Norm estimates for operators from $H^p$ to $\ell^q$.

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Abstract

We give upper and lower estimates of the norm of a bounded linear operator from the Hardy space $H^p$ to $\ell^q$ in terms of the norm of the rows and the columns of its associated matrix in certain vector-valued sequence spaces.

Key words: Hardy spaces, vector-valued sequence spaces, vector-valued BMO, absolutely summing operators.

1 Introduction

Let $1 \leq p, q \leq \infty$ and let $T : H^p \to \ell^q$ be a linear and bounded operator where $H^p$ denote the Hardy space in the unit disc. To such an operator we associate the matrix $(t_{kn})_{k,n}$, defined by

$$T(u_n) = \sum_{k \in \mathbb{N}} t_{kn} e_k$$

where $u_n(z) = z^n$, $n \geq 0$, and $(e_k)_{k \in \mathbb{N}}$ stands for the canonical basis of $\ell^q$. We denote by $T_k = (t_{kn})_{n \geq 0}$ and $x_n = (t_{kn})_{k \in \mathbb{N}}$ its rows and its columns respectively. Although explicitly computing the norm is not possible (even for $p = q = 2$) several theorems concerning upper and lower estimates of the norm $\|T\|$ in terms of

$$\|(T_k)\|_{\ell^r \to \ell^s} = \left( \sum_{k=1}^{\infty} \left( \sum_{n=0}^{\infty} |t_{kn}|^s \right)^{r/s} \right)^{1/r}$$

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for different values of $r$ and $s$ were proved by B. Osikiewicz in [23]. The
following results are the content of Theorems 2.1, 2.2, 2.3 and 2.4 in [23]: If
$1 \leq p \leq 2$, $1 \leq q \leq \infty$ and $1/r = (1/q - 1/2) +$ then
$$
\| (T_k) \|_{\ell^r(\ell^2)} \leq \| T \| \leq \| (T_k) \|_{\ell^q(\ell^r)}.
$$
(1)

If $2 \leq p < \infty$, $1 \leq q \leq \infty$ and $1/s = (1/q - 1/p') +$ then
$$
\| (T_k) \|_{\ell^s(\ell^p)} \leq \| T \| \leq \| (T_k) \|_{\ell^q(\ell^r)}.
$$
(2)

Whilst the upper estimates were shown to be sharp in the scale of $\ell^r(\ell^s)$ spaces,
it was left open whether the values of $r$ and $s$ in the lower estimates could be
improved.

The reader is referred to [8] for some results in the same spirit in the cases
$0 < p < 1$. In this paper we shall see (1) and (2) can actually be improved
in different directions. On the one hand we shall use not only the norm of
the rows $(T_k)$ but also the norm of the columns $(x_n)$, which, sometimes gives
better estimates. On the other hand we shall consider $\ell(p, q)$-spaces instead
of $\ell^q$-spaces to produce more precise estimates. Our main tool will be the
description of the boundedness of operators between $H^p$ and $\ell^q$ by means of
vector-valued functions which will allow us to use results from vector-valued
Hardy spaces and absolutely summing operators to get our theorems.

Let $X$ be a complex Banach space with dual space $X^*$. We denote by $\ell^s(X)$
and $\ell^s_{\text{weak}}(X)$ the spaces of bounded sequences in $X$ for $s = \infty$, and, for
$1 \leq s < \infty$, the spaces of sequences $(A_j) \subset X$ such that
$$
\| (A_j) \|_{\ell^s(X)} = (\sum_j \| A_j \|^s)^{1/s} < \infty
$$
and
$$
\| (A_j) \|_{\ell^s_{\text{weak}}(X)} = \sup_{\| x^* \| = 1} \left( \sum_j |\langle A_j, x^* \rangle|^s \right)^{1/s} < \infty.
$$

It is easy to see that, for $1 \leq p \leq \infty$, $1/p + 1/p' = 1$,
$$
\| (A_j) \|_{\ell^s_{\text{weak}}(X)} = \sup \left\{ \left\| \sum_j \beta_j A_j \right\| : \| (\beta_j) \|_{\ell^p} = 1 \right\}.
$$

Hence $\ell^s_{\text{weak}}(X)$ can be identified with $L(\ell^{p'}, X)$ for $1 < p < \infty$ and $L(c_0, X)$
for $p = 1$. Also, for reflexive Banach spaces $X$ and $1 \leq p < \infty$, $\ell^p_{\text{weak}}(X)$
can be identified with $L(X^*, \ell^p)$ by defining $T(x^*) = (\langle A_j, x^* \rangle)_j$ and $\| T \| = \| (A_j) \|_{\ell^p_{\text{weak}}(X)}$. 
We denote by $\ell(s, r, X)$, $0 < r, s \leq \infty$, the space of sequences $(x_n)_{n \geq 0} \subset X$ such that

$$
\| (x_n) \|_{\ell(s, r, X)} = \max\{\| x_0 \|, \sup_{k \in \mathbb{N}} \left( \sum_{n=2^{k-1}}^{2^k-1} \| x_n \|^r \right)^{1/r} \} < \infty,
$$
or

$$
\| (x_n) \|_{\ell(s, r, X)} = (\| x_0 \|^r + \sum_{k \in \mathbb{N}} \left( \sum_{n=2^{k-1}}^{2^k-1} \| x_n \|^r \right)^{1/r} ) < \infty.
$$

In particular, $\ell(s, s, X) = \ell^s(X)$.

We denote by $H^p(X)$ (resp. $H^p_{weak}(X)$) the vector-valued Hardy spaces consisting of analytic functions $F : \mathbb{D} \rightarrow X$ such that

$$
\| F \|_{H^p(X)} = \sup_{0 < r < 1} (\int_0^{2\pi} \| F(re^{it}) \|^p \frac{dt}{2\pi})^{1/p} < \infty,
$$

(resp.

$$
\| F \|_{H^p_{weak}(X)} = \sup_{\| x^* \| = 1} \| \langle F, x^* \rangle \|_{H^p} < \infty.
$$

As usual we write $M_p(F, r) = (\int_0^{2\pi} \| F(re^{it}) \|^p \frac{dt}{2\pi})^{1/p}$.

We shall use the notation $\ell^p = \ell^p(\mathbb{C})$, $\ell(p, q) = \ell(p, q, \mathbb{C})$, $L^p = L^p(T)$ and $H^p = H^p(\mathbb{C})$ where $H^p$ will be sometimes understood as functions in $L^p$ using the fact that $H^p$ isometrically embeds into $L^p$ for $1 \leq p \leq \infty$. We also make use of the duality results $(H^1)^* = BMOA$ (see [17]) and $(H^p)^* = H^{p'}$ (see [16]) for $1 < p < \infty$.

We shall prove, among other things, the following estimates.

**Theorem 1** Let $1 < p < \infty$, $1 \leq q < \infty$ and let $T : H^p \rightarrow \ell^q$ be a bounded operator. Then, for $p_1 = \min\{p, 2\}$, $p_2 = \max\{p, 2\}$, $1/r = (1/q - 1/p_1) + 1/s_u = (1/q - 1/p_2) - (1/u - 1/2)^+$, we have

$$
\| T \| \leq \min\{\| (T_k) \|_{\ell^r(\ell^{p_1})}, \| (x_n) \|_{\ell^r(\ell^{p_1})}\}. \tag{3}
$$

For each $u \geq q$ there exists $C > 0$ such that

$$
\max\{\| (T_k) \|_{\ell^r(\ell^{p_2})}, \| (x_n) \|_{\ell^u(\ell^{p_1})}\} \leq C \| T \|. \tag{4}
$$

**Remark 2** Note that the use of columns in Theorem 1 provides sometimes better results than the use of rows. Indeed, taking into account that, for $q \leq p$,

$$
\left( \sum_{n=0}^{\infty} \left( \sum_{k=1}^{\infty} |a_{kn}|^p \right)^{q/p} \right)^{1/p} \leq \left( \sum_{k=1}^{\infty} \left( \sum_{n=0}^{\infty} |a_{kn}|^p \right)^{q/p} \right)^{1/q},
$$
we obtain, for instance, in the case \( p > 2, q = 1 \) and \( u = 2 \), that \( s_u = p \) and (4) improves (2) because
\[
\| (T_k) \|_{(p)} \leq \| (x_n) \|_{( wh(p))}.
\]

Also in the case \( 1 < p < \infty, 1 \leq q \leq \min\{ p, 2 \} = p_1 \) we obtain that (3) improves (1) because
\[
\| (x_n) \|_{( w(p_1))} \leq \| (T_k) \|_{( w( p_1))}.
\]

Selecting special values of \( u \) in Theorem 1 we obtain some new lower estimates of \( \| T \| \).

**Corollary 3** Let \( 1 \leq q \leq 2 \) and let \( T : H^p \to \ell^q \) be a bounded operator.

(i) If \( 1 \leq q \leq p \leq 2, 1/r = 1/q - 1/p \) and \( 1/s = 1/q - 1/2 \) then
\[
C^{-1} \max \{ \| (T_k) \|_{(w(p))}, \| (x_n) \|_{(w(p))}, \| (x_n) \|_{(w(p_1))} \} \leq \| T \|.
\]

(ii) Let \( 1 \leq q \leq p' \leq 2 \leq p < \infty \) such that \( 1/q - 1/p' \geq 1/p' - 1/2 \). If \( 1/r = 1/q - 1/2, 1/s = 1/q - 1/p' \) and \( 1/t = 1/q - 2/p' + 1/2 \) then
\[
C^{-1} \max \{ \| (T_k) \|_{(w(p))}, \| (x_n) \|_{(w(p))}, \| (x_n) \|_{(w(p'))} \} \leq \| T \|.
\]

In particular, for \( 1 \leq q \leq 2, p = 2 \) and \( 1/r = 1/q - 1/2 \), we have
\[
\max \{ \| (T_k) \|_{(w(p))}, \| (x_n) \|_{(w(p))} \} \leq C \| T \|.
\]

**Proof.** (i) Let \( 1 \leq q \leq p \leq 2 \). For each \( p \leq u \leq 2 \), we write \( 1/u = (1 - \theta)/p + \theta/2 \) for some \( 0 \leq \theta \leq 1 \). Hence the values in Theorem 1 become \( p_1 = p, p_2 = 2, 1/r = 1/q - 1/p \) and \( 1/s = 1/q - 1/u = 1/r + \theta (1/p - 1/2) \). Now select \( \theta = 0 \) and \( \theta = 1 \) and apply (4) to get the desired estimates.

(ii) Let \( 1 \leq q \leq p' \leq 2 \leq p < \infty \) such that \( 1/q - 1/p' \geq 1/p' - 1/2 \). For each \( p' \leq u \leq 2 \) now we obtain \( p_1 = 2, p_2 = p, 1/r = 1/q - 1/2 \) and \( 1/s = (1/q - 1/p' - (1/u - 1/2))^{+} \). Our assumption implies that \( s_u = t \) for \( u = p' \) and \( s_u = s \) for \( u = 2 \). Apply again (4) to finish the proof. \( \square \)

**Remark 4** Assume \( 1 \leq q \leq p' < 2 < p < \infty \). Then (ii) in Corollary 3 gives \( \| (T_k) \|_{(w(p))} \leq C \| T \| \) for \( 1/r = 1/q - 1/2 \) (which produces a better lower estimate than (2) since \( r \leq s \) for \( 1/s = 1/q - 1/p' \)).

Actually, for \( p \geq 2 \), the value \( v = r \) given by \( 1/r = 1/q - 1/2 \) is the smallest value in the scale \( \ell^v(p) \) to get the estimate \( \| (T_k) \|_{(w(p))} \leq C \| T \| \) as the
following example shows: Consider a lacunary multiplier $T : H^p \to \ell^q$ given by
\[
T(f)(z) = \sum_{k=0}^{\infty} \lambda_k a_{2^k} e_{2^k},
\]
where $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

In such a case $\| (T_k) \|_{\ell^q(\ell^p)} = \| (\lambda_k) \|_{\ell^p}$ and $\| \sum_k a_{2^k} z^{2^k} \|_{H^p} \approx (\sum_k |a_{2^k}|^2)^{1/2}$. This shows that $\| T \| \approx \| (\lambda_k) \|_{\ell^q}$ for $1/r = 1/q - 1/2$.

To present further improvements we shall replace the scale of $\ell^p$-spaces by the $\ell(p, q)$-spaces (see [19]) when computing the norm of the rows and the columns of the matrix associated to the operator.

Our first result will be the following extension of Theorem 1.

**Theorem 5** Let $1 < p < \infty$, $1 \leq q < \infty$, $p_1 = \min\{p, 2\}$ and $p_2 = \max\{p, 2\}$ and let $T : H^p \to \ell^q$ be a bounded operator. Then
\[
\| T \| \leq \min\{\| (T_k) \|_{\ell^q(\ell(p_1, 2))}, \| (x_n) \|_{\ell(p_1, 2, \ell^q)}\}.
\]  

For each $u \geq q$ there exists $C > 0$ such that
\[
\max\{\| (T_k) \|_{\ell^q(\ell(p_2, 2))}, \| (x_n) \|_{\ell(s_u, 2, \ell^q)}\} \leq C \| T \|,
\]  

where $1/r = (1/q - 1/p_1)^+$ and $1/s_u = (1/q - 1/p_2' - (1/u - 1/2)^+)^+$.

Of course, Theorem 1 follows from Theorem 5 using the inclusions $\ell^q(\ell^p_1) \subset \ell^q(\ell(p_1, 2))$, $\ell^p_1(\ell^q) \subset \ell(p_1, 2, \ell^q)$, $\ell^q(\ell(p_2, 2)) \subset \ell^q(\ell^p_2)$ and, since $s_u \geq 2$, also $\ell(s_u, 2, \ell^q) \subset \ell^{s_u}(\ell^u)$.

Using the inequalities (see Lemma 13 below)
\[
\| (x_n) \|_{\ell(p, q, \ell^q)} \leq \| (T_k) \|_{\ell^q(\ell(p, q))}, \quad \min\{p, q\} \geq r,
\]
\[
\| (x_n) \|_{\ell^q(\ell(p, q))} \leq \| (T_k) \|_{\ell^q(\ell(p, q))}, \quad \max\{p, q\} \leq r,
\]
we can formulate the following corollaries of Theorem 5.

**Corollary 6** Let $1 \leq q < p \leq 2$ and $T : H^p \to \ell^q$ be a bounded operator. If $1/s = 1/q - 1/p$ then there exists $C > 0$ such that
\[
C^{-1} \| (x_n) \|_{\ell(s, 2, \ell^q)} \leq \| T \| \leq \| (x_n) \|_{\ell(p, 2, \ell^q)}.
\]  

**Corollary 7** Let $1 \leq q \leq p' \leq 2 \leq p < \infty$ and $T : H^p \to \ell^q$ be a bounded operator. If $1/r = 1/q - 1/2$ and $1/s = 1/q - 1/p'$ then there exists $C > 0$ such that
\[
C^{-1} \max\{\| (T_k) \|_{\ell^q(\ell(p, 2))}, \| (x_n) \|_{\ell(s, 2, \ell^q)}\} \leq \| T \| \leq \| (x_n) \|_{\ell^q(\ell^r)}.
\]
Theorem 5 will follow from very general arguments valid for many other spaces relying upon some geometrical properties which are shared by other spaces. However in the case $1 \leq p < 2$ other tools are at our disposal and allow us to get better estimates. For instance, in the case $p = 1$ we can produce new upper estimates using results on Taylor coefficients of functions in BMOA.

Theorem 8 Let $T : H^1 \rightarrow \ell^q$ be a bounded operator.

(i) For $q = 1$ we have
\[
\|T\| \leq C \min\{(x_n)\|\ell(1,2,e_1), \|(n+1)^{1/2}x_n\|\ell(2,\infty,e_1)\}.
\]

(ii) For $1 \leq q \leq 2$ we have
\[
\|T\| \leq C \min\{(T_k)\|\ell(1,2), \|(x_n)\|\ell(1,2,e_r), \|(n+1)^{1/2}x_n\|\ell(2,\infty,e_r)\}.
\]

(iii) For $q \geq 2$ we have
\[
\|T\| \leq C \min\{(T_k)\|\ell(1,2), \|(A_k)\|\ell(1,2), \|(n+1)^{1/2}x_n\|\ell(2,\infty,e_1)\},
\]
where $A_k = ((n+1)^{1/2}t_{kn})n$.

Also new lower estimates can be achieved for $1 < p < 2$ using the factorization $H^p = H^2H^t$ where $1/2 + 1/t = 1/p$.

Theorem 9 Let $1 \leq p < 2$, $1 \leq q \leq 2$, $1/r = 1/q - 1/2$ and $1/t = 1/p - 1/2$ and let $T : H^p \rightarrow \ell^q$ be a bounded operator. Then there exists $C > 0$ such that
\[
\sup_{\|\alpha\|_{\ell(1,2)}} \max\{\left(\sum_{l=0}^{\infty} \alpha t_{k,l+n}\right)\|\ell(1,2), \left(\sum_{l=0}^{\infty} \alpha t_{k,l+n}\right)\|\ell(1,2)\} \leq C \|T\|.
\]

Finally the special behavior of the inclusion map $\ell^1 \rightarrow \ell^2$ allows to get further extensions in the case $q = 1$.

Theorem 10 Let $1 \leq p < 2$, $1/t = 1/p - 1/2$ and $T : H^p \rightarrow \ell^1$ be a bounded operator. There exists $C > 0$ such that
\[
\max\{\sup_{\|\alpha\|_{\ell(1,2)}} \left(\sum_{l=0}^{\infty} \alpha t_{k,l+n}\right)\|\ell(1,2), \sup_{\|\alpha\|_{\ell(1,2)}} \left(\sum_{l=0}^{\infty} \alpha t_{k,l+n}\right)\|\ell(1,2)\} \leq C \|T\|.
\]

As a simple application of Theorem 8 and Theorem 10 (selecting sequences $\alpha_j = \frac{1}{\sqrt{N}}$ for $0 \leq j \leq N$ and $\alpha_j = 0$ for $j \geq N + 1$) we get the following new estimates, that can be compared with the known ones for particular types of operators such as multipliers, composition operators and so on.
Corollary 11 Let $T : H^1 \to \ell^1$ be a bounded operator. There exists $C > 0$ such that

$$
\sup_{N \in \mathbb{N}} \| (\frac{1}{\sqrt{N}} \sum_{l=n}^{n+N} x_l)_n \|_{\ell^2(\ell^1)} \leq C \| T \| \leq C \min\{\| (n + 1)^{1/2} x_n \|_{\ell(2,\infty,\ell^1)}, \| (x_n)\|_{\ell(1,2,\ell^1)}\}.
$$

The paper is organized as follows. Section 2 contains some preliminary results concerning the reformulation of the boundedness of operators from $H^p$ to $\ell^q$ and some facts on the spaces $\ell(p,q,X)$ to be used in the sequel. Some tools from the theory of vector-valued Hardy and BMOA spaces are presented in Section 3. The proof of Theorem 5 is postponed to Section 4. Last section is devoted to the case $1 \leq p < 2$ and to present the proofs of Theorems 8, 9 and 10.

Throughout the paper, as usual, $L(X,Y)$ stands for the space of bounded linear operators, $a^+ = \max\{a,0\}$, $p'$ for the conjugate exponent of $p$ and $C$ denotes a constant that may vary from line to line.

2 Preliminary results

As it was mentioned in the introduction for each $1 \leq p,q \leq \infty$ and each bounded operator $T : H^p \to \ell^q$ we define the matrix $(a_{kn}(T)) = (t_{kn})$ given by

$$
T(u_n) = (t_{kn})_{k \in \mathbb{N}} \quad \text{for} \quad u_n(z) = z^n, n \geq 0. \quad (10)
$$

Observe that for each $k \in \mathbb{N}$ the functional $\xi_k T(f) = \langle T(f), e_k \rangle$, which belongs to $(H^p)^*$, is represented by an analytic function, say $g_k = g_k(T)$. We denote by $F_T(z) = (g_k(z))_{k \in \mathbb{N}}$ the $\ell^q$-valued analytic function associated to $T$.

Clearly each row $T_k = (t_{kn})_{n \geq 0}$ coincides with the sequence of Taylor coefficients of the function $g_k$, that is

$$
g_k(z) = \sum_{n=0}^{\infty} t_{kn} z^n \quad (11)
$$

and each column $x_n = (t_{kn})_{k \in \mathbb{N}}$ coincides with the $n$-Taylor coefficient of the vector-valued analytic function $F_T : \mathbb{D} \to \ell_q$ given by

$$
F_T(z) = \sum_{n=0}^{\infty} x_n z^n, \quad x_n = \sum_{k=1}^{\infty} t_{kn} e_k. \quad (12)
$$
With this notation, for a polynomial $f(z)$ with Taylor coefficients $(a_n)$, we have the expressions

$$T(f) = \sum_{n=0}^{\infty} a_n x^n = \lim_{r \to 1} \int_{-\pi}^{\pi} f(re^{i\theta}) \frac{d\theta}{2\pi}.$$  \hspace{1cm} (13)

$$T(f) = (\sum_{n=0}^{\infty} a_n t_k)_{k \in \mathbb{N}} = (\lim_{r \to 1} \int_{-\pi}^{\pi} g_k(re^{i\theta}) \frac{d\theta}{2\pi})_{k \in \mathbb{N}}.$$  \hspace{1cm} (14)

Let us make explicit the conditions describing that a function belongs to the vector-valued Hardy spaces for $X = \ell^s$. If $1 \leq r, s < \infty$, $(f_k)$ is a sequence in $H^r$ and $\sum_k |f_k(z)|^s < \infty$, $|z| < 1$, then $F(z) = (f_k(z))_{k \in \mathbb{N}}$ is a well defined $\ell^s$-valued analytic function in the unit disc. Moreover

$$\|F\|_{H^r_{\text{weak}}(\ell^s)} = \sup \{ \| \sum_{k=0}^{\infty} \lambda_k f_k \|_{H^r} : \|(\lambda_k)\|_{\ell^s} = 1 \}$$  \hspace{1cm} (15)

and

$$\|F\|_{H^r(\ell^s)} = \|(\sum_{k=0}^{\infty} |f_k|^s)^{1/s}\|_{L^r},$$  \hspace{1cm} (16)

where in (16) $f_k$ stands also for the boundary values of the same analytic function. Note that (16) follows from the fact that $\ell^s$ has the Radon-Nikodym property (see [15] and [9]) and therefore functions in $H^r(\ell^s)$ have radial boundary values in $L^r(\ell^s)$.

The following useful reformulation of the boundedness of operators from $H^p$ to $\ell^q$ is straightforward.

**Proposition 12** Let $1 < p < \infty, 1 \leq q < \infty$ and let $T : H^p \to \ell^q$ be a linear operator. The following are equivalent:

(i) $T$ is bounded.

(ii) $F_T \in H^{p'}_{\text{weak}}(\ell^q)$.

(iii) $(g_k(T))_{k} \in \ell^q_{\text{weak}}(H^{p'})$.

Moreover

$$\|T\| = \|F_T\|_{H^{p'}_{\text{weak}}(\ell^q)} = \|(g_k(T))\|_{\ell^q_{\text{weak}}(H^{p'})}. $$  \hspace{1cm} (17)
Let us now mention some facts about the spaces $\ell(p, q, X)$ which will be needed later on: If $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, $1/p = (1/p_2 - 1/p_1)^+$ and $1/q = (1/q_2 - 1/q_1)^+$ then

$$\ell(p, q) = \{(\lambda_n) : (\lambda_n, \beta_n) \in \ell(p_2, q_2) \text{ for any } (\beta_n) \in \ell(p_1, q_1)\}. \quad (18)$$

Let $q, \beta > 0$. Then (see [16] and [4,21] respectively)

$$\|((n + 1)^{-\beta} \alpha_n)_n\|_{\ell(1, \infty)} \approx \sup_{0 < r < 1} (1 - r)^{\beta} \left(\sum_n |\alpha_n r^n|\right), \quad (19)$$

$$\|((n + 1)^{-\beta} \alpha_n)_n\|_{\ell(1, q)} \approx \left(\int_0^1 (1 - r)^{\beta q - 1} \left(\sum_n |\alpha_n r^n| q^r dr\right)^{1/q}\right). \quad (20)$$

For any Banach space $X$ and $1 \leq p, q < \infty$ we have

$$\ell(p, q, X)^* = \ell(p', q', X^*). \quad (21)$$

We finish the section with the following application of Minkowski’s inequality.

**Lemma 13** Let $(a_{kn})_{k,n} \subset \mathbb{C}$ and write $A_k = (a_{kn})_{n \geq 0}$ and $B_n = (a_{kn})_{k \in \mathbb{N}}$. Then

$$\|(A_k)\|_{\ell(q, s, \ell(p)} \leq \|(B_n)\|_{\ell(p, \ell(q, s))}, \quad 1 \leq p \leq \min\{q, s\} \leq \infty. \quad (22)$$

$$\|(A_k)\|_{\ell(p, \ell(q, s))} \leq \|(B_n)\|_{\ell(q, s, \ell(p)}, \quad 1 \leq \max\{q, s\} \leq p < \infty. \quad (23)$$

**Proof.** Assume $1 \leq p \leq \min\{q, s\} \leq \infty$. Since $\ell(q/p, s/p)$ is a normed space (because $q/p \geq 1$ and $s/p \geq 1$) using Minkowski’s inequality we have

$$\|(A_k)\|_{\ell(q, s, \ell(p)} = \|(\sum_{n=0}^{\infty} |a_{kn}|^p)_k\|_{\ell(q/p, s/p)}^{1/p} \leq \left(\sum_{n=0}^{\infty} \|(a_{kn})^p_\ell(q/p, s/p)\|^{1/p}ight) \leq \left(\sum_{n=0}^{\infty} \|B_n\|_{\ell(q, s)}^p\right)^{1/p} = \|(B_n)\|_{\ell(p, \ell(q, s))}. \quad (24)$$

Assume now that $1 \leq \max\{q, s\} \leq p < \infty$. Observe that applying (22) to the adjoint matrix, we conclude that for any matrix $(a'_{kn})$ we also have

$$\|(B'_n)\|_{\ell(q', s', \ell(p')} \leq \|(A'_k)\|_{\ell(p', \ell(q', s')}. \quad (25)$$

Now use (21) to conclude (23). □
3 Some results for vector-valued Hardy and BMOA

One of the first uses of Hausdorff-Young’s inequality for vector-valued Lebesgue spaces goes back to [25]. The next lemma is well known and its proof is sketched here for completeness.

Lemma 14 Let $1 < p \leq 2$, $p \leq q \leq p'$ and $F(z) = \sum_{n=0}^{\infty} x_n z^n \in H^p(\ell^q)$. Then

$$\left( \sum_{n=0}^{\infty} \|x_n\|_{\ell^q}^{p'/p} \right)^{1/p'} \leq \|F\|_{H^p(\ell^q)}.$$ 

Proof. For $p = 2$ and $q = 2$ Plancherel’s theorem holds and gives

$$\left( \sum_{n=0}^{\infty} \|x_n\|_{\ell^2}^2 \right)^{1/2} = \|F\|_{L^2(\ell^2)}.$$ 

On the other hand for $q = 1$ or $q = \infty$ we trivially have

$$\sup_{n \geq 0} \|x_n\|_{\ell^q} \leq \|F\|_{L^1(\ell^q)}.$$ 

Hence it follows, by interpolation, that

$$\left( \sum_{n=0}^{\infty} \|x_n\|_{\ell^q}^{p'/p'} \right)^{1/p'} \leq \|F\|_{H^p(\ell^s)}$$

for $s = p$ or $s = p'$. Now interpolating again between $\ell^p$ and $\ell^{p'}$ we get the general case. \hfill \Box

Actually there exists a generalization of Hausdorff-Young’s inequalities to the setting on $\ell(p, q, X)$ spaces valid for some Banach spaces $X$. We present here a self contained proof of the following result, although the reader should be aware that the proof relies upon certain vector-valued Littlewood-Paley inequalities (see [6,5]) and it can be extended to other spaces.

Lemma 15 Let $1 \leq p, q < \infty$ and $F(z) = \sum_{n=0}^{\infty} x_n z^n \in H^p(\ell^q)$.

(i) If $1 < p \leq 2$ and $p \leq q \leq 2$ then $\|(x_n)\|_{\ell(p', 2, \ell^q)} \leq \|F\|_{H^p(\ell^q)}$.

(ii) If $2 \leq p < \infty$ and $2 \leq q \leq p$ then $\|F\|_{H^p(\ell^q)} \leq \|(x_n)\|_{\ell(p', 2, \ell^q)}$.

Proof. (i) It was shown in [1, Proposition 1.4] that, for $1 \leq p \leq q \leq 2$, we have

$$\left( \int_0^1 (1 - r) M^2_p(F', r) dr \right)^{1/2} \leq C \|F\|_{H^p(\ell^q)}.$$
Using Lemma 14 we obtain

\[ \int_0^1 (1 - r) \left( \sum_{n=1}^{\infty} n^{r''} \| x_n \|_\ell^{r'(n-1)r''} \right)^{2/r'} dr \leq C \| F \|^2_{H^p(\ell^q)}. \]

Applying now (20) to \( \alpha_n = n^{r''} \| x_n \|_\ell^{r'(n-1)r''}, \beta = p' \) and \( q = 2/p' \) we get

\[ \int_0^1 (1 - r) \left( \sum_{n=1}^{\infty} n^{r''} \| x_n \|_\ell^{r'(n-1)r''} \right)^{2/r'} dr \approx \| \left( \| x_n \|_\ell^{r''} \right) \|_{\ell^2(p',2)}, \]

which finishes this part.

(ii) follows from the dualities \( (H^p(\ell^q))^* = H^{p'}(\ell^{q'}) \) for \( 1 < p, q < \infty \) and \( \ell(r, s, X)^* = \ell(r', s', X^*) \) for \( 1 < r, s < \infty \).

Let us now use the embedding \( \ell^1 \to \ell^2 \) and its properties.

**Lemma 16** Let \( 1 \leq p < \infty \). If \( F \in H^p_{\text{weak}}(\ell^1) \) then \( F \in H^p(\ell^2) \) and

\[ \| F \|_{H^p(\ell^2)} \leq C \| F \|_{H^p_{\text{weak}}(\ell^1)}. \]

**Proof.** Write \( F(z) = (f_k(z))_{k \in \mathbb{N}} \) where \( f_k \in H^p \) and

\[ \sup_{\| \epsilon_k \| = 1} \left\| \sum_{k=1}^{\infty} \epsilon_k f_k \right\|_{H^p} = \| F \|_{H^p_{\text{weak}}(\ell^1)}. \]

Now considering \( \epsilon_k = r_k(t) \) for \( t \in [0, 1] \) where \( r_k \) are the Rademacher functions, we obtain

\[ \int_0^1 \left\| \sum_{k=1}^{\infty} r_k(t) f_k \right\|_{L^p} dt \leq \sup_{t \in [0,1]} \left\| \sum_{k=1}^{\infty} r_k(t) f_k \right\|_{L^p}. \]

Hence Kintchine’s inequality implies

\[ \left\| \left( \sum_{k=1}^{\infty} |f_k|^2 \right)^{1/2} \right\|_{L^p} \leq C \| F \|_{H^p_{\text{weak}}(\ell^1)}. \]

The result now follows from (16). \( \square \)

Let us now introduce the vector-valued versions of \( BMOA \) that we shall use in the paper. The reader is referred to [5,?] for other possible definitions and their connections. We write \( BMOA_c(X) \) (resp. \( BMOA_{\text{weak}}(X) \)) for the space of analytic functions \( F : \mathbb{D} \to X \) such that

\[ \| F \|_{BMOA_c(X)} = \| F(0) \| + \sup_{|z| < 1} \left( \int (1 - |w|^2) \| F'(w) \|^2 P_z(w) dA(w) \right)^{1/2} < \infty, \]
\[ \|F\|_{BMOA_{\text{weak}}(\Omega)} = \sup_{\|x^*\|=1} \|\langle F, x^* \rangle\|_{BMOA} < \infty \]

where, as usual, \( P_z(w) = \frac{1-|z|^2}{|1-wz|^2} \) is the Poisson kernel and \( dA \) stands for the normalized Lebesgue measure on the unit disc \( \mathbb{D} \).

Note that \( BMOA_{\text{weak}}(X) = L(H^1, X) \). Therefore if \( T : H^1 \to \ell^q \) is a bounded linear operator for \( 1 < q < \infty \) we have
\[
\|(g_k(T))\|_{\ell^q_{\text{weak}}(BMOA)} = \|T^*\| = \|T\| = \|F_T\|_{BMOA_{\text{weak}}(\ell^q)}. \tag{24}
\]

In the case \( q = 1 \) we have that if \( T : H^1 \to \ell^1 \) is bounded then
\[
\|(g_k(T))\|_{\ell^1_{\text{weak}}(BMOA)} \leq \|T^*\| = \|T\| = \|F_T\|_{BMOA_{\text{weak}}(\ell^1)}. \tag{25}
\]

Let us see that the following limiting case for \( p = \infty \) of Lemma 16 also holds.

**Lemma 17** If \( F \in BMOA_{\text{weak}}(\ell^1) \) then \( F \in BMOA_C(\ell^2) \). Moreover
\[
\|F\|_{BMOA_C(\ell^2)} \leq C\|F\|_{BMOA_{\text{weak}}(\ell^1)}.
\]

**Proof.** Recall first that the inclusion map \( i : \ell^1 \to \ell^2 \) is 2-summing (it is even 1-summing from Grothendieck’s theorem [14,27]), i.e. if \((A_n) \in \ell^2_{\text{weak}}(\ell^1)\) then \((A_n) \in \ell^2(\ell^2)\) with \(\|(A_n)\|_{\ell^2(\ell^2)} \leq C\|(A_n)\|_{\ell^2_{\text{weak}}(\ell^1)}\). This implies (see [27]) that there exists \( C > 0 \) such that, for any finite measure space \((\Omega, \Sigma, \mu)\), if \( f : \Omega \to \ell^1 \) is measurable and \( \sup_{\|x^*\|=1} \|\langle f, x^* \rangle\|_{L^2(\mu)} \leq 1 \) then \( f \in L^2(\mu, \ell^2) \) and \( \|f\|_{L^2(\mu, \ell^2)} \leq C \).

Let us fix \( z \in \mathbb{D} \) and consider the probability measure on \( \mathbb{D} \) given by \( d\mu_z(w) = P_z(w)dA(w) \). Consider now \( f(w) = (1-|w|^2)^{1/2}F'(w) \) and note that, since \( F \in BMOA_{\text{weak}}(\ell^1) \), we have
\[
\sup_{|z| < 1} \|\langle f(w), x^* \rangle\|_{L^2(d\mu_z)} \leq \|F\|_{BMOA_{\text{weak}}(\ell^1)}.
\]

Hence \( f \in L^2(d\mu_z, \ell^1) \) for all \( z \in \mathbb{D} \) with \( \|f\|_{L^2(d\mu_z, \ell^1)} \leq C\|F\|_{BMOA_{\text{weak}}(\ell^1)} \). This implies \( F \in BMOA_C(\ell^2) \) and \( \|F\|_{BMOA_C(\ell^2)} \leq C\|F\|_{BMOA_{\text{weak}}(\ell^1)} \).

\[ \square \]

### 4 Proof of Theorem 5

We start by showing the following general fact.
Proposition 18 Let $1 < p < \infty, 1 \leq q < \infty$, $p_1 = \min\{2, p\}$ and $1/r = (1/q - 1/p_1)^+$. Let $T : H^p \to \ell^q$ be a bounded linear operator and $g_k = g_k(T)$ be given by (11). Then there exists $C > 0$ such that

$$
\|T\| \leq \min\{\sum_{k=0}^{\infty} \|g_k\|^q_{H^{p'}}^{1/q}, \sum_{k=0}^{\infty} \|g_k\|_{L^p}^{1/r}\}. \tag{26}
$$

$$
C^{-1} \max\left\{\sup_{\|\lambda\|_{p'} = 1} \left(\sum_{k=0}^{\infty} |\lambda_k|^2 \|g_k\|_{L^p}^{1/2}\right)^{1/2}, \sum_{k=0}^{\infty} \|g_k\|_{H^{p'}}^r\right\} \leq \|T\| \tag{27}
$$

Proof. (26) follows by Proposition 12 using (16) and the facts $\|(g_k)\|_{\ell^1} \leq \|(g_k)\|_{\ell^r(X)}$ and $\|F\|_{H^p(X)} \leq \|F\|_{H^p(X)}$. Let us show (27). For each $\lambda = (\lambda_k) \in \ell^q'$, denote $T_\lambda : H^p \to \ell^1$ given by

$$
T_\lambda(f) = \sum_{k=0}^{\infty} \lambda_k \langle f, e_k \rangle.
$$

Since $\|T\| = \sup\{\|T_\lambda\| : \|\lambda\|_{q'} = 1\}$, and $g_k(T_\lambda) = \lambda_k g_k(T)$, from (17), we have to get lower estimates of $\|(g_k(T))\|_{\ell^1}^{\ell^1_{\text{weak}}(H^p')}$. Using that $H^p'$ has cotype $u = \max\{p', 2\}$ (see for instance [14]), we have

$$
\sum_{k=0}^{\infty} |\lambda_k|^u \|g_k\|^u_{H^{p'}} \leq C \|(g_k(T))\|_{\ell^1_{\text{weak}}(H^p')}^{\ell^1_{\text{weak}}(H^p')}
$$

and, taking the supremum over $\lambda_k$ in the unit ball of $\ell^q'$, we obtain that $\|(g_k)\|_{\ell^r_{\text{weak}}(H^p')} \leq C \|T\|$ for $1/r = (1/u - 1/q')^+ = (1/q - 1/p_1)^+$. On the other hand, Khinchine’s inequality implies that

$$
\left(\sum_{k=0}^{\infty} |\lambda_k|^2 \|g_k\|^2_{L^p}\right)^{1/2} \leq C \|(\lambda_k g_k)\|_{\ell^1_{\text{weak}}(H^p')},
$$

and the proof of the proposition is finished. \qed

We now proceed to the proof of Theorem 5. Let $1 < p < \infty, 1 \leq q < \infty$, $p_1 = \min\{p, 2\}$ and $p_2 = \max\{p, 2\}$. Let $T : H^p \to \ell^q$ be a bounded linear operator and $F_T(z) = (g_k(z))_k = \sum_{n=0}^{\infty} x_n z^n$ be defined by the formulas (11) and (12).

Let us first show that $\|T\| \leq \min\{\|(T_k)\|_{\ell^1(T_1, 2)}, \|(x_n)\|_{\ell^1(T_2, 2)}\}$. Our proof will be based upon the following extension of Hausdorff-Young’s inequalities (see [19]): If $p_1 = \min\{p, 2\}$ and $p_2 = \max\{p, 2\}$ then

$$
\|g\|_{H^{p'}} \leq \|(\alpha_n)\|_{\ell^1_{\text{weak}}(p_1, 2)}, \quad \|(\alpha_n)\|_{\ell_{\text{weak}}(p_2, 2)} \leq \|g\|_{H^{p'}}
$$
for any \( g(z) = \sum_{n=0}^{\infty} \alpha_n z^n \).

Therefore (26) in Proposition 18 implies
\[
\|T\| \leq C \| (g_k) \|_{\ell^r(H^{p'})} \leq C \| (T_k) \|_{\ell^r(\ell(p_1, 2))}.
\]

On the other hand, \( \|T\| = \| F_T \|_{\ell^r(H^{p'})} \) and we have
\[
\| F_T \|_{\ell^r(H^{p'})} = \sup_{\|\lambda_k\|_{q'}=1} \| \langle \lambda, F_T \rangle \|_{H^{p'}}
\leq \sup_{\|\lambda_k\|_{q'}=1} \| \langle \lambda, x_n \rangle \|_{\ell(p_1, 2)}
\leq \sup_{\|\lambda_k\|_{q'}=1} \| \sum_{k=1}^{\infty} \lambda_k t_k n \|_{\ell(p_1, 2)}
\leq \| \sum_{k=1}^{\infty} |t_k n|^q \|_{\ell(p_1, 2)}^{1/q}
\leq \| (x_n) \|_{\ell(p_1, 2, \ell^q)}.
\]

Let us now show that for each \( u \geq q \) there exists \( C > 0 \) such that
\[
\max\{\| (T_k) \|_{\ell^r(\ell(p_2, 2))}, \| (x_n) \|_{\ell(s_u, 2, \ell^q)}\} \leq C \|T\|,
\]
where \( 1/r = (1/q - 1/p_1)^+ \) and \( 1/s_u = (1/q - 1/p'_2 - (1/u - 1/2)^+)^+ \).

Note that (27) in Proposition 18 together with the Hausdorff-Young’s inequalities give
\[
\| (T_k) \|_{\ell^r(\ell(p_2, 2))} \leq \| (g_k) \|_{\ell^r(H^{p'})} \leq C \|T\|.
\]

On the other hand, as above \( \|T\| = \| F_T \|_{\ell^r(H^{p'})} \) and combining Hausdorff-Young and (18), we obtain
\[
\| F_T \|_{\ell^r(H^{p'})} = \sup_{\|\lambda_k\|_{q'}=1} \| \langle \lambda, F_T \rangle \|_{H^{p'}}
\geq \sup_{\|\lambda_k\|_{q'}=1} \| \langle \lambda, x_n \rangle \|_{\ell(p_2, 2)}
= \sup_{\|\lambda_k\|_{q'}=1, \|\beta_n\|_{\ell(p'_2, 2)}=1} \sum_{n=0}^{\infty} \beta_n \sum_{k=1}^{\infty} \lambda_k t_k n
\geq \sup_{\|\lambda_k\|_{q'}=1, \|\beta_n\|_{\ell(p'_2, 2)}=1} \| \sum_{n=0}^{\infty} \langle \beta_n x_n, \lambda \rangle \|.
\]
Therefore \((\beta_n x_n) \in \ell^1_{\text{weak}}(\ell^q)\) for any \((\beta_n) \in \ell(p_2', 2)\) and
\[
\sup_{\|\beta_n\|_{\ell(p_2', 2)} = 1} \| (\beta_n x_n) \|_{\ell^1_{\text{weak}}(\ell^q)} \leq \| F_T \|_{H^p_{\text{weak}}(\ell^q)} = \| T \|.
\]

We now use the fact (due to B. Carl in [12] and G. Bennett in [3] independently) that the inclusion map \(\ell^q \rightarrow \ell^u\) is \((a, 1)\)-summing for \(1/a = 1/q - (1/u - 1/2)^+\) (see [14, pg. 209]) to conclude that \((\beta_n x_n) \in \ell^a(\ell^u)\) for any \((\beta_n) \in \ell(p_2', 2)\). Now (18) implies \((x_n) \in \ell(s, 2, \ell^u)\) for \(1/s = (1/a - 1/p_2')^+\). The proof is then complete.

5 Improvements for \(1 \leq p < 2\)

We first recall some known facts about \(BMOA\)-functions. It was shown in [10] that \(M_2(f', r) = O\left(\frac{1}{(1-r)^{1/2}}\right)\) implies \(f \in BMOA\). Moreover
\[
\| f \|_{BMOA} \leq C(\| f(0) \| + \sup_{0 < r < 1} (1 - r)^{1/2} M_2(f', r)).
\]

Using this estimate and (19) we conclude that
\[
\| g \|_{BMOA} \leq C\| (n + 1)^{1/2} \alpha_n \|_{\ell(2, \infty)}
\]
which we need to prove.

Also, using duality together with Paley’s inequality for functions in \(H^1\) (see [16]) we obtain
\[
\| g \|_{BMOA} \leq C\| (\alpha_n) \|_{\ell(1, 2)}.
\]

The reader should notice that these two sufficient conditions on the Taylor coefficients to define \(BMOA\)-function are of independent nature. It suffices to take \(\alpha_n = \frac{1}{n+1}\) to have an example satisfying \((n + 1)^{1/2} \alpha_n) \in \ell(2, \infty)\) but \((\alpha_n) \notin \ell(1, 2)\) and to take \(\alpha_{2^k} = \frac{1}{k}\) and zero otherwise to have \((\alpha_n) \in \ell(1, 2)\) but \((n + 1)^{1/2} \alpha_n) \notin \ell(2, \infty)\).

Proof of Theorem 8

Using (28) and (29) together with (24) we have the estimate
\[
\| T \| \leq \| (g_k) \|_{\mathcal{L}(BMOA)} \leq C \min\{\| (T_k) \|_{\mathcal{L}(\ell(1, 2))}, \| (A_k) \|_{\mathcal{L}(\ell(2, \infty))}\}.
\]

On the other hand
\[
\| T \| = \| F_T \|_{BMOA_{\text{weak}}(\ell^p)}
\]
\[
\sup_{\|(\lambda_k)\|_{\ell^q}} \|\langle \lambda, F_T \rangle\|_{BMO} \\
\leq \sup_{\|(\lambda_k)\|_{\ell^q}} \min\{\|(\lambda, x_n)\|_{\ell(1,2)}, \|(\lambda, (n+1)^{1/2}x_n)\|_{\ell(2,\infty)}\} \\
\leq \min\{\|(x_n)\|_{\ell(1,2,\ell^q)}, \|(n+1)^{1/2}x_n\|_{\ell(2,\infty,\ell^q)}\}.
\]

Invoking Lemma 13 we obtain the following estimates

\[
\|(x_n)\|_{\ell(1,2,\ell^q)} \leq \|(T_k)\|_{\ell(1,2,\ell^q)}, \\
\|(n+1)^{1/2}x_n\|_{\ell(2,\infty,\ell^q)} \leq \|(A_k)\|_{\ell(2,\infty)}, \quad q \leq 2, \\
\|(T_k)\|_{\ell(1,2,\ell^q)} \leq \|(x_n)\|_{\ell(1,2,\ell^q)}, \quad q \geq 2.
\]

Hence (i), (ii) and (iii) follow from these estimates.

**Proof of Theorem 9**

Take \( t \geq 2 \) such that \( 1/t + 1/2 = 1/p \) and \( \phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in H^t \) with \( \|\phi\|_{H^t} = 1 \). Define \( T_\phi : H^2 \to \ell^q \) given by

\[ T_\phi(f) = T(\phi f). \]

Due to the factorization result (see [16]) \( H^p = H^2 H^t \) we can write

\[ \|T\| = \sup\{\|T_\phi\| : \|\phi\|_{H^t} = 1\}. \]

Observe that

\[ x_n(T_\phi) = T(u_n \phi) = \sum_{l=0}^{\infty} \alpha_l T(u_{n+l}) = \sum_{l=0}^{\infty} \alpha_l x_{n+l}. \]

Therefore the matrix associated to \( T_\phi \) is given by \( a_{kn}(T_\phi) = (t'_{kn}) \) where

\[ t'_{kn} = \sum_{l \geq n} \alpha_{l-n} t_{kl} = \sum_{l=0}^{\infty} \alpha_l t_{k,l+n}. \]

Now using (5) one can write, for \( 1/r = 1/q - 1/2 \),

\[
\max\{\|(T_\phi)_k\|_{\ell^r(\ell^2)}, \|(x_n(T_\phi))\|_{\ell^r(\ell^2)}\} \leq C\|T_\phi\| \\
\leq C\|T\|\|\phi\|_{H^t} \\
\leq C\|T\|\|\alpha_l\|_{\ell^r(\ell^r,\ell^q)}. 
\]

This shows the result.

**Proof of Theorem 10**
Assume $1 \leq p < 2$ and let $T : H^p \to \ell^1$ be bounded. The estimate
\[
\sup_{\|\alpha\|_{\ell^1} = 1} \left\| \left( \sum_{l=0}^{\infty} \alpha_l t_{k,n+l} \right)_k \right\|_{\ell^2} \leq C \|T\|
\]
was obtained in Theorem 9 in the case $q = 1$.

Let us show
\[
\sup_{\|\alpha\|_{\ell^1} = 1} \left\| \sum_{l=0}^{\infty} \alpha_l z^{l,k,n} \right\|_{\ell^1} \leq C \|T\|. \tag{30}
\]

In the case $1 < p < 2$, we can use (17) to conclude that $F_T \in H^p_{weak}(\ell^1)$ and, due to Lemma 16, $F_T \in H^p(\ell^2)$.

In the case $p = 1$, we can use (25) to obtain $F_T \in BMOA_{weak}(\ell^1)$ and Lemma 17 to conclude that $F_T \in BMOA_{C}(\ell^2)$.

Using the dualities $(H^p(\ell^2))^* = H^{p'}(\ell^2)$ for $1 < p < 2$ and $(H^1(\ell^2))^* = BMOA_{C}(\ell^2))$ for $p = 1$, we can write, for $1 \leq p < 2$, that
\[
\sup \left\{ \| \sum_{n=0}^{\infty} \langle x_n, x'_n \rangle \| : G(z) = \sum_{j=0}^{\infty} x'_n z^n, \|G\|_{H^p(\ell^2)} = 1 \right\} \leq C \|T\|.
\]

In particular, for each $g(z) = \sum_{n=0}^{\infty} y_n z^n \in H^2(\ell^2)$ and $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in H^t$ where $1/t + 1/2 = 1/p$, the function $G(z) = g(z)\phi(z) = \sum_{n=0}^{\infty} x'_n z^n \in H^p(\ell^2)$ satisfies $x'_n = \sum_{n=0}^{\infty} y_n \alpha_{n-t}$ and $\|G\|_{H^p(\ell^2)} \leq \|g\|_{H^2(\ell^2)} \|\phi\|_{H^t}$. Therefore, in such a case, we obtain
\[
\sum_{n=0}^{\infty} \langle x_n, x'_n \rangle = \sum_{l=0}^{\infty} \sum_{n=l}^{\infty} \langle x_n, y_l \alpha_{n-l} \rangle = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \alpha_l \langle x_{n+l}, y_l \rangle.
\]

Finally, taking the supremum over $\|\langle y_j \rangle\|_{\ell^2} = 1$ and $\|\phi\|_{H^t} = 1$ we get (30). □

References


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