# Composition operators on the minimal space invariant under Möbius transformations

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Abstract. It is shown that if $\Phi : \mathbb{D} \to \mathbb{D}$ is an analytic function such that $M_p(\Phi'', r) \in L^{p'}(dr)$ for some $1 < p < \infty$ and $1/p + 1/p' = 1$ then $C_\Phi(f) = f \circ \Phi$ defines a bounded composition operator on the space $B_1$, the minimal space invariant under Möbius transformations. This was conjectured by J. Arazy, S. Fisher and J. Peetre in [**AFP**].

## 1. Introduction

Let us denote by $G$ the group of holomorphic automorphisms on the unit disk $\mathbb{D}$, i.e. the set of functions $\phi \in \mathcal{H}(\mathbb{D})$ such that $\phi = \lambda\varphi_a$ for $|\lambda| = 1$, $|a| < 1$ and

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z} = -a + (1-|a|^2)\sum_{k=1}^{\infty} \bar{a}^{k-1} z^k.$$

The fact $|\phi'(z)| = \frac{1-|\phi(z)|^2}{1-|z|^2}$ for $\phi \in G$ guarantees that the measure $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$ is invariant under Möbius transformations, where $dA(z)$ stands for the normalized area measure $dA(z) = \frac{dxdy}{\pi}$.

The paper where a systematic study of spaces of invariant under Möbius transformations was started is [**AFP**], and we also refer the reader to [**AF, F, RT, T1, T2**] for further considerations.

Although the precise definition may vary from author to author we shall say that a complete space $X \subset \mathcal{H}(\mathbb{D})$ with a semi-norm $\rho$ is $G$-invariant (or invariant under Möbius transformations) if for all $f \in X$ and $\phi \in G$ one has that $f \circ \phi \in X$ and there exists $C > 0$ such that

$$\sup_{\phi\in G} \rho(f\circ\phi) \le C\rho(f). \tag{1.1}$$

The basic examples of $G$-invariant spaces are the following:

- The space $H^\infty$: The space of bounded analytic functions.

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Note that for $1 \leq p < \infty$ the spaces

$$H^p = \{f \in \mathcal{H}(\mathbb{D}) : \sup_{0<r<1} \Big(\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi}\Big)^{1/p} < \infty\}$$

are not $G$-invariant in our sense.

The reader should be aware that they however become $G$-invariant under the action $f \to (f \circ \phi)(\phi')^{1/p}$.

- The space $BMOA$:

$$BMOA = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_* = \sup_{|a|<1} \|f \circ \varphi_a - f(a)\|_{H^2} < \infty\}.$$

- The Dirichlet space:

$$\begin{aligned} \mathcal{D}_2 &= \{f \in \mathcal{H}(\mathbb{D}) : (\sum_{n=1}^{\infty} n|a_n|^2)^{1/2} < \infty\} \\ &= \{f \in \mathcal{H}(\mathbb{D}) : f' \in L^2(\mathbb{D}, dA)\}. \end{aligned}$$

- The Besov spaces for $1 < p < \infty$:

$$B_p = \{f \in \mathcal{H}(\mathbb{D}) : \Big(\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z)\Big)^{1/p} < \infty\}.$$

In particular $B_2 = \mathcal{D}_2$.

- The Bloch space: $\mathcal{B} = \{f \in \mathcal{H}(\mathbb{D}) : \sup_{|z|<1}(1-|z|^2)|f'(z)| < \infty\}$.

We write $\|f\|_{\mathcal{B}} = \max\{|f(0)|, \sup_{|z|<1}(1-|z|^2)|f'(z)|\}$.

Let us also consider the invariant pairing on $\mathbb{D}$ given by

$$\langle f, g\rangle = \lim_{r\to 1} \int_{|z|<r} f'(z)\overline{g'(z)} dA(z).$$

Under such a pairing one has, for each $\phi \in G$,

$$\langle f \circ \phi, g \circ \phi\rangle = \langle f, g\rangle.$$

Using the Bergman projection one sees that

$$\langle f, \varphi_a\rangle = (1-|a|^2)f'(a). \tag{1.2}$$

It is not difficult to see that the previously mentioned examples are $G$-invariant. Note, for instance, that (1.2) implies that $f \in B_p$ if and only if

$$\int_{\mathbb{D}} |\langle f, \varphi_z\rangle|^p d\lambda(z) < \infty$$

and $f \in \mathcal{B}$ if and only if $\sup_{|z|<1} |\langle f, \varphi_z\rangle| < \infty$.

By the work by Rubel and Timoney (see [**RT**]) one has that $\mathcal{B}$ becomes a maximal space among the decent $G$-invariant ones. We say that a $G$-invariant space $X$ is "decent" if

(1.3) There exists $0 \neq x^* \in X^*$ which is also continuous in $\mathcal{H}(\mathbb{D})$.

For decent $G$-invariant spaces (see [**RT, F**]) one has that $X \subset \mathcal{B}$ continuously.

To find out which is the corresponding limiting case of the Besov spaces $B_p$ for $p = 1$ just recall the following well-known facts (see [**Z**]): Let $1 < p < \infty$ and denote $M_p(f,r) = \Big(\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi}\Big)^{1/p}$. The following are equivalent:

(i) $f \in B_p$.

(ii) $\int_0^1 M_p^p(f', r)(1-r)^{p-2} dr < \infty$.

(iii)$\int_0^1 M_p^p(f'', r)(1-r)^{p-1}dr < \infty$.
In the case $p = 1$ (iii) becomes $\int_0^1 M_1(f'', r)dr < \infty$. Thus one defines

$$B_1 = \{f \in \mathcal{H}(\mathbb{D}) : \rho_1(f) = \int_{\mathbb{D}} |f''(z)| dA(z) < \infty\}.$$

We can define a norm by considering

$$\|f\|_{B_1} = \max\{|f(0)|, |f'(0)|, \rho_1(f)\}.$$

This space is known to be minimal among $G$-invariant spaces with some extra properties (see [**AFP, T2**]). We shall see here that this is also the case when assuming certain measurability condition on the map $\phi \to f \circ \phi$ from $G$ to $X$.

The reader should be aware that the space $B_1$ was denoted by $\mathcal{M}$ in [**AFP**] and coincides with the space consisting in those functions $f \in \mathcal{H}(\mathbb{D})$ such that $f = \sum_{z=1}^{\infty} \lambda_k \varphi_{a_k}$ where $|a_k| < 1$ and $\sum_k |\lambda_k| < \infty$.

It was shown in [**AFP**, Theorems 18 and 19] respectively that

$$\Phi'' \in H^1 \Longrightarrow C_\Phi : B_1 \to B_1 \text{ is bounded} \tag{1.4}$$

and

$$\sup_\theta |\Phi''(re^{i\theta})| \in L^1(dr) \Longrightarrow C_\Phi : B_1 \to B_1 \text{ is bounded .} \tag{1.5}$$

Observe that (1.4) and (1.5) means $M_1(\Phi'', r) \in L^\infty(dr)$ and $M_\infty(\Phi'', r) \in L^1(dr)$ respectively. It was conjectured in the Arazy-Fisher-Peetre paper that $M_p(\Phi'', r) \in L^{p'}(dr)$ for some $1 < p < \infty$ and $1/p + 1/p'$ should be sufficient for $C_\Phi$ to be bounded on $B_1$.

In this direction it was even shown in [**AFP**, Theorem 20] that $C_\Phi$ was bounded on $B_1$ whenever $M_p(\Phi'', r) \in L^s(dr)$ for $s = \frac{2p}{p-1}$ (note that in this case $L^s(dr) \subset L^{p'}(dr)$).

Our main result is Theorem 3.7 where we show that the conjecture is true.

The paper is organized as follows: Section 1 contains some basic facts on $B_1$, in particular that $B_1 \subset X$ for a relatively wide class of $G$-invariant spaces. In Section 2 we apply a general theorem on the characterization of the boundedness of operators from $B_1$ into a Banach space to the particular case of composition operators from $B_1$ into $B_1$ and give a proof of the conjecture mentioned above.

As usual $p'$ stands for the conjugate exponent of $p$, $1/p + 1/p' = 1$ and $C$ denotes a constant that may vary from line to line.

## 2. The minimal space invariant under Möbius transformations

We may consider $G \subset \mathbb{T} \times \mathbb{D}$ by the mapping $(\lambda, a) \to \phi = \lambda\varphi_a$ or as a subspace of $H^\infty(\mathbb{D})$ or simply $G \subset \mathcal{H}(\mathbb{D})$ with the locally convex topology of the convergence over compact sets. Let us mention that all these topologies on $G$ are actually equivalent.

Proposition 2.1. Let $\phi_n = \lambda_n \varphi_{a_n}$ and $\phi = \lambda\varphi_a$ for some $|\lambda_n| = |\lambda| = 1$ and $a, a_n \in \mathbb{D}$. The following are equivalent:

(1) $\phi_n(z)$ converges to $\phi(z)$ for all $z \in \mathbb{D}$.
(2) $\lambda_n$ converges to $\lambda$ and $a_n$ converges to $a$.
(3) $\phi_n$ converges to $\phi$ in $H^\infty$.
(4) $\phi_n$ converges to $\phi$ in $\mathcal{H}(\mathbb{D})$, i.e. uniformly on compact subsets of $\mathbb{D}$.

PROOF. (1) $\Longrightarrow$ (2) Assume that $\phi_n(z)$ converges to $\phi(z)$ for all $z \in \mathbb{D}$.

Note that $\lambda_n \varphi_{a_n}(z) \to \lambda \varphi_a(z)$ is equivalent to $\bar{\lambda}\lambda_n \varphi_{a_n}((\varphi_a)^{-1}(w)) \to w$ for all $w \in \mathbb{D}$. For $w = 0$ one gets $\varphi_{a_n}(a) \to 0$ and then $a_n \to a$. In particular now $\varphi_{a_n}(z) \to \varphi_a(z)$ for all $z \in \mathbb{D}$ which together with $\phi_n(z) \to \phi(z)$ implies that $\lambda_n \to \lambda$.

(2) $\Longrightarrow$ (3)

$$\begin{aligned} |\phi_n(z) - \phi(z)| &= |(\lambda_n - \lambda)\varphi_{a_n} + \lambda(\varphi_{a_n} - \varphi_a)| \\ &\leq |\lambda_n - \lambda| + |\varphi_{a_n} - \varphi_a)| \\ &\leq |\lambda_n - \lambda| + \frac{2|a_n - a| + |\bar{a}_n a - a_n \bar{a}|}{(1 - |a_n|)(1 - |a|)} \end{aligned}$$

Hence $\|\phi_n - \phi\|_\infty \to 0$.

(3) $\Longrightarrow$ (4) Immediate.

(4) $\Longrightarrow$ (1) Immediate. □

Let us now give one characterization of the space $B_1$ (see [**AFP**]). Let us point out the following easy fact that we shall need for such a purpose.

PROPOSITION 2.2. If $f \in B_1$ and $f'(0) = 0$ then $F(z) = zf(z) \in B_1$ and $\rho_1(F) \leq 3\rho_1(f)$.

PROOF. $F'(z) = f(z) + zf'(z)$ and $F''(z) = 2f'(z) + zf''(z)$.

Therefore it suffices to see that $\int_{\mathbb{D}} |f'(z)| dA(z) \leq \int_{\mathbb{D}} |f''(z)| dA(z)$.

Since $f'(re^{it}) = \int_0^1 f''(rse^{it}) ds$, we can conclude that $M_1(f', r) \leq \int_0^1 M_1(f'', rs) ds$ and

$$\int_{\mathbb{D}} |f'(z)| dA(z) = \int_0^1 M_1(f', r) r dr \leq \int_0^1 \int_0^1 M_1(f'', rs) r dr ds \leq \int_{\mathbb{D}} |f''(z)| dA(z).$$

□

THEOREM 2.3. *Let $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = f'(0) = 0$. $f \in B_1$ if and only if there exists a complex Borel measure $\nu$ of bounded variation on $\mathbb{D}$ such that*

$$f(z) = \int_{\mathbb{D}} \varphi_a(z) d\nu(a).$$

*Moreover,*

$$\rho_1(f) \approx \inf\{\|\nu\|_1 : f = \int_{\mathbb{D}} \varphi_a d\nu(a).\}$$

PROOF. Assume $\nu$ is a measure of bounded variation with

$$f(z) = \int_{\mathbb{D}} \varphi_a(z) d\nu(a).$$

Then

$$f''(z) = \int_{\mathbb{D}} \varphi_a''(z) d\nu(a).$$

It suffices to use the standard estimate (see [**Z**, Page 53])

$$\|\varphi_a''\|_{L^1} = \int_{\mathbb{D}} \frac{2|a|(1 - |a|^2)}{|1 - \bar{a}z|^3} dA(z) \leq C \tag{2.1}$$

and Fubini's theorem to conclude that $\int_D |f''(z)| dA(z) \leq C\|\nu\|_1$. This shows that $\rho_1(f) \leq C \inf\{\|\nu\|_1\}$.

Conversely, observe that if $F(z)=\sum_{n=0}^\infty a_n z^n$ belong to $L^1(\mathbb{D})$ then

$$\int_{\mathbb{D}} \frac{F(a)}{1-\bar{a}z} dA(a) = \sum_{n=0}^\infty \frac{a_n}{n+1} z^n.$$

This shows that

$$\int_D F(a)\varphi_a(z) dA(a) = \sum_{n=1}^\infty \frac{a_{n-1}}{n(n+1)} z^n.$$

Now if $f(z)=\sum_{n=2}^\infty b_n z^n \in B_1$ we define $G(z)=zf(z)=\sum_{n=1}^\infty b_{n+1}z^{n+2}$. Hence, applying the formula above to

$$F(z)=G''(z)=\sum_{n=1}^\infty (n+2)(n+1)b_{n+1}z^n$$

we obtain

$$f(z)=\int_{\mathbb{D}} \varphi_a(z) G''(a) dA(a).$$

Using Proposition 2.2 one has that $G\in B_1$ and $\rho_1(G)\le 3\rho_1(f)$. Taking $d\nu(a)=G''(a)dA(a)$ one gets $\|\nu\|_1=\rho_1(G)\le 3\rho_1(f)$. □

Proposition 2.4. $B_1$ is $G$-invariant and $a\to\varphi_a$ is continuous from $\mathbb{D}$ to $B_1$.

Proof. Let $f\in B_1$ with $f(0)=f'(0)=0$ and $\phi=\lambda\varphi_b\in G$. Using Theorem 2.3 one can write $f=\int_{\mathbb{D}}\varphi_a(z)d\nu(a)$ for some $\nu$ with $\|\nu\|_1\le C\rho_1(f)$. Hence

$$f\circ\phi(z)=\int_{\mathbb{D}}\varphi_a(\phi(z))d\nu(a)=\int_{\mathbb{D}}\lambda\varphi_{\bar{\lambda}a}(\varphi_b(z))d\nu(a)=\int_{\mathbb{D}}\lambda\varphi_{\varphi^{-1}_{\bar{\lambda}a}(b)}(z)d\nu(a).$$

Now take the second derivative and use (2.1) to get that $\sup_{|c|<1}\|(\varphi_c)''\|_{L^1(\mathbb{D})}<\infty$ and $\rho_1(f\circ\phi)\le C\rho_1(f)$.

Note also that if $a_n\to a$ then $\varphi''_{a_n}(z)\to\varphi''_a(z)$ for all $z\in\mathbb{D}$. Now applying the dominated convergence theorem one concludes $\rho_1(\varphi_{a_n}-\varphi_a)\to 0$. □

It was shown in [**AFP**] that $(B_1)^*=\mathcal{B}$. Let us now show the minimal character of $B_1$ . The reader should note that we did not assume the map $\phi\to f\circ\phi$ to be continuous from $G$ to $X$ in the definition of $G$-invariant. This allowed to have more examples, as $\mathcal{B}$ or $H^\infty$, in this category.

Proposition 2.5. Let $(X,\|.\|)$ be a non-trivial (i.e there exists $f\in X$ non constant) $G$-invariant Banach space.

(1) If the map $\Gamma_f: G\to X$ defined by $\phi\to f\circ\phi$ is Borel measurable and bounded for all $f\in X$ then $X$ contains the space of polynomials, $B_1\subset X$ and there exists $C>0$ such that $\|f\|_X\le C\|f\|_{B_1}$ for all $f\in B_1$.

(2) If the map $\Gamma_f: G\to X$ defined by $\phi\to f\circ\phi$ is continuous for all $f\in X$ then the space of polynomials is dense in $X$.

Proof. (1) Let $f(z)=\sum_{n=0}^\infty a_n z^n\in X$ be non constant, that is $a_k\ne 0$ for some $k\ge 1$. Consider the bounded measurable map $\mathbb{T}\to X$ defined by $e^{it}\to f_t(z)=f(e^{it}z)$. Now one can use the Bochner integral to obtain that, for $n\ge 0$,

$$\int_0^{2\pi} f_t e^{-int}\frac{dt}{2\pi}=a_n u_n\in X$$

where $u_n(z)=z^n$. Therefore $u_k\in X$.

Now we can conclude that $(\varphi_a(z))^k \in X$ for all $|a| < 1$. We can repeat the previous argument for $f(z) = (\frac{2z-1}{2-z}))^k$ whose Taylor coefficients are all different from zero to get $u_n \in X$ for all $n \geq 0$.

Note that if $f$ is a polynomial with $f(0) = f'(0) = 0$ and $\nu$ a measure of bounded variation such that $f = \int_{\mathbb{D}} \varphi_a d\nu(a)$. Using that $a \to \varphi_a$ is bounded and measurable with values in $X$ one obtains

$$\|f\|_X \leq \int_{\mathbb{D}} \|\varphi_a\|_X d|\nu|(a) \leq \sup_{|a|<1} \|\varphi_a\|_X \|\nu\|_1.$$

This shows that $\|f\|_X \leq C\rho_1(f)$ for all polynomial $f$ with $f(0) = f'(0) = 0$. Hence, for a general polynomial $f$, writting $f = (f - f(0) - f'(0)z) + f(0) + f'(0)z$ one gets

$$\|f\|_X \leq \|f - f(0) - f'(0)z\|_X + |f(0)|\|u_0\|_X + |f'(0)|\|u_1\|_X \leq C\|f\|_{B_1}.$$

Now extend the result for functions in $B_1$ using the density of polynomials in $B_1$.

(2) Assume $\phi \to f \circ \phi$ is continuous from $G \to X$. Denote $f_r(z) = f(rz)$ for $0 < r < 1$ and observe that

$$f_r(z) - f(z) = \int_0^{2\pi} \Big(f(e^{it}z) - f(z)\Big) P_r(e^{-it}) \frac{dt}{2\pi}$$

where, as usual, $P_r(e^{it})$ stands for the Poisson kernel. This shows that

$$\|f_r - f\|_X \leq \int_0^{2\pi} \|f_t - f\|_X P_r(e^{-it}) \frac{dt}{2\pi}.$$

Using that $e^{it} \to f_t$ is continuous standard arguments imply that $f_r$ converges to $f$ in $X$. Using that polynomials are dense in $B_1$ and $f_r \in B_1$ for each $0 < r < 1$ one shows the density of polynomials in $X$.

□

## 3. Operators on $B_1$

Definition 3.1. Let $Y$ be a complex Banach space and $F : \mathbb{D} \to Y$ be analytic function. $F$ is said to be a vector-valued Bloch function, say $F \in \mathcal{B}(Y)$, if

$$\sup_{|a|<1} (1 - |a|^2)\|F'(a)\|_Y < \infty.$$

Write the norm $\|F\|_{\mathcal{B}(Y)} = \|F(0)\| + \sup_{|a|<1}(1 - |a|^2)\|F'(a)\|_Y$.

Theorem 3.2. *Let $Y$ be a complex Banach space and let $T : B_1 \to Y$ be a linear operator. Denote $x_n = T(u_n)$ for $u_n(z) = z^n$, $n \geq 0$, and assume that* $\limsup \sqrt[n]{\|x_n\|} \leq 1$. *The following are equivalent:*

(1) *$T$ is bounded.*
(2) *$g_T(a) = T(\varphi_a)$ is bounded and continuous from $\mathbb{D}$ to $Y$.*
(3) *$F_T(a) = \sum_{n=0}^{\infty} \frac{x_n}{n+1} z^{n+1} \in \mathcal{B}(Y)$.*

*Moreover*

$$\|T\| \approx \sup_{|a|<1} \|g_T(a)\|_Y \approx \|F_T\|_{\mathcal{B}(Y)}.$$

Proof. (1) $\Longrightarrow$ (2) Since $g_T(a) = T(\varphi_a)$ the result follows by composing $T \circ J$ where $J : \mathbb{D} \to B_1$ is the the continuous map given by $a \to \varphi_a$ according to Proposition 2.4.

(2) $\Longrightarrow$ (1) Let $f$ be a polynomial. Hence one has $f-f(0)-f'(0)z = \int_{\mathbb{D}} \varphi_a d\nu(a)$ for some measure $\nu$. Now, using linearity,

$$T(f) = f(0)x_0 + f'(0)x_1 + \int_{\mathbb{D}} g_T(a)d\nu(a).$$

Now from the assumption one obtains

$$\|T(f)\| \leq |f(0)|\|x_0\| + |f'(0)|\|x_1\| + \sup_{|a|<1} \|g_T(a)\|\|\nu\|_1.$$

This gives $\|T(f)\| \leq C\|f\|_{B_1}$ for any polynomial. Now use the density of polynomials in $B_1$ to extend to a bounded operator from $B_1$ into $Y$.

(1) $\Longrightarrow$ (3) From the assumption on $(x_n)$ the map $F_T$ is holomorphic (at least) on the unit disc and takes values in $Y$. Note that

$$F_T'(a) = \sum_{k=0}^{\infty} T(u_k)a^k = T(u_0) + T(\sum_{k=1}^{\infty} u_k a^k).$$

Since $\varphi_a = -a + (1-|a|^2)\sum_{k=1}^{\infty} \bar{a}^k u_k$. This shows that

$$(1-|a|^2)F_T'(a) = T(\varphi_{\bar{a}}) + (\bar{a} + (1-|a|^2))T(u_0).$$

Now use that $T$ is bounded and (2.1).

(3) $\Longrightarrow$ (2) Use the formula

$$g_T(a) = (1-|a|^2)F_T'(\bar{a}) - (a + (1-|a|^2))F(0).$$

□

Corollary 3.3. Let $X$ be a $G$-invariant space and let $\Phi : \mathbb{D} \to \mathbb{D}$ be a non constant analytic function. Then $C_\Phi : B_1 \to X$ defined by $C_\Phi(f) = f \circ \Phi$ is a bounded operator if and only if

$$\sup_{|a|<1} \|\varphi_a \circ \Phi\|_X < \infty.$$

Let us now apply this result to several cases.

Corollary 3.4. Let $\Phi : \mathbb{D} \to \mathbb{D}$ a non constant analytic function. Then $C_\Phi : B_1 \to \mathcal{D}_2$ is bounded if and only if

$$\sup_{|a|<1} \int_{\mathbb{D}} \frac{(1-|a|^2)^2 n_\Phi(z)}{|1-a\bar{z}|^4} dA(z) < \infty,$$

where

$$n_\Phi(z) = \#\{w \in \mathbb{D} : \Phi(w) = z\}.$$

In particular $C_\Phi$ is bounded from $B_1$ to $\mathcal{D}_2$ for univalent functions $\Phi$.

Proof. From Corollary 3.3 the boundedness is characterized by

$$\int_{\mathbb{D}} |\varphi_a'(\Phi(z))|^2 |\Phi'(z)|^2 dA(z) < \infty.$$

Now using that $\Phi$ is locally univalent and the usual change of variables formula one has

$$\int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-a\bar{z}|^4} n_\Phi(z) dA(z) < \infty.$$

□

Next result was shown in [**AFP**, Proposition 17]:

THEOREM 3.5. *Let $\Phi : \mathbb{D} \to \mathbb{D}$ be a non constant analytic function. Then $C_\Phi : B_1 \to B_1$ is bounded operator if and only if*

$$\sup_{|a|<1} \int_{\mathbb{D}} \frac{(1-|a|^2)}{|1-a\bar{z}|^3} n_\Phi(z) dA(z) < \infty, \tag{3.1}$$

$$\sup_{|a|<1} \int_{\mathbb{D}} \frac{(1-|a|^2)|\Phi''(z)|}{|1-\bar{a}\Phi(z)|^2} dA(z) < \infty. \tag{3.2}$$

Let us now prove the Arazy-Fisher-Peetre conjecture.

PROPOSITION 3.6. Let $1 < p < \infty$ and $F \in \mathcal{H}(\mathbb{D})$ with $M_p(F', r) \in L^{p'}(dr)$. Then $F \in BMOA$ and $\|F\|_* \leq C\Big(\int_0^1 M_p^{p'}(F', r)dr\Big)^{1/p'}$.

PROOF. First notice that, since the map $s \mapsto M_p^{p'}(F', s)$ is continuous and non-decreasing, one has

$$M_p^{p'}(F', r)(1-r) \leq \int_r^1 M_p^{p'}(F', s)ds.$$

Hence $M_p(F', r) \in L^{p'}(dr)$ implies $M_p(F', r) = o(\frac{1}{(1-r)^{1/p'}})$ as $r \to 1$. Now use the fact that $M_p(F', r) = O(\frac{1}{(1-r)^{1/p'}})$ can be described in terms of Lipschitz functions (see [**D**]) and then use the result in [**BSS**] to obtain that $F \in BMOA$.

$\square$

THEOREM 3.7. *Let $1 < p < \infty$ and $\Phi : \mathbb{D} \to \mathbb{D}$ a non constant analytic function. If $M_p(\Phi'', r) \in L^{p'}(dr)$ then $C_\Phi : B_1 \to B_1$ is bounded.*

PROOF. Let us show that condition (3.1) holds. Recall that

$$\int_{\mathbb{D}} \frac{n_\Phi(z)}{|1-\bar{a}z|^3} dA(z) = \int_{\mathbb{D}} \frac{|\Phi'(z)|^2}{|1-\bar{a}\Phi(z)|^3} dA(z).$$

Given a polynomial $h$ we write

$$\int_{\mathbb{D}} \frac{\Phi'(z)}{(1-a\overline{\Phi(z)})^{3/2}} \overline{h(z)} dA(z) = \int_0^1 \int_0^{2\pi} \frac{\Phi'(re^{it})}{(1-a\overline{\Phi(re^{it})})^{3/2}} \overline{h(re^{it})} r dr \frac{dt}{\pi}.$$

From Proposition 3.6 and the duality $(H^1)^* = BMOA$ (see [**Z**]), we have

$$|\int_{\mathbb{D}} \frac{\Phi'(z)}{(1-a\overline{\Phi(z)})^{3/2}} \overline{h(z)} dA(z)| \leq \int_0^1 \|\Phi'\|_* M_1(\frac{h}{(1-\bar{a}\Phi)^{3/2}}, r)dr$$

We now recall that Littlewood subordination principle (see [**Z**, Theorem 10.1.3]) implies that for $1 \leq q < \infty$ and $\alpha > 0$ we have

$$M_q(\frac{1}{(1-\bar{a}\Phi(z))^\alpha}, r) \leq M_q(\frac{1}{(1-\bar{a}z)^\alpha}, r). \tag{3.3}$$

Also it is well known that if $0 < \alpha, q < \infty$ and $\alpha q > 1$ then there exists a constant $C > 0$ such that

$$M_q(\frac{1}{(1-\bar{a}z)^\alpha}, r) \leq C \frac{1}{(1-|a|^2r^2)^{\alpha - 1/q}} \tag{3.4}$$

Therefore Cauchy-Schwartz, (3.3) and (3.4) give

$$\begin{aligned} M_1(\frac{h}{(1-\bar{a}\Phi)^{3/2}}, r) &\leq M_2(h,r) M_2(\frac{1}{(1-\bar{a}\Phi(z))^{3/2}}, r) \\ &\leq M_2(h,r) M_2(\frac{1}{(1-\bar{a}z)^{3/2}}, r) \\ &\leq C\frac{M_2(h,r)}{1-|a|^2 r^2}. \end{aligned}$$

Integrating over $[0,1]$ the previous estimates and, using the Cauchy-Schwartz inequality, one has

$$|\int_{\mathbb{D}} \frac{\Phi'(z)}{(1-\bar{a}\Phi(z))^{3/2}} \overline{h(z)} dA(z)| \leq C\|\Phi'\|_* \|h\|_{L^2(\mathbb{D})} \frac{1}{(1-|a|^2)^{1/2}}.$$

(3.1) now follows taking supremum over polynomials with $\|h\|_{L^2(\mathbb{D})} \leq 1$.

Let us now show (3.2). Hence, using again (3.3) and (3.4), we obtain

$$\begin{aligned} \int_{\mathbb{D}} \frac{(1-|a|^2)|\Phi''(z)|}{|1-a\bar{\Phi}(z)|^2} dA(z) &\leq (1-|a|^2) \int_0^1 M_{p'}(\frac{1}{(1-\bar{a}\Phi(z))^2}, r) M_p(\Phi'', r) dr \\ &\leq \Big(\int_0^1 M_{p'}^p(\frac{(1-|a|^2)}{(1-\bar{a}\Phi(z))^2}, r) dr\Big)^{1/p} \Big(\int_0^1 M_p^{p'}(\Phi'', r) dr\Big)^{1/p'} \\ &\leq C\Big(\int_0^1 M_{p'}^p(\frac{1-|a|^2}{(1-\bar{a}z)^2}, r) dr\Big)^{1/p} \\ &\leq C\Big(\int_0^1 \frac{(1-|a|^2)^p}{(1-|a|^2 r^2)^{2p-p/p'}} dr\Big)^{1/p} \\ &\leq C(1-|a|^2) \frac{1}{(1-|a|^2)^{2-1/p'-1/p}} \leq C. \end{aligned}$$

□

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