ such that $h_{j,\ell} = h'_{j,\ell} + h''_{j,\ell}$, with $\text{card}(\text{supp} h'_{j,\ell}) = k_s$ and $\|h''_{j,\ell}\|_\infty \leq \frac{\alpha_{s+1}}{k_{s+1}}$. Therefore we can decompose $f$ as $f = f_1 + f_2$ where

$$f_1 = \sum_{\ell=0}^{s} 2^{-\ell} \sum_{j \leq m^\ell_r} \zeta_{j,\ell} h'_{j,\ell}$$

and

$$f_2 = \sum_{\ell=0}^{s} 2^{-\ell} \sum_{j \leq m^\ell_r} \zeta_{j,\ell} h''_{j,\ell} + \sum_{\ell=s+1}^{\infty} 2^{-\ell} \sum_{j \leq m^\ell_r} \zeta_{j,\ell} h_{j,\ell}.$$

Now, the support of $f_1$ has at most $2k_s m_*^s$ points. Indeed, since $m_1 \geq 2$ and $(m_s)_{s=1}^\infty$ is a strictly increasing sequence we have that

$$\sum_{\ell=0}^{s} m^\ell_r \leq \left( m^s_r \sum_{\ell=0}^{\infty} \left( \frac{1}{m^s_r} \right)^\ell \right) = m^s_r \frac{1}{1 - \frac{1}{m^s_r}} \leq \frac{2m^s_r}{1 - \frac{1}{2}} = 2m^s_*.$$

Therefore

$$\text{card}(\text{supp} f_1) \leq k_s \sum_{\ell=0}^{s} m^\ell_r \leq 2k_s m_*^s.$$

On the other hand, by $\sum_{i=1}^{n_s} h''_{j,\ell}(i) \leq n_s \frac{\alpha_{s+1}}{k_{s+1}}$, $n_s \leq k_{s+1}$ and (1) in Proposition 2.1,

$$\sum_{i=1}^{n_s} f_2(i) \leq n_s \frac{\alpha_{s+1}}{k_{s+1}} \sum_{\ell=0}^{s} 2^{-\ell} + \sum_{\ell=s+1}^{\infty} 2^{-\ell} \left( \sum_{j=2}^{s+1} \alpha_j \right).$$

Finally, by $(\ast)$ we get

$$\sum_{i=1}^{n_s} f_2(i) \leq \alpha_{s+1} \frac{n_s}{k_{s+1}} + C\alpha_{s+1} 2^{-s}$$

and conclude the proof of the lemma. \qed

As a consequence we have:
Lemma 4.2. Let \( s, r \in \mathbb{N} \) with \( r \leq s \), and let \((n_s)_{s=1}^{\infty}\) be an increasing sequence of natural numbers with \( n_1 = 1 \), such that \( n_s \leq k_{s+1} \) for every \( s \geq 1 \). Finally assume that the sequence \((\alpha_s)_{s=1}^{\infty}\) satisfies (\(\ast\)). If \((f_r)_{r=1}^{s}\) are functions in \( \mathcal{F}_r \) and \( \gamma_r \in \mathbb{R}^+ \) so that \( \sum_{r \geq 1} \gamma_r = 1 \), then there exist \( f' \) and \( f'' \) functions of \( \mathfrak{F} \) so that

\[
\sum_{r=1}^{s} \gamma_r f_r = f' + f''
\]

with

\[
\text{card}(\text{supp}f') \leq 2k_s \left( \sum_{r=1}^{s} m_r^s \right) \quad \text{and} \quad \sum_{i=1}^{n_s} f''(i) \leq \alpha_{s+1} \left( \frac{n_s}{k_{s+1}} + \frac{C}{2^s} \right).
\]

The new assumption on the sequence \((\alpha_s)_{s=0}^{\infty}\) that will be needed is the following:

(\(\ast\ast\)) There exists a constant \( K \geq 0 \) such that \( \frac{\alpha_{s+1}}{\alpha_s} \leq K \) for all \( s \geq 2 \).

Proposition 4.3. Let \((n_s)_{s=1}^{\infty}\) be a 2-lacunary sequence of natural numbers, i.e. \( 2n_s \leq n_{s+1}, \, n_1 = 1 \), such that \( k_s \leq n_s \leq k_{s+1} \) and assume that the sequence \((\alpha_s)_{s=0}^{\infty}\) satisfies (\(\ast\ast\)). Let \( \tau > 0 \) be a fixed integer, \( 1 < q < \infty \) and let \( x \) and \( y \) be the vectors belonging to \( X_q(\mathfrak{F}) \) defined by

\[
x = \sum_{s=2}^{\tau} \frac{1}{\sqrt{\alpha_s} \sqrt{k_s}} \chi[k_{s-1}, k_s) \quad \text{and} \quad y = \sum_{s=2}^{\tau} \frac{1}{\sqrt{\alpha_{s+1}} \sqrt{n_s}} \chi[n_{s-1}, n_s).
\]

Then, there exists a finite number of permutations of the set \( \mathbb{N} \), \( \{\sigma_1, \ldots, \sigma_N\} \), such that if we set \( x_j = x \sigma_j \) then

\[
\frac{1}{N} \sum_{j=1}^{N} x'^q(j)(i) \leq 2(2K^{q-1} + 1)y^q(i), \quad \text{for all } i \in \mathbb{N}.
\]

Proof. Let \( N = n_r - n_{r-1} \) and let \( \sigma \in \Pi(\mathbb{N}) \) be defined as

\[
\begin{cases}
\sigma(n_s - 1) = n_{s-1}, & s \geq 2, \\
\sigma(i) = i + 1, & \text{otherwise}.
\end{cases}
\]
We take \( x_j = x\sigma^j, j = 1, \ldots, N \). Then for \( i \in [n_{s-1}, n_s), s \geq 2 \), we have

\[
\frac{1}{N} \sum_{j=1}^{N} x_j^q(i) \leq \frac{1}{N} \left( \sum_{n_{s-1} \leq j < n_s} x^q(j) \right) \left( E \left[ \frac{N}{n_s - n_{s-1}} \right] + 1 \right)
\]

\[
\leq \frac{2}{n_s - n_{s-1}} \left( \sum_{n_{s-1} \leq j < k_s} x^q(j) + \sum_{k_s \leq j < n_s} x^q(j) \right)
\]

\[
= 2 \frac{1}{\alpha_{s-1}^q} (k_s - n_{s-1}) + \frac{1}{\alpha_{s+1}^q} (n_s - k_s)
\]

Let \( s \geq 2 \) and \( i \in [n_{s-1}, n_s) \). Since \( k_s \leq n_s \leq k_{s+1}, n_s \geq 1, n_s - n_{s-1} \geq \frac{1}{2} n_s \) and \( (\alpha_s)_{s=0}^\infty \) satisfies \((*)\) we conclude that

\[
\frac{1}{N} \sum_{j=1}^{N} x_j^q(i) \leq 2 \left( \frac{k_s}{\alpha_{s-1}^q} \frac{1}{k_s (n_s - n_{s-1})} + \frac{1}{\alpha_{s+1}^q} \frac{1}{1 (n_s - n_{s-1})} \right)
\]

\[
\leq 2 \left( \frac{K^{q-1}}{\alpha_{s+1}^q (n_s - n_{s-1})} + \frac{1}{\alpha_{s+1}^q} \right)
\]

\[
\leq 2 \left( \frac{2K^{q-1}}{\alpha_{s+1}^q n_s} + \frac{1}{\alpha_{s+1}^q} \right) = 2(2K^{q-1} + 1) y^q(i).
\]

\[\square\]

The main theorem of this section is the following:

**Theorem 4.4.** Let \( 1 < q < \infty \) and let \( (n_s)_{s=1}^\infty \) be a sequence of natural numbers with \( n_1 = 1 \). Assume that the sequence \( (\alpha_s)_{s=0}^\infty \) satisfies \((*)\) and \((**)*\), and that the sequences \( (n_s)_{s=1}^\infty \) and \( (k_s)_{s=1}^\infty \) are \( 2 \)-lacunary and satisfy that \( k_s \leq n_s \leq k_{s+1} \) for all \( s \geq 1 \). Assume further that the sequences \( (k_s)_{s=1}^\infty, (n_s)_{s=1}^\infty \) and \( (m_r)_{r=1}^\infty \) satisfy

\[
\sum_{s=1}^{\infty} \sqrt[n_s]{k_{s+1}} < \infty \quad \text{and} \quad \sum_{s=1}^{\infty} \sqrt[n_s]{k_s (\sum_{r=1}^{s} m_r)} < \infty.
\]

Then, the space \( X_q(\mathcal{F}) \) fails to be \( q \)-concave.

**Proof.** Let \( \tau > 0 \) be a fixed integer and let \( x, y \) and \( x_j, j = 1, \ldots, N \), be the vectors defined in Proposition 4.3. We know that \( X_q(\mathcal{F}) \) is a rearrangement invariant space, \( h \in \mathcal{F} \) and \( (k_s)_{s=1}^\infty \) is a lacunary sequence.
Therefore \( \|x_j\| = \|x\| \) for all \( j = 1, \ldots, N \) and

\[
\|x\| \geq \langle |x|, \sqrt{h} \rangle = \sum_{s=2}^{\tau} \frac{(k_s - k_{s-1}) \sqrt{\alpha_s}}{\sqrt[\alpha_s]{k_s} \sqrt[\alpha_{s+1}]{k_{s+1}}} = \sum_{s=2}^{\tau} \frac{(k_s - k_{s-1})}{k_s} \geq \frac{1}{2}(\tau - 1).
\]

Thus,

\[
\sum_{j=1}^{N} \|x_j\|^q = N \|x\|^q \geq \frac{N}{2q}(\tau - 1)^q.
\]

In order to show that \( \left( \sum_{j=1}^{N} \|x_j\|^q \right)^{\frac{1}{q}} \bigg/ \left( \sum_{j=1}^{N} |x_j|^q \right)^{\frac{1}{q}} \) is arbitrarily large we are going to find an upper bound for the denominator in the last expression. By Proposition 4.3 we know that \( \frac{1}{N} \sum_{j \leq N} x_j^q(i) \leq 2(2K^{q-1} + 1)y^q(i) \) for all \( i \in \mathbb{N} \), and hence it is enough to estimate \( \|y\| \).

Let \( f \in \mathcal{F} \) and assume that \( f \leq \sum_{r \geq 1} \gamma_r f_r \) with \( f_r \in \mathcal{F}_r, \gamma_r \geq 0 \) and \( \sum_{r \geq 1} \gamma_r = 1 \). Then

\[
\langle |y|, \sqrt[q]{f} \rangle = \sum_{i=1}^{\infty} |g(i)| \sqrt[q]{f(i)} \leq \sum_{s=2}^{\tau} I(s) + II(s) + III(s)
\]

where for \( s \geq 2 \)

\[
I(s) = \frac{1}{\sqrt[\alpha_{s+1}]{\sqrt[\alpha_s]{n_s}}} \sum_{n_{s-1} \leq i < n_s} \sqrt[q]{\sum_{r=1}^{s} \gamma_r f_r(i)},
\]

\[
II(s) = \frac{1}{\sqrt[\alpha_{s+1}]{\sqrt[\alpha_s]{n_s}}} \sum_{n_{s-1} \leq i < n_s} \sqrt[q]{\gamma_{s+1} f_{s+1}(i)},
\]

\[
III(s) = \frac{1}{\sqrt[\alpha_{s+1}]{\sqrt[\alpha_s]{n_s}}} \sum_{n_{s-1} \leq i < n_s} \sqrt[q]{\sum_{r \geq s+2} \gamma_r f_r(i)}.
\]
We shall first estimate $II(s)$. We observe that Hölder’s inequality and (1) in Proposition 2.2 give us

$$II(s) \leq \frac{1}{\sqrt[4]{\alpha_{s+1} \sqrt{n_s}}} \sum_{i=1}^{n_s} \sqrt[n_s]{\gamma_{s+1} f_{s+1}(i)} \leq \frac{1}{\sqrt[4]{\alpha_{s+1} \sqrt{n_s}}} \sqrt[n_s]{n_s^{q/4} \sqrt[n_s]{\sum_{i=1}^{n_s} f_{s+1}(i)}}$$

$$\leq \frac{\gamma_{s+1}^{1/q'}}{\sqrt[4]{\alpha_{s+1}}} \sqrt[n_s]{\sum_{i=1}^{n_s} f_{s+1}(i)} \leq \frac{\gamma_{s+1}^{1/q'}}{\sqrt[4]{\alpha_{s+1}}} \sqrt[n_s]{\sum_{\ell=1}^{s+1} \alpha_{\ell}}.$$  

And by (∗) we have

$$II(s) \leq \sqrt[n_s]{\frac{C_{\alpha_{s+1}}}{\alpha_{s+1}}} = \sqrt[n_s]{\gamma_{s+1}^{1/q'}} \sqrt[n_s]{n_s}.$$  

Thus, again, using Hölder’s inequality, we have

$$\sum_{s=2}^{\tau} II(s) \leq \sqrt[n_s]{\gamma_{s+1}^{1/q'}} \sqrt[n_s]{\frac{C_{\alpha_{s+1}}}{\alpha_{s+1}}} \leq \sqrt[n_s]{\gamma_{s+1}^{1/q'}} \sqrt[n_s]{n_s} \sum_{s=2}^{\tau} \gamma_{s+1} \leq \sqrt[n_s]{\gamma_{s+1}^{1/q'}} \sqrt[n_s]{\gamma_{s+1}^{1/q'}} \sqrt[n_s]{n_s} \gamma_{s+1}^{1/q'}.$$

To bound $III(s)$, we observe that by Hölder’s inequality

$$III(s) \leq \frac{1}{\sqrt[4]{\alpha_{s+1} \sqrt{n_s}}} \sqrt[n_s]{\sum_{i=1}^{n_s} \sum_{r \geq s+2} \gamma_r f_r(i)} \leq \frac{1}{\sqrt[4]{\alpha_{s+1} \sqrt{n_s}}} \sqrt[n_s]{\sum_{i=1}^{n_s} \sum_{r=1}^{s} \gamma_r f_r(i)} \leq \frac{1}{\sqrt[4]{\alpha_{s+1} \sqrt{n_s}}} \sqrt[n_s]{\sum_{i=1}^{n_s} \sum_{r=1}^{s} \gamma_r f_r(i)} \leq \frac{1}{\sqrt[4]{\alpha_{s+1} \sqrt{n_s}}} \sqrt[n_s]{\sum_{i=1}^{n_s} \sum_{r=1}^{s} \gamma_r f_r(i)} \leq \frac{1}{\sqrt[4]{\alpha_{s+1} \sqrt{n_s}}} \sqrt[n_s]{n_s} \gamma_{s+1}^{1/q'} \sqrt[n_s]{\gamma_{s+1}^{1/q'}} \sqrt[n_s]{n_s} \gamma_{s+1}^{1/q'}.$$

where in the last step we used that $\|f_r\|_{\ell_{\infty}} \leq \frac{\alpha_{r-1}}{k_{r-1}} \leq \frac{\alpha_{s+1}}{k_{s+1}}$ for $r \geq s+2$.

Finally, we shall estimate $I(s)$. Let us fix $s \geq 2$. By Lemma 4.2 we can find functions $f'$ and $f''$ such that $\sum_{r=1}^{s} \gamma_r f_r = f' + f''$ with

$$\text{card}(\text{supp} f') \leq 2k_s \left( \sum_{r=1}^{s} m_r \right)$$

and

$$\sum_{i=1}^{n_s} f''(i) \leq \alpha_{s+1} \left( \frac{n_s}{k_{s+1}} + \frac{C}{2s} \right).$$

This allows us to split $I(s)$ as $I(s) \leq IV(s) + V(s)$ for all $s \geq 2$ where

$$IV(s) = \frac{1}{\sqrt[4]{\alpha_{s+1} \sqrt{n_s}}} \sum_{n_{s-1} \leq i < n_s} \sqrt[n_s]{f'(i)}$$

and

$$V(s) = \frac{1}{\sqrt[4]{\alpha_{s+1} \sqrt{n_s}}} \sum_{n_{s-1} \leq i < n_s} \sqrt[n_s]{f''(i)}.$$
By Hölder’s inequality,

\[ IV(s) \leq \frac{1}{\sqrt[\alpha+1]{n_s}} \left( \sum_{i=1}^{n_s} f''(i) \chi \supp f'(i) \right)^{\frac{1}{q'}} \left( \sum_{i=1}^{n_s} f'(i) \right)^{\frac{1}{q'}} \]

\[ \leq \frac{1}{\sqrt[\alpha+1]{n_s}} \left( \sum_{i=1}^{n_s} \chi \supp f'(i) \right)^{\frac{1}{q'}} \left( \sum_{i=1}^{n_s} f'(i) \right)^{\frac{1}{q'}} \]

\[ \leq \frac{1}{\sqrt[\alpha+1]{n_s}} (\text{card(} \supp f'))^{\frac{1}{q'}} \left( \sum_{i=1}^{n_s} f'(i) \right)^{\frac{1}{q'}}. \]

Since \( \text{card(} \supp f') \leq 2k_s(\sum_{r=1}^{s} m_r^s) \), \((*)\), \((1)\) in Proposition 2.2 yields

\[ IV(s) \leq \sqrt{2k_s(\sum_{r=1}^{s} m_r^s)} = \sqrt[\alpha+1]{n_s} \frac{k_s(\sum_{r=1}^{s} m_r^s)}{n_s}. \]

On the other hand, Hölder’s inequality and the fact that \( \sum_{i=1}^{n_s} f''(i) \leq \alpha_{s+1}(\frac{n_s}{k_s+1} + \frac{C_2}{2^s}) \) imply

\[ V(s) \leq \frac{1}{\sqrt[\alpha+1]{n_s}} \left( \sum_{i=1}^{n_s} f''(i) \right)^{\frac{1}{q'}} \leq \frac{n_s}{k_s+1} + C2^{-s}. \]

It follows from these relations that

\[ \langle |y|, \sqrt{f} \rangle \leq \sqrt{C} \sqrt[\tau-1]{2} + 2 \sum_{s=2}^{\tau} \sqrt{\frac{n_s}{k_{s+1}}} + \sqrt{C} \sum_{s=2}^{\tau} \frac{1}{2^{s/q'}} \]

\[ + \sqrt{C} \sqrt[2]{2} \sum_{s=2}^{\tau} \sqrt{\frac{k_s(\sum_{r=1}^{s} m_r^s)}{n_s}}. \]

Since we are assuming that \( A = \sum_{s=1}^{\infty} \sqrt{\frac{n_s}{k_{s+1}}} \) and \( B = \sum_{s=1}^{\infty} \sqrt{\frac{k_s(\sum_{r=1}^{s} m_r^s)}{n_s}} \) are finite we have

\[ \langle |y|, \sqrt{f} \rangle \leq \sqrt{C} \sqrt[\tau-1]{2} + 2A + \frac{\sqrt{C}}{2^{1/q'} - 1} + \sqrt{2} \sqrt{C} B \leq \sqrt{C} \sqrt[\tau-1]{2} + S \]
where $S$ is a constant independent of $\tau$. Putting this altogether we have
\[
\left(\sum_{j=1}^{N} \|x_j\|^q\right)^{\frac{1}{q}} \geq \frac{\frac{1}{2} \sqrt{N}(\tau - 1)}{\sqrt{2}(2Kq^{-1} + 1)\sqrt{N}(\sqrt{C/\sqrt{\tau - 1}} + S)} = \frac{(\tau - 1)}{2\sqrt{2}(2Kq^{-1} + 1)(\sqrt{C/\sqrt{\tau - 1}} + S)}.
\]
This expression goes to infinity as $\tau$ goes to infinity.

\[\Box\]

**Proof of the Theorem 1.1.** Let $1 < q < \infty$ and take $(k_s)_{s=0}^\infty$ to be
the sequence of natural numbers defined by
\[
k_0 = k_1 = 1,
\]
\[
k_{s+1} = 3^{2^s + 2s^2} (E[g]+1)^{k_1+s(E[g]+1)}
\]
and the sequences
\[
\alpha_s = 3^{2^s}, \quad (\alpha_0 = 9), \quad m_s = (3^{2^s k_s})^{E[g]+1}, \quad \eta_s = 3^s, \quad n_s = \frac{k_{s+1}}{3^s}
\]
for all $s \geq 1$. These sequences satisfy the assumptions in Theorem 3.7 and in Theorem 4.4, hence $X_q(\vec{\beta})$ satisfies a lower $q$-estimate and is not $q$-concave. $\Box$

**References**


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