MULTIPLIERS ON WEIGHTED BESOV SPACES OF ANALYTIC FUNCTIONS

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Abstract. We characterize the space of multipliers between certain weighted Besov spaces of analytic functions. This extend and give a new proof of a result of Wojtaszczyk about multipliers between Bergman spaces.

Introduction.

P. Wojtaszczyk [W], using certain factorization theorems due to Maurey and Grothendieck, proved the following results:
Let $\alpha > 0$, $0 < p \leq 2 \leq q < \infty$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$.

$$(B_q, B_p) = \{\lambda_n : \sup_{2^n \leq k < 2^{n+1}} (k^{-1/r}|\lambda_k|) \in l^r\} \tag{0.1}$$

$$(X_\alpha, B_p) = \{\lambda_n : \sup_{2^n \leq k < 2^{n+1}} (k^{\alpha-1/p}|\lambda_k|) \in l^p\} \tag{0.2}$$

where B_p and X_α stand for the spaces

$$B_p = \{f : D \to \mathbb{C} \text{ analytic } : \Big(\int_D |f(z)|^p d\sigma(z)\Big)^{1/p} < \infty\} \text{ and}$$

$$X_\alpha = \{f : D \to \mathbb{C} \text{ analytic } : |f(z)| = O(\frac{1}{(1-|z|)^\alpha})\}.$$

The main objective of this paper is to extend such results to a much more general situation of general weighted Bergman and Besov spaces. We shall present a proof based simply on Kintchine's inequality for the analogue to (0.1) and then we shall use the previous case combined with some duality arguments to get the analogue to (0.2).

1991 *Mathematics Subject Classification.* 42A45.
Key words and phrases. Besov spaces, multipliers.
This research has been partially supported by the Spanish DGICYT, Proyecto PS89-0106

Given be a nondecreasing function on $(0,1)$, say ρ, such that $\frac{\rho(t)}{t} \in L^1\big((0,1)\big)$, we denote by $B_p(\rho)$, $0 < p < \infty$, the space of analytic function f on the unit disc such that

$$\|f\|_{p,\rho} = \Big(\int_D \frac{\rho(1-|z|)}{(1-|z|)}|f(z)|^p d\sigma(z)\Big)^{1/p} < \infty.$$

For certain weights the spaces $B_p(\rho)$ have been extensively studied in the literature. They can be regarded as extensions of the classical Bergman spaces ($\rho(t) = t$). Although the condition appearing in the case $p = 1$ and $\rho(t) = t^{1/q-1}$ for $q < 1$ goes back to the work of Hardy and Littlewood (see [HL1, Theorem 3.1], the corresponding spaces were first studied as Banach spaces by P. Duren, B.W. Romberg and A.L. Shield in [DRS] and by A.L. Shields and D.L. Williams [SW] for certain weights. Later J. Shapiro [Sh] considered the spaces, denoted by A^p_α and called weighted Bergman spaces, for $0 < p \le \infty$ and $\rho(t) = t^{\alpha+1}$.

The extension of (0.1) and (0.2) for weighted Bergman spaces can be achieved from similar arguments as those used by Wojtaszczyk, as it was pointed out in [W], when we deal with weights of the type considered in [SW]. In fact they can be obtained, as we shall prove, not only for these weights but for bit more general weights, defined by Janson [J], under Dini and b_α condition (see definitions below).

In [B1] the author gave a procedure to characterize multipliers acting on $B_p(\rho)$ when $0 < p \le 1$ and ρ verifies the previous conditions (see [B1, section 3]). Here we shall complete the cases when $2 \le p < \infty$, and not only for weighted Bergman spaces but for weighted Besov classes of analytic functions.

For $0 < p \le \infty$, $0 < q < \infty$, we denote by $H(p,q,\rho)$ the spaces formed by analytic functions on the unit disc D satisfying

$$\|f\|_{p,q,\rho} = \Big(\int_0^1 \frac{\rho(1-r)}{1-r} M_p^q(f,r)dr\Big)^{1/q} < \infty,$$

and by H^p_ρ the spaces of analytic functions such that

$$M_p(f,r) = O\Big(\frac{\rho(1-r)}{1-r}\Big) \quad (r \to 1).$$

The definition of these classes for the particular case of $\rho(t) = t^\alpha$, goes back to Hardy and Littlewood (see [HL1] [HL2]) and they were extensively studied for different reasons and by different authors (see [F1, F2, M, MP1, MP2, S]).

Given two sequence spaces X, Y we denote by (X, Y) the space of multipliers from X to Y, that is $(X, Y) = \{(\lambda_n) : (\lambda_n a_n) \in Y \text{ for every } (a_n) \in X\}$.

We identify $H(p,q,\rho)$ and H^p_ρ with sequence spaces by associating with each analytic function the sequence of its Taylor coefficients. Hence λ will stand for either the sequence (λ_n) or the function $\lambda(z) = \sum_{n=0}^\infty \lambda_n z^n$.

Theorem 1. *Let* $0 < p_2 \le 2 \le p_1 < \infty$, $0 < q_1, q_2 < \infty$, *and* ρ_1, ρ_2 *weight functions verifying Dini and* b_{α_1} *and* b_{α_2} *respectively for some* $\alpha_1, \alpha_2 > 0$. *Let* $\frac{1}{r} = \frac{1}{min(q_1,q_2)} - \frac{1}{q_1}$. *Then*

$$\Big(H(p_1,q_1,\rho_1), H(p_2,q_2,\rho_2)\Big) = \{\lambda_n : \sup_{2^n \le k < 2^{n+1}} \big(\rho_2(k^{-1})^{1/q_2}\rho_1(k^{-1})^{-1/q_1}|\lambda_k|\big) \in l^r\}.$$

Theorem 2. *Let $2 \le p_1 \le \infty$, $1 < p_2 \le 2$, $1 < q_2 < \infty$ and ρ_1, ρ_2 weight functions verifying Dini and b_1 and b_{α_2} respectively for some $\alpha_2 > 0$. Then*

$$\Big(H^{p_1}_{\rho_1}, H(p_2,q_2,\rho_2)\Big) = \{\lambda_n : \sup_{2^n \le k < 2^{n+1}} \big(k\rho_2(k^{-1})^{1/q_2}\rho_1(k^{-1})|\lambda_k|\big) \in l^{q_2}\}.$$

The reader is referred to [B2] for these and more results about multipliers when $\rho(t) = t^\alpha$.

The key point for these results is that for $p \ge 2$ multipliers on $H(p,q,\rho)$ depend on those on $H(2,q,\rho)$ and this space is isomorphic to $l(2,q)$ which makes them very easy to deal with.Then some duality arguments allow us to get Theorem 2 from Theorem 1.

Throughout the paper all functions f will be analytic on the unit disc and $M_p(f,r)$ stands for $\big(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\big)^{1/p}$. A weight function ρ will be a nondecreasing function on $(0,1)$ with $\rho(0^+) = 0$ such that $\frac{\rho(t)}{t} \in L^1\big((0,1)\big)$ and C will be a numerical constant not necessarily the same in each instance.

§1. Definitions and Lemmas.

There are several types of weight functions that have been considered to extend those results valid for $\rho(t) = t^\alpha$ to more general situations (see [SW, BS, BTS, MP2, J]). We shall be dealing with conditions considered in [J], which cover the other cases and have been found adequated to many other situations.

Definition 1. *A weight function will be a non negative non-decreasing function on $(0,1)$ such that $\rho(t)/t \in L^1\big((0,1)\big)$. ρ is said to be a Dini-weight if*

$$\int_0^s \frac{\rho(t)}{t} dt \le C\rho(s) \tag{1.1}$$

Given $0 < \alpha < \infty$, ρ is said to be a b_α-weight, $\rho \in b_\alpha$, if

$$\int_s^1 \frac{\rho(t)}{t^{\alpha+1}} dt \le C\frac{\rho(s)}{s^\alpha} \tag{1.2}$$

Remark 1. The main example of a Dini-weight ρ such that $\rho \in b_{\alpha+\varepsilon}$ for any $\varepsilon > 0$ is

$$\rho(t) = C_{\alpha,\beta} t^\alpha (log\frac{e}{t})^\beta.$$

Remark 2. It is not difficult to see that if $\rho \in b_\alpha$ for some $\alpha > 0$, then the Dini condition is equivalent to

$$\int_0^s \frac{\rho(t)}{t} dt = O\big(\rho(s)\big) \qquad (s \to 0) \tag{1.1$'$}$$

It is clear that b_α condition can be written

$$\int_s^1 \frac{\rho(t)}{t^{\alpha+1}} dt \approx \frac{\rho(s)}{s^\alpha}, \tag{1.2$'$}$$

which implies that there is a constant $C < 1$ such that

$$C\rho(2s) \le \rho(s) \le \rho(2s) \qquad (0 < s < \frac{1}{2}). \tag{1.3}$$

Let us include now some results on weights to be used later on.

Lemma 1. *Let ρ be a Dini weight such that $\rho \in b_\alpha$. Then*

$$(1.4)\qquad \rho(2^{-n}) \in l^1 \text{ and } \sum_{m=n}^{\infty} \rho(2^{-m}) = O\big(\rho(2^{-n})\big) \quad (n \to \infty)$$

$$(1.5)\qquad \int_0^1 \frac{\rho(1-r)}{(1-r)}\, r^n dr = O(\rho(\frac{1}{n})) \qquad (n \to \infty) \qquad (\textit{see } [\text{B1, Lemma 1.1 }])$$

Proof. Note that from (1.3)

$$\rho(2^{-n}) \approx \int_{1-2^{-n}}^{1-2^{-(n+1)}} \frac{\rho(1-r)}{1-r} dr.$$

Therefore $\rho(2^{-n}) \in l^1$ follows from $\frac{\rho(t)}{t} \in L^1\big((0,1)\big)$ and

$$\sum_{m=n}^{\infty} \rho(2^{-m}) \approx \int_{1-2^{-n}}^{1} \frac{\rho(1-r)}{1-r} dr \approx \int_0^{2^{-n}} \frac{\rho(t)}{t} dt = O\big(\rho(2^{-n})\big). \quad \square$$

Definition 2. *Let $0 < p \leq \infty$, $0 < q < \infty$ and ρ a weight function. $H(p,q,\rho)$ will denote the space of analytic functions on the unit disc D satisfying*

$$\|f\|_{p,q,\rho} = \Big(\int_0^1 \frac{\rho(1-r)}{1-r} M_p^q(f,r) dr \Big)^{1/q} < \infty.$$

Let ρ be a weight function with $inf \frac{\rho(1-r)}{1-r} > 0$. H^p_ρ will denote the space of analytic functions on the unit disc D satisfying

$$M_p(f,r) = O(\frac{\rho(1-r)}{1-r}).$$

We define the norm by

$$\|f\|_{p,\rho} = \inf\{C : M_p(f,r) \leq C \frac{\rho(1-r)}{1-r}, 0 < r < 1\}.$$

Definition 3. *Let $0 < p \leq \infty, 0 < q < \infty$ and $\bar{\gamma} = (\gamma_n)$ a sequence of positive real numbers. Denote by $I_n = \{k \in \mathbb{N} : 2^{n-1} \leq k < 2^n\}$ and $I_0 = \{0\}$.*

$$l(p,q,\bar{\gamma}) = \{(a_n) \in \mathbb{C} : \|(a_n)\|_{p,q,\bar{\gamma}} = \Big(\sum_n (\sum_{k \in I_n} |a_n|^p)^{q/p} \gamma_n\Big)^{1/q} < \infty\}$$

We denote by $l(p,q)$, with the obvious modification for $q = \infty$, the case where $\gamma_n = 1$ for all $n \in \mathbb{N}$.

Let us recall a very useful example of multipliers to be used later on.

Lemma 2. *([K]) Let $0 < p_1, q_1, p_2, q_2 \leq \infty$ Then*

$$\Big(l(p_1, q_1), l(p_2, q_2)\Big) = l(p, q)$$

where $\frac{1}{p} = \frac{1}{min(p_1,p_2)} - \frac{1}{p_1}$ *and* $\frac{1}{q} = \frac{1}{min(q_1,q_2)} - \frac{1}{q_1}$.

Note that Plancherel's theorem implies

$$\begin{aligned}\|f\|_{2,2,\rho} &= \Big(\int_0^1 \frac{\rho(1-r)}{1-r}\sum_{n=0}^{\infty}|a_n|^2 r^{2n} dr\Big)^{1/2}\\ &= \Big(\sum_{n=0}^{\infty}|a_n|^2\int_0^1 \frac{\rho(1-r)}{1-r} r^{2n} dr\Big)^{1/2} \approx \Big(\sum_{n=0}^{\infty}|a_n|^2 \rho(n^{-1})\Big)^{1/2}.\end{aligned}$$

Hence, for ρ verifying (1.3), we have

$$H(2,2,\rho) = \{\lambda_n : \lambda_n \in l\big(2,2,\rho(2^{-n})\big)\} \tag{1.6}$$

To extend this to other values of $0 < q < \infty$ (see [MP1, S] for the case $\rho(t) = t^\alpha$) we shall use the following lemma.

Lemma 3. *Let $0 < q < \infty$, ρ a Dini weight with $\rho \in b_\alpha$ for some $\alpha > 0$ and $\alpha_n \geq 0$. Then*

$$\int_0^1 \frac{\rho(1-r)}{1-r}(\sum_{n=0}^{\infty}\alpha_n r^n)^q dr \approx \sum_{n=0}^{\infty}(\sum_{k\in I_n}\alpha_k)^q \rho(2^{-n})$$

Proof.

$$\begin{aligned}\int_0^1 \frac{\rho(1-r)}{1-r}(\sum_{n=0}^{\infty}\alpha_n r^n)^q dr &= \sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} \frac{\rho(1-r)}{1-r}(\sum_{n=0}^{\infty}\alpha_n r^n)^q dr\\ &\geq C\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} \frac{\rho(1-r)}{1-r}(\sum_{k\in I_n}\alpha_k)^q r^{2^n q} dr\\ &\geq C\sum_{n=0}^{\infty}(\sum_{k\in I_n}\alpha_k)^q \rho(2^{-n}).\end{aligned}$$

Let us now show the converse inequality. Assume first $0 < q \leq 1$.

$$\begin{aligned}\int_0^1 \frac{\rho(1-r)}{1-r}(\sum_{n=0}^{\infty}\alpha_n r^n)^q dr &\leq \int_0^1 \frac{\rho(1-r)}{1-r}\Big(\sum_{n=0}^{\infty}(\sum_{k\in I_n}\alpha_k) r^{2^n-1}\Big)^q dr\\ &\leq \int_0^1 \frac{\rho(1-r)}{1-r}\Big(\sum_{n=0}^{\infty}(\sum_{k\in I_n}\alpha_k)^q r^{(2^n-1)q}\Big) dr\\ &\leq \sum_{n=0}^{\infty}(\sum_{k\in I_n}\alpha_k)^q\int_0^1 \frac{\rho(1-r)}{1-r} r^{(2^n-1)q} dr\\ &\leq C\sum_{n=0}^{\infty}(\sum_{k\in I_n}\alpha_k)^q \rho(2^{-n})\end{aligned}$$

where the last inequality follows from (1.5).

Let us then asumme $q>1$.

$$\int_0^1 \frac{\rho(1-r)}{1-r}(\sum_{n=0}^\infty \alpha_n r^n)^q dr = \sum_{n=0}^\infty \int_{1-2^{-n}}^{1-2^{-(n+1)}} \frac{\rho(1-r)}{1-r}(\sum_{n=0}^\infty \alpha_n r^n)^q dr$$
$$\leq C\sum_{n=0}^\infty \int_{1-2^{-n}}^{1-2^{-(n+1)}} \frac{\rho(1-r)}{1-r}\big(\sum_{m=0}^n \sum_{k\in I_m} \alpha_k\big)^q dr$$
$$+ C\sum_{n=0}^\infty \int_{1-2^{-n}}^{1-2^{-(n+1)}} \frac{\rho(1-r)}{1-r}\big(\sum_{m=n+1}^\infty (\sum_{k\in I_m} \alpha_k) r^{2^{m-1}}\big)^q dr$$
$$= (I) + (II).$$

In order to estimate (I) note that for any sequence $\beta_n \geq 0$ we have

$$\sum_{n=0}^\infty (\sum_{m=0}^n \beta_m)^q \rho(2^{-n}) \leq C \sum_{n=0}^\infty \beta_n^q \rho(2^{-n}) \tag{1.7}$$

Indeed, put $\gamma_{n+1} = \sum_{m=0}^n \beta_m$, $\gamma_0 = 0$ and $\mu(A) = \sum_{n\in A} \rho(2^{-n})$, then

$$\sum_{n=0}^\infty (\sum_{m=0}^n \beta_m)^q \rho(2^{-n}) = q\int_0^\infty \lambda^{q-1} \mu(\{n : \sum_{m=0}^n \beta_m > \lambda\}) d\lambda$$
$$= q\sum_{n=0}^\infty \int_{\gamma_n}^{\gamma_{n+1}} \lambda^{q-1}\mu(\{k : \sum_{m=0}^k \beta_m > \lambda\}) d\lambda$$
$$\leq C\sum_{n=0}^\infty (\sum_{m=0}^n \beta_m)^{q-1}\big(\sum_{m=n}^\infty \rho(2^{-m})\big)(\gamma_{n+1}-\gamma_n)$$
$$\leq C\sum_{n=0}^\infty (\sum_{m=0}^n \beta_m)^{q-1}\beta_n \rho(2^{-n})$$
$$\leq C\Big(\sum_{n=0}^\infty (\sum_{m=0}^n \beta_m)^q \rho(2^{-n})\Big)^{1/q'}\Big(\sum_{n=0}^\infty \beta_m^q \rho(2^{-n})\Big)^{1/q}$$

This gives (1.7), what allows us to get by writing $\beta_n = \sum_{k\in I_n} \alpha_k$

$$(I) \leq C\sum_{n=0}^\infty \sum_{m=0}^n (\sum_{k\in I_m} \alpha_k)^q \rho(2^{-n}) \leq C\sum_{n=0}^\infty (\sum_{k\in I_n} \alpha_k)^q \rho(2^{-n}).$$

In order to estimate (II), first observe that for any $\beta_n \geq 0$

$$\big(\sum_{m=n+1}^\infty \beta_m r^{2^{m-1}}\big)^q \leq \big(\sum_{m=n+1}^\infty 2^{-m\alpha}\beta_m^q\big)\big(\sum_{m=n+1}^\infty 2^{m\alpha(q'-1)} r^{(2^{m-1})q'}\big)^{q-1}$$
$$\leq C\big(\sum_{m=n+1}^\infty 2^{-m\alpha}\beta_m^q\big)\big(\sum_{m=2^n}^\infty m^{\alpha(q'-1)-1} r^{mq'}\big)^{q-1}$$
$$\leq C\big(\sum_{m=n+1}^\infty 2^{-m\alpha}\beta_m^q\big)\frac{1}{(1-r)^\alpha}.$$

Hence

$$
\begin{aligned}
(II) &\le C\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}}\frac{\rho(1-r)}{1-r}\Big(\sum_{m=n+1}^{\infty}\big(\sum_{k\in I_m}\alpha_k\big)r^{2^{m-1}}\Big)^q dr\\
&\le C\sum_{n=0}^{\infty}\Big(\sum_{m=n+1}^{\infty}2^{-m\alpha}\big(\sum_{k\in I_m}\alpha_k\big)^q\Big)\int_{1-2^{-n}}^{1-2^{-(n+1)}}\frac{\rho(1-r)}{(1-r)^{\alpha+1}}dr\\
&\le C\sum_{m=0}^{\infty}\big(\sum_{k\in I_m}\alpha_k\big)^q 2^{-m\alpha}\sum_{n=0}^{m}\int_{1-2^{-n}}^{1-2^{-(n+1)}}\frac{\rho(1-r)}{(1-r)^{\alpha+1}}dr\\
&= C\sum_{m=0}^{\infty}\big(\sum_{k\in I_m}\alpha_k\big)^q 2^{-m\alpha}\int_{0}^{1-2^{-(m+1)}}\frac{\rho(1-r)}{(1-r)^{\alpha+1}}dr\\
&\le C\sum_{m=0}^{\infty}\big(\sum_{k\in I_m}\alpha_k\big)^q \rho(2^{-m}). \quad \square
\end{aligned}
$$

Corollary 1. *Let* $0 < q < \infty$, ρ *a Dini weight,* $\rho \in b_\alpha$ *and* $f(z) = \sum_{n=1}^{\infty} a_n z^n$. *Then*

$$f \in H(2,q,\rho) \ \text{ if and only if } \ a_n \in l\big(2,q,\rho(2^{-n})\big).$$

Corollary 2. *Let* $1 < q < \infty$, $\frac{1}{q}+\frac{1}{q'} = 1$, ρ *a Dini weight and* $\rho \in b_\alpha$. *Then*

$$\Big(H(2,q,\rho)\Big)^* = H(2,q',\rho).$$

under the duality pair $< f, g >= \sum_{n=0}^{\infty} a_n \bar{b}_n \rho(\frac{1}{n})$.

Remark 3. For other values of $1 \le p \le \infty$ using (1.5)

$$
\begin{aligned}
\sum_{n=0}^{\infty} a_n\bar{b}_n\rho(n^{-1}) &\approx \int_0^1 \frac{\rho(1-r)}{1-r}\sum_{n=0}^{\infty} a_n\bar{b}_n r^n dr\\
&= \int_0^1 \frac{\rho(1-r)}{1-r}\int_0^{2\pi} f(re^{it})\bar{g}(re^{it})\frac{dt}{2\pi}\\
&\le C\int_0^1 \frac{\rho(1-r)}{1-r}M_p(f,r)M_{p'}(g,r)dr\\
&\le C\|f\|_{p,q,\rho}\|g\|_{p',q',\rho}.
\end{aligned}
$$

This shows that for $1 \le p \le \infty$ and $1 < q < \infty$

$$H(p',q',\rho) \subset \Big(H(p,q,\rho)\Big)^*. \tag{1.8}$$

Actually for the case $1 < p < \infty$ it can be shown that $H(p',q',\rho)$ is the dual of $H(p,q,\rho)$. Altough we shall not need the duality result the reader is referred to [S] for a proof that can be easily extended to the weighted case.

Let us finally mention the duality for the case $q = 1$.

Theorem A. *(See* [BS, Theorem 3.2]. *Let* $1 \le p < \infty$ *and* ρ *a Dini weight such that* $\rho \in b_1$. *Then*

$$\Big(H(p,1,\rho)\Big)^* = H^{p'}_\rho$$

under the pairing $< f, g >_\alpha = lim_{r\to 1} \sum_{n=0}^\infty (n+1)^{-1} a_n b_n r^n$.

Remark 4. This can be extended to weights having b_α condition by means of fractional derivatives (see [B2]).

§4 The Theorems and their proofs.

Next result follows easily from Kintchine's inequality.

Lemma 4. *Let* $0 < p, q < \infty$. *Let* $f(z) = \sum_{n=0}^\infty a_n z^n$. *Let us denote by* $f_t(z) = \sum_{n=0}^\infty r_n(t) a_n z^n$ *where* r_n *are the Rademacher functions.*

$$M_2^q(f,r) \approx \int_0^1 M_p^q(f_t,r)dt \tag{2.1}$$

Proposition 1. *Let* $0 < p_2 \le 2 \le p_1 < \infty$, $0 < q_1, q_2 < \infty$ *and* ρ_1, ρ_2 *weight functions. Then*

$$\Big((H(p_1,q_1,\rho_1), H(p_2,q_2,\rho_2)\Big) = \Big((H(2,q_1,\rho_1), H(2,q_2,\rho_2)\Big).$$

Proof. It is immediate that

$$\Big((H(2,q_1,\rho_1), H(2,q_2,\rho_2)\Big) \subset \Big((H(p_1,q_1,\rho_1), H(p_2,q_2,\rho_2)\Big).$$

Take now $\lambda_n \in \Big((H(p_1,q_1,\rho_1), H(p_2,q_2,\rho_2)\Big)$ and $f \in H(p_1,q_1,\rho_1)$. Denote by $(f*\lambda)(z) = \sum_{n=0}^\infty a_n\lambda_n z^n$ and $(f*\lambda)_t(z) = \sum_{n=0}^\infty r_n(t)a_n\lambda_n z^n$ where $r_n(t)$ stand for the Rademacher functions.

An application of (2.1) and Fubini's Theorem give

$$\begin{aligned}
\int_0^1 \frac{\rho_2(1-r)}{1-r} M_2^{q_2}(f*\lambda,r)dr &\le C\int_0^1 \frac{\rho_2(1-r)}{1-r}\int_0^1 M_{p_2}^{q_2}\big((f*\lambda)_t,r\big)dt\ dr\\
&= C\int_0^1\int_0^1 \frac{\rho_2(1-r)}{1-r} M_{p_2}^{q_2}\big((f*\lambda)_t,r\big)dr\ dt\\
&\le C\int_0^1 \Big(\int_0^1 \frac{\rho_1(1-r)}{1-r} M_{p_1}^{q_1}(f_t,r)dr\Big)^{q_2/q_1} dt = A
\end{aligned}$$

Now if $q_2 \le q_1$ then from Fubini and (2.1) again

$$\begin{aligned}
A &\le \Big(\int_0^1\int_0^1 \frac{\rho_1(1-r)}{1-r} M_{p_1}^{q_1}(f_t,r)dr\ dt\Big)^{q_2/q_1}\\
&\le \Big(\int_0^1 \frac{\rho_1(1-r)}{1-r}\int_0^1 M_{p_1}^{q_1}(f_t,r)dt\ dr\Big)^{q_2/q_1}\\
&\le C\Big(\int_0^1 \frac{\rho_1(1-r)}{1-r} M_2^{q_1}(f,r)dr\Big)^{q_2/q_1}
\end{aligned}$$

If $q_2 > q_1$ put $s = \frac{q_2}{q_1}$ and apply duality

$$\begin{aligned} A &= \sup_{\|h\|_{s'}=1} \Big(\int_0^1 \big(\int_0^1 \frac{\rho_1(1-r)}{1-r} M_{p_1}^{q_1}(f_t, r) dr \big) h(t) dt \Big)^{q_2/q_1} \\ &= \sup_{\|h\|_{s'}=1} \Big(\int_0^1 \Big(\int_0^1 M_{p_1}^{q_1}(f_t, r) h(t) dt \Big) \frac{\rho_1(1-r)}{1-r} dr \Big)^{q_2/q_1} \\ &\le \Big(\int_0^1 \Big(\int_0^1 M_{p_1}^{q_2}(f_t, r) dt \Big)^{q_1/q_2} \frac{\rho(1-r)}{1-r} dr \Big)^{q_2/q_1} \\ &\le C \Big(\int_0^1 \frac{\rho(1-r)}{1-r} M_2^{q_1}(f, r) dr \Big)^{q_2/q_1}. \quad \Box \end{aligned}$$

Theorem 1. *Let $0 < p_2 \le 2 \le p_1 < \infty$, $0 < q_1, q_2 < \infty$, and ρ_1, ρ_2 weight functions verifying Dini and b_{α_1} and b_{α_2} respectively for some $\alpha_1, \alpha_2 > 0$. Let $\frac{1}{r} = \frac{1}{min(q_1,q_2)} - \frac{1}{q_1}$. Then*

$$\Big(H(p_1, q_1, \rho_1), H(p_2, q_2, \rho_2)\Big) = \{\lambda_n : \sup_{2^n \le k < 2^{n+1}} \big(\rho_2(k^{-1})^{1/q_2} \rho_1(k^{-1})^{-1/q_1} |\lambda_k|\big) \in l^r\}.$$

Proof. Using Proposition 1 and Corollary 1 we simply have to find

$$\Big(l\big(2, q_1, \rho_1(2^{-n})\big), l\big(2, q_2, \rho_2(2^{-n})\big)\Big).$$

Note that $(\lambda_n) \in \Big(l\big(2, q_1, \rho_1(2^{-n})\big), l\big(2, q_2, \rho_2(2^{-n})\big)\Big)$ is equivalent to

$$\rho_2(n^{-1})^{1/q_2} \lambda_n \rho_1(n^{-1})^{-1/q_1} \in \Big(l(2, q_1), l(2, q_2)\Big).$$

Hence the proof is completed by invoking Lemma 2. $\Box$

Lemma 5. *(see* [B1, Lemma 5.1 and 5.2]*) Let ρ be a Dini weight and $\rho(t) \in b_\alpha$ for some $\alpha > 0$. Let $(\alpha_n) \ge 0$. Then*

$$\sum_{n=0}^\infty \alpha_n r^n = O\big(\frac{\rho(1-r)}{(1-r)^\alpha}\big) \qquad \text{if and only if} \qquad \sum_{m \in I_n} n^{-\alpha} \alpha_n = O\big(\rho(2^{-n})\big).$$

Theorem 2. *Let $2 \le p_1 \le \infty$, $1 < p_2 \le 2$, $1 < q_2 < \infty$ and ρ_1, ρ_2 weight functions verifying Dini and b_1 and b_{α_2} respectively for some $\alpha_2 > 0$. Then*

$$\Big(H_{\rho_1}^{p_1}, H(p_2, q_2, \rho_2)\Big) = \{\lambda_n : \sup_{2^n \le k < 2^{n+1}} \big(k\rho_2(k^{-1})^{1/q_2} \rho_1(k^{-1}) |\lambda_k|\big) \in l^{q_2}\}.$$

Proof. It is clear that $\Big(H_{\rho_1}^2, H(2, q_2, \rho_2)\Big) \subset \Big(H_{\rho_1}^{p_1}, H(p_2, q_2, \rho_2)\Big)$.

Let $(\lambda_n) \in \Big(H^{p_1}_{\rho_1}, H(p_2, q_2, \rho_2)\Big)$. It is not hard to see from duality (see (1.8) and Theorem A) that then

$$(n+1)\lambda_n \rho_2(n^{-1}) \in \Big(H(p_2', q_2', \rho_2), H(p_1', 1, \rho_1)\Big).$$

Using Proposition 1 and duality again this implies $(\lambda_n) \in \Big(H^2_{\rho_1}, H(2, q_2, \rho_2)\Big)$, which, from Corollary 1 and Lemma 5, is equivalent to

$$\rho_2(n^{-1})^{1/q_2} \lambda_n \rho_1(n^{-1})(n+1) \in \Big(l(2,\infty), l(2, q_1)\Big).$$

The proof is again finished from invoking Lemma 2. □

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