Composition operators on the minimal space invariant under Möbius transformations

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Abstract. It is shown that if $\Phi : \mathbb{D} \rightarrow \mathbb{D}$ is an analytic function such that $M_p(\Phi'' , r) \in L^{p'} (dr)$ for some $1 < p < \infty$ and $1/p + 1/p' = 1$ then $C_{\Phi}(f) = f \circ \Phi$ defines a bounded composition operator on the space $B_1$, the minimal space invariant under Möbius transformations. This was conjectured by J. Arazy, S. Fisher and J. Peetre in [AFP].

1. Introduction

Let us denote by $G$ the group of holomorphic automorphisms on the unit disk $\mathbb{D}$, i.e. the set of functions $\phi \in H(\mathbb{D})$ such that $\phi = \lambda \varphi_a$ for $|\lambda| = 1, |a| < 1$ and

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a}z} = -a + (1 - |a|^2) \sum_{k=1}^{\infty} \bar{a}^k z^k.$$ 

The fact $|\phi'(z)| = \frac{1 - |\phi(z)|^2}{1 - |z|^2}$ for $\phi \in G$ guarantees that the measure $d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^{1/2}}$ is invariant under Möbius transformations, where $dA(z)$ stands for the normalized area measure $dA(z) = \frac{dz \, d\bar{z}}{\pi}$. The paper where a systematic study of spaces of invariant under Möbius transformations was started is [AFP], and we also refer the reader to [AF, F, RT, T1, T2] for further considerations. Although the precise definition may vary from author to author we shall say that a complete space $X \subset H(\mathbb{D})$ with a semi-norm $\rho$ is $G$-invariant (or invariant under Möbius transformations) if for all $f \in X$ and $\phi \in G$ one has that $f \circ \phi \in X$ and there exists $C > 0$ such that

$$\sup_{\phi \in G} \rho(f \circ \phi) \leq C \rho(f).$$

The basic examples of $G$-invariant spaces are the following:

- The space $H^\infty$: The space of bounded analytic functions.

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Note that for $1 \leq p < \infty$ the spaces

$$H^p = \{ f \in \mathcal{H} (\mathbb{D} ) : \sup_{0<r<1} \left( \int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} \right)^{1/p} < \infty \}$$

are not $G$-invariant in our sense.

The reader should be aware that they however become $G$-invariant under the action $f \to (f \circ \phi)(g')^{1/p}$.

- The space $BMOA$:

$$BMOA = \{ f \in \mathcal{H} (\mathbb{D} ) : \| f \|_* = \sup_{|a|<1} \| f \circ \varphi_a - f(a) \|_{H^2} < \infty \}.$$ 

- The Dirichlet space:

$$\mathcal{D}_2 = \{ f \in \mathcal{H} (\mathbb{D} ) : \left( \sum_{n=1}^{\infty} n|a_n|^2 \right)^{1/2} < \infty \}$$

$$= \{ f \in \mathcal{H} (\mathbb{D} ) : f' \in L^2 (\mathbb{D} , dA) \}.$$ 

- The Besov spaces for $1 < p < \infty$:

$$B_p = \{ f \in \mathcal{H} (\mathbb{D} ) : \left( \int_{\mathbb{D} } |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) \right)^{1/p} < \infty \}.$$ 

In particular $B_2 = \mathcal{D}_2$.

- The Bloch space: $\mathcal{B} = \{ f \in \mathcal{H} (\mathbb{D} ) : \sup_{|z|<1} (1 - |z|^2) |f'(z)| < \infty \}.$

We write $\| f \|_\mathcal{B} = \max \{ \| f(0) \| , \sup_{|z|<1} (1 - |z|^2) |f'(z)| \}$.

Let us also consider the invariant pairing on $\mathbb{D}$ given by

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{|z|<r} f'(z) \overline{g'(z)} dA(z).$$

Under such a pairing one has, for each $\phi \in G$,

$$\langle f \circ \phi, g \circ \phi \rangle = \langle f, g \rangle.$$ 

Using the Bergman projection one sees that

$$(1.2) \quad \langle f, \varphi_a \rangle = (1 - |a|^2) f'(a).$$

It is not difficult to see that the previously mentioned examples are $G$-invariant.

Note, for instance, that (1.2) implies that $f \in B_p$ if and only if

$$\int_{\mathbb{D} } |\langle f, \varphi_z \rangle|^p d\lambda (z) < \infty$$

and $f \in \mathcal{B}$ if and only if $\sup_{|z|<1} |\langle f, \varphi_z \rangle| < \infty$.

By the work by Rubel and Timoney (see [RT]) one has that $\mathcal{B}$ becomes a maximal space among the decent $G$-invariant ones. We say that a $G$-invariant space $X$ is "decent" if

$$(1.3) \quad \text{There exists } 0 \neq x^* \in X^* \text{ which is also continuous in } \mathcal{H} (\mathbb{D}) .$$

For decent $G$-invariant spaces (see [RT, F]) one has that $X \subset \mathcal{B}$ continuously.

To find out which is the corresponding limiting case of the Besov spaces $B_p$ for $p = 1$ just recall the following well-known facts (see [Z]): Let $1 < p < \infty$ and denote $M_p (f, r) = \left( \int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} \right)^{1/p}$. The following are equivalent:

(i) $f \in B_p$,

(ii) $\int_0^1 M_p (f', r)(1 - r)^{p-2} dr < \infty$. 

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(iii) \( \int_0^1 M_p^p(f'', r)(1 - r)^{p-1} dr < \infty. \)

In the case \( p = 1 \) (iii) becomes \( \int_0^1 M_1(f'', r) dr < \infty. \) Thus one defines

\[ B_1 = \{ f \in \mathcal{H}(\mathbb{D}) : \rho_1(f) = \int_{\mathbb{D}} |f''(z)| dA(z) < \infty \}. \]

We can define a norm by considering

\[ \|f\|_{B_1} = \max\{|f(0)|, |f'(0)|, \rho_1(f)\}. \]

This space is known to be minimal among \( G \)-invariant spaces with some extra properties (see [AFP, T2]). We shall see here that this is also the case when assuming certain measurability condition on the map \( \phi \to f \circ \phi \) from \( G \) to \( X \).

The reader should be aware that the space \( B_1 \) was denoted by \( \mathcal{M} \) in [AFP] and coincides with the space consisting in those functions \( f \in \mathcal{H}(\mathbb{D}) \) such that \( f = \sum_{k=1}^{\infty} \lambda_k \varphi_{a_k} \) where \( |a_k| < 1 \) and \( \sum_k |\lambda_k| < \infty. \)

It was shown in [AFP, Theorems 18 and 19] respectively that

\[ \Phi'' \in H^1 \implies C_\Phi : B_1 \to B_1 \text{ is bounded} \]

and

\[ \sup_{\theta} |\Phi''(re^{i\theta})| \in L^1(dr) \implies C_\Phi : B_1 \to B_1 \text{ is bounded}. \]

Observe that (1.4) and (1.5) means \( M_1(\Phi'', r) \in L^\infty(dr) \) and \( M_\infty(\Phi'', r) \in L^1(dr) \) respectively. It was conjectured in the Arazy-Fisher-Peetre paper that \( M_p(\Phi'', r) \in L^p(dr) \) for some \( 1 < p < \infty \) and \( 1/p + 1/p' = 1 \) should be sufficient for \( C_\Phi \) to be bounded on \( B_1 \).

In this direction it was even shown in [AFP, Theorem 20] that \( C_\Phi \) was bounded on \( B_1 \) whenever \( M_p(\Phi'', r) \in L^s(dr) \) for \( s = \frac{2p}{p+1} \) (note that in this case \( L^s(dr) \subset L^{p'}(dr) \)).

Our main result is Theorem 3.7 where we show that the conjecture is true.

The paper is organized as follows: Section 1 contains some basic facts on \( B_1 \), in particular that \( B_1 \subset X \) for a relatively wide class of \( G \)-invariant spaces. In Section 2 we apply a general theorem on the characterization of the boundedness of operators from \( B_1 \) into a Banach space to the particular case of composition operators from \( B_1 \) into \( B_1 \) and give a proof of the conjecture mentioned above.

As usual \( p' \) stands for the conjugate exponent of \( p \), \( 1/p + 1/p' = 1 \) and \( C \) denotes a constant that may vary from line to line.

2. The minimal space invariant under Möbius transformations

We may consider \( G \subset T \times \mathbb{D} \) by the mapping \( (\lambda, a) \to \phi = \lambda \varphi_a \) or as a subspace of \( H^\infty(\mathbb{D}) \) or simply \( G \subset H(\mathbb{D}) \) with the locally convex topology of the convergence over compact sets. Let us mention that all these topologies on \( G \) are actually equivalent.

Proposition 2.1. Let \( \phi_n = \lambda_n \varphi_{a_n} \) and \( \phi = \lambda \varphi_a \) for some \( |\lambda_n| = |\lambda| = 1 \) and \( a, a_n \in \mathbb{D} \). The following are equivalent:

1. \( \phi_n(z) \) converges to \( \phi(z) \) for all \( z \in \mathbb{D} \).
2. \( \lambda_n \) converges to \( \lambda \) and \( a_n \) converges to \( a \).
3. \( \phi_n \) converges to \( \phi \) in \( H^\infty \).
4. \( \phi_n \) converges to \( \phi \) in \( H(\mathbb{D}) \), i.e. uniformly on compact subsets of \( \mathbb{D} \).
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PROOF. (1) $\Rightarrow$ (2) Assume that $\phi_n(z)$ converges to $\phi(z)$ for all $z \in \mathbb{D}$.

Note that $\lambda_n\varphi_{a_n}(z) \to \lambda\varphi_a(z)$ is equivalent to $\lambda\lambda_n\varphi_{a_n}((\varphi_a)^{-1}(w)) \to w$ for all $w \in \mathbb{D}$. For $w = 0$ one gets $\varphi_{a_n}(a) \to 0$ and then $a_n \to a$. In particular now $\varphi_{a_n}(z) \to \varphi(z)$ for all $z \in \mathbb{D}$ which together with $\phi_n(z) \to \phi(z)$ implies that $\lambda_n \to \lambda$.

(2) $\Rightarrow$ (3)

$|\phi_n(z) - \phi(z)| = |(\lambda_n - \lambda)\varphi_{a_n} + \lambda(\varphi_{a_n} - \varphi_a)|$

$\leq |\lambda_n - \lambda| + |\varphi_{a_n} - \varphi_a|$

$\leq |\lambda_n - \lambda| + \frac{2|a_n - a| + |\bar{a}_n - a|}{(1 - |a_n|)(1 - |a|)}$

Hence $\|\phi_n - \phi\|_\infty \to 0$.

(3) $\Rightarrow$ (4) Immediate.

(4) $\Rightarrow$ (1) Immediate. $\square$

Let us now give one characterization of the space $B_1$ (see [AFP]). Let us point out the following easy fact that we shall need for such a purpose.

PROPOSITION 2.2. If $f \in B_1$ and $f'(0) = 0$ then $F(z) = zf(z) \in B_1$ and $\rho_1(F) \leq 3\rho_1(f)$.

PROOF. $F'(z) = f(z) + zf'(z)$ and $F'''(z) = 2f'(z) + zf''(z)$.

Therefore it suffices to see that $\int_D |F'(z)|dA(z) \leq \int_D |F'''(z)|dA(z)$.

Since $f'(re^{i\theta}) = \int_0^1 f''(rse^{i\theta})ds$, we can conclude that $M_1(f', r) \leq \int_0^1 M_1(f'', rs)ds$ and

$\int_D |F'(z)|dA(z) = \int_0^1 M_1(f', r)rdr \leq \int_0^1 \int_0^1 M_1(f'', rs)rdrds \leq \int_D |f''(z)|dA(z)$. $\square$

THEOREM 2.3. Let $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = f'(0) = 0$. $f \in B_1$ if and only if there exists a complex Borel measure $\nu$ of bounded variation on $\mathbb{D}$ such that

$f(z) = \int_D \varphi_a(z)d\nu(a)$.

Moreover,

$\rho_1(f) \approx \inf\{\|\nu\|_1 : f = \int_D \varphi_a d\nu(a)\}$

PROOF. Assume $\nu$ is a measure of bounded variation with

$f(z) = \int_D \varphi_a(z)d\nu(a)$.

Then

$f''(z) = \int_D \varphi_a''(z)d\nu(a)$.

It suffices to use the standard estimate (see [Z, Page 53])

(2.1) $\|\varphi_a''\|_{L^1} = \int_D 2|a||1 - |a|^2|dA(z) \leq C$

and Fubini’s theorem to conclude that $\int_D |f''(z)|dA(z) \leq C\|\nu\|_1$. This shows that $\rho_1(f) \leq C\inf\{\|\nu\|_1\}$. 

Conversely, observe that if \( F(z) = \sum_{n=0}^{\infty} a_n z^n \) belong to \( L^1(\mathbb{D}) \) then
\[
\int_{\mathbb{D}} \frac{F(a)}{1 - a z} dA(a) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^n.
\]
This shows that
\[
\int_{\mathbb{D}} F(a) \varphi_a(z) dA(a) = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n(n+1)} z^n.
\]
Now if \( f(z) = \sum_{n=2}^{\infty} b_n z^n \in B_1 \) we define \( G(z) = z f(z) = \sum_{n=1}^{\infty} b_{n+1} z^{n+2} \). Hence, applying the formula above to
\[
F(z) = G''(z) = \sum_{n=1}^{\infty} (n+2)(n+1) b_{n+1} z^n
\]
we obtain
\[
f(z) = \int_{\mathbb{D}} \varphi_a(z) G''(a) dA(a).
\]
Using Proposition 2.2 one has that \( G \in B_1 \) and \( \rho_1(G) \leq 3 \rho_1(f) \). Taking \( d\nu(a) = G''(a) dA(a) \) one gets \( \|\nu\|_1 = \rho_1(G) \leq 3 \rho_1(f) \). \( \square \)

**Proposition 2.4.** \( B_1 \) is \( G \)-invariant and \( a \to \varphi_a \) is continuous from \( \mathbb{D} \) to \( B_1 \).

**Proof.** Let \( f \in B_1 \) with \( f(0) = f'(0) = 0 \) and \( \phi = \lambda \varphi_b \in G \). Using Theorem 2.3 one can write \( f = \int_{\mathbb{D}} \varphi_a(z) d\nu(a) \) for some \( \nu \) with \( \|\nu\|_1 \leq C \rho_1(f) \). Hence
\[
f \circ \phi(z) = \int_{\mathbb{D}} \varphi_a(\phi(z)) d\nu(a) = \int_{\mathbb{D}} \lambda \varphi_b(\varphi_a(z)) d\nu(a) = \int_{\mathbb{D}} \lambda \varphi^{-1}_{\varphi_a(\phi)}(a) d\nu(a).
\]
Now take the second derivative and use (2.1) to get that \( \sup_{|z| < 1} \| (\varphi_a)'' \|_{L^1(\mathbb{D})} < \infty \) and \( \rho_1(f \circ \phi) \leq C \rho_1(f) \).

Note also that if \( a_n \to a \) then \( \varphi''_{a_n}(z) \to \varphi''_a(z) \) for all \( z \in \mathbb{D} \). Now applying the dominated convergence theorem one concludes \( \rho_1(\varphi_{a_n} - \varphi_a) \to 0 \). \( \square \)

It was shown in [AFP] that \((B_1)^* = B\). Let us now show the minimal character of \( B_1 \). The reader should note that we did not assume the map \( \phi \to f \circ \phi \) to be continuous from \( G \) to \( X \) in the definition of \( G \)-invariant. This allowed to have more examples, as \( B \) or \( H^\infty \), in this category.

**Proposition 2.5.** Let \( (X, \|\cdot\|) \) be a non-trivial (i.e there exists \( f \in X \) non constant) \( G \)-invariant Banach space.

1. If the map \( \Gamma_f : G \to X \) defined by \( \phi \to f \circ \phi \) is Borel measurable and bounded for all \( f \in X \) then \( X \) contains the space of polynomials, \( B_1 \subset X \) and there exists \( C > 0 \) such that \( \|f\|_X \leq C \|f\|_{B_1} \) for all \( f \in B_1 \).

2. If the map \( \Gamma_f : G \to X \) defined by \( \phi \to f \circ \phi \) is continuous for all \( f \in X \) then the space of polynomials is dense in \( X \).

**Proof.** (1) Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in X \) be non constant, that is \( a_k \neq 0 \) for some \( k \geq 1 \). Consider the bounded measurable map \( T \to X \) defined by \( e^{it} \to f_t(z) = f(e^{it} z) \). Now one can use the Bochner integral to obtain that, for \( n \geq 0 \),
\[
\int_0^{2\pi} f_t e^{-int} dt = a_n u_n \in X
\]
where \( u_n(z) = z^n \). Therefore \( u_k \in X \).
Now we can conclude that \((\varphi_a(z))^k \in X\) for all \(|a| < 1\). We can repeat the previous argument for \(f(z) = (2^{n-1}z)^k\) whose Taylor coefficients are all different from zero to get \(u_n \in X\) for all \(n \geq 0\).

Note that if \(f\) is a polynomial with \(f(0) = f'(0) = 0\) and \(\nu\) a measure of bounded variation such that \(f = \int_D \varphi_a d\nu(a)\). Using that \(a \to \varphi_a\) is bounded and measurable with values in \(X\) one obtains

\[
\|f\|_X \leq \int_D \|\varphi_a\|_X d|\nu| \leq \sup_{|a| < 1} \|\varphi_a\|_1.
\]

This shows that \(\|f\|_X \leq C \rho_1(f)\) for all polynomial \(f\) with \(f(0) = f'(0) = 0\). Hence, for a general polynomial \(f\), writing \(f = (f - f(0) - f'(0)z) + f(0) + f'(0)z\) one gets

\[
\|f\|_X \leq \|f - f(0) - f'(0)z\|_X + |f(0)||u_0\|_X + |f'(0)||u_1\|_X \leq C\|f\|_{B_1}.
\]

Now extend the result for functions in \(B_1\) using the density of polynomials in \(B_1\).

(2) Assume \(\phi \to f \circ \phi\) is continuous from \(G \to X\). Denote \(f_t(z) = f(rz)\) for \(0 < r < 1\) and observe that

\[
f_t(z) - f(z) = \int_0^{2\pi} \left( f(e^{it}z) - f(z) \right) P_t(e^{-it}) \frac{dt}{2\pi}
\]

where, as usual, \(P_t(e^{it})\) stands for the Poisson kernel. This shows that

\[
\|f_t - f\|_X \leq \int_0^{2\pi} \|f_t - f\|_X P_t(e^{-it}) \frac{dt}{2\pi}.
\]

Using that \(e^{it} \to f_t\) is continuous standard arguments imply that \(f_t\) converges to \(f\) in \(X\). Using that polynomials are dense in \(B_1\) and \(f_t \in B_1\) for each \(0 < r < 1\) one shows the density of polynomials in \(X\).

\(\square\)

3. Operators on \(B_1\)

**Definition 3.1.** Let \(Y\) be a complex Banach space and \(F : \mathbb{D} \to Y\) be analytic function. \(F\) is said to be a vector-valued Bloch function, say \(F \in B(Y)\), if

\[
\sup_{|a| < 1} (1 - |a|^2)\|F'(a)\|_Y < \infty.
\]

Write the norm \(\|F\|_{B(Y)} = \|F(0)\| + \sup_{|a| < 1} (1 - |a|^2)\|F'(a)\|_Y\).

**Theorem 3.2.** Let \(Y\) be a complex Banach space and let \(T : B_1 \to Y\) be a linear operator. Denote \(x_n = T(u_n)\) for \(u_n(z) = z^n, n \geq 0\), and assume that

\[
\limsup \sqrt[n]{\|x_n\|} \leq 1.
\]

The following are equivalent:

1. \(T\) is bounded.
2. \(g_T(a) = T(\varphi_a)\) is bounded and continuous from \(\mathbb{D}\) to \(Y\).
3. \(F_T(a) = \sum_{n=0}^{\infty} \frac{2^n}{n!} a^{n+1} \in B(Y)\).

Moreover

\[
\|T\| \approx \sup_{|a| < 1} \|g_T(a)\|_Y \approx \|F_T\|_{B(Y)}.
\]

**Proof.** (1) \(\implies\) (2) Since \(g_T(a) = T(\varphi_a)\) the result follows by composing \(T \circ J\) where \(J : \mathbb{D} \to B_1\) is the the continuous map given by \(a \to \varphi_a\) according to Proposition 2.4.
Let $f$ be a polynomial. Hence one has $f - f(0) - f'(0)z = \int_D \varphi_a d\nu(a)$ for some measure $\nu$. Now, using linearity,

$$T(f) = f(0)x_0 + f'(0)x_1 + \int_D g_T(a) d\nu(a).$$

Now from the assumption one obtains

$$\|T(f)\| \leq |f(0)||x_0| + |f'(0)||x_1| + \sup_{|a|<1} \|g_T(a)\||\nu||1.$$

This gives $\|T(f)\| \leq C\|f\|_{B_1}$ for any polynomial. Now use the density of polynomials in $B_1$ to extend to a bounded operator from $B_1$ into $Y$.

From the assumption on $(x_n)$ the map $F_T$ is holomorphic (at least) on the unit disc and takes values in $Y$. Note that

$$F_T'(a) = \sum_{k=0}^{\infty} T(u_k) a^k = T(u_0) + T(\sum_{k=1}^{\infty} u_k a^k).$$

Since $\varphi_a = -a + (1 - |a|^2) \sum_{k=1}^{\infty} a^k u_k$. This shows that

$$(1 - |a|^2) F_T'(a) = T(\varphi_a) + (\bar{a} + (1 - |a|^2)) T(u_0).$$

Now use that $T$ is bounded and (2.1).

(3) $\implies$ (2) Use the formula

$$g_T(a) = (1 - |a|^2) F_T'(\bar{a}) - (a + (1 - |a|^2)) F(0).$$

**Corollary 3.3.** Let $X$ be a $G$-invariant space and let $\Phi : D \to D$ be a non constant analytic function. Then $C_\Phi : B_1 \to X$ defined by $C_\Phi(f) = f \circ \Phi$ is a bounded operator if and only if

$$\sup_{|a|<1} \|\varphi_a \circ \Phi\|_X < \infty.$$

Let us now apply this result to several cases.

**Corollary 3.4.** Let $\Phi : D \to D$ a non constant analytic function. Then $C_\Phi : B_1 \to D_2$ is bounded if and only if

$$\sup_{|a|<1} \int_D \frac{(1 - |a|^2)^2 n_\Phi(z)}{|1 - az|^4} dA(z) < \infty,$$

where

$$n_\Phi(z) = \# \{ w \in D : \Phi (w) = z \}.$$

In particular $C_\Phi$ is bounded from $B_1$ to $D_2$ for univalent functions $\Phi$.

**Proof.** From Corollary 3.3 the boundedness is characterized by

$$\int_D |\varphi_a(\Phi(z))|^2 |\Phi'(z)|^2 dA(z) < \infty.$$

Now using that $\Phi$ is locally univalent and the usual change of variables formula one has

$$\int_D \frac{(1 - |a|^2)^2}{|1 - az|^4} n_\Phi(z) dA(z) < \infty.$$

**Next result was shown in [AFP, Proposition 17]:**
Theorem 3.5. Let $\Phi : \mathbb{D} \to \mathbb{D}$ be a non constant analytic function.
Then $C_{\Phi} : B_1 \to B_1$ is bounded operator if and only if

\[ \sup_{|a| < 1} \int_D \frac{(1 - |a|^2)}{|1 - az|^3} n_{\Phi}(z) dA(z) < \infty, \]

(3.1)

\[ \sup_{|a| < 1} \int_D \frac{(1 - |a|^2)|\Phi''(z)|}{|1 - a\Phi(z)|^2} dA(z) < \infty. \]

(3.2)

Let us now prove the Arazy-Fisher-Peetre conjecture.

Proposition 3.6. Let $1 < p < \infty$ and $F \in \mathcal{H}(\mathbb{D})$ with $M_p(F', r) \in L^{p'}(dr)$.
Then $F \in BMOA$ and $\|F\|_* \leq C \left( \int_0^1 M_p(F', r) dr \right)^{1/p'}$.

Proof. First notice that, since the map $s \mapsto M_p(F', s)$ is continuous and non-decreasing, one has

\[ M_p(F', r) (1 - r) \leq \int_r^1 M_p(F', s) ds. \]

Hence $M_p(F', r) \in L^{p'}(dr)$ implies $M_p(F', r) = o \left( \frac{1}{(1 - r)^{1/p'}} \right)$ as $r \to 1$. Now use the fact that $M_p(F', r) = O \left( \frac{1}{(1 - r)^{1/p}} \right)$ can be described in terms of Lipschitz functions (see [D]) and then use the result in [BSS] to obtain that $F \in BMOA$.

\[ \square \]

Theorem 3.7. Let $1 < p < \infty$ and $\Phi : \mathbb{D} \to \mathbb{D}$ a non constant analytic function. If $M_p(\Phi'', r) \in L^{p'}(dr)$ then $C_{\Phi} : B_1 \to B_1$ is bounded.

Proof. Let us show that condition (3.1) holds. Recall that

\[ \int_D \frac{n_{\Phi}(z)}{|1 - az|^3} dA(z) = \int_D \frac{|\Phi'(z)|^2}{|1 - a\Phi(z)|^2} dA(z). \]

Given a polynomial $h$ we write

\[ \int_D \frac{\Phi'(z)}{(1 - a\Phi(z))^{3/2}} h(z) dA(z) = \int_0^{2\pi} \int_0^1 \frac{\Phi'(re^{i\theta})}{(1 - a\Phi(re^{i\theta}))^{3/2}} h(re^{i\theta}) rdr d\theta. \]

From Proposition 3.6 and the duality $(H^1)^* = BMOA$ (see [Z]), we have

\[ \int \frac{\Phi'(z)}{(1 - a\Phi(z))^{3/2}} h(z) dA(z) \leq \int_0^1 \|\Phi\|_* M_1 \left( \frac{h}{(1 - a\Phi)^{3/2}}, r \right) dr. \]

We now recall that Littlewood subordination principle (see [Z, Theorem 10.1.3]) implies that for $1 \leq q < \infty$ and $\alpha > 0$ we have

\[ M_q \left( \frac{1}{(1 - a\Phi(z))^\alpha}, r \right) \leq M_q \left( \frac{1}{(1 - az)^\alpha}, r \right). \]

Also it is well known that if $0 < \alpha, q < \infty$ and $\alpha q > 1$ then there exists a constant $C > 0$ such that

\[ M_q \left( \frac{1}{(1 - az)^\alpha}, r \right) \leq C \frac{1}{(1 - |a|^2r^2)^{\alpha - 1/q}}. \]

(3.4)
Therefore Cauchy-Schwartz, (3.3) and (3.4) give
\[
M_1\left(\frac{h}{(1-\overline{a}\Phi)^{3/2}}, r\right) \leq \frac{1}{M_2(h, r)M_2\left(\frac{1}{(1-\overline{a}\Phi(z))^{3/2}}, r\right)}
\]
\[
\leq M_2(h, r)M_2\left(\frac{1}{(1-az)^{3/2}}, r\right)
\]
\[
\leq C M_2(h, r)\frac{1}{1-|a|^{2r^2}}.
\]

Integrating over [0, 1] the previous estimates and, using the Cauchy-Schwartz inequality, one has
\[
|\int_{D} \Phi'(z) \frac{\Phi'(z)}{(1-\overline{a}\Phi(z))^{3/2}} dA(z)| \leq C\|\Phi'\|_\infty\|h\|_{L^2(D)} \frac{1}{(1-|a|^2)^{1/2}}.
\]

(3.1) now follows taking supremum over polynomials with \(\|h\|_{L^2(D)} \leq 1\).

Let us now show (3.2). Hence, using again (3.3) and (3.4), we obtain
\[
\left(\int_0^1 M_p^p\left(\frac{1-|a|^2}{(1-\overline{a}\Phi(z))^{2}}, r\right) dr\right)^{1/p'} \leq C\left(\int_0^1 M_p^p(\Phi'', r) dr\right)^{1/p'}
\]
\[
\leq C\left(\int_0^1 \frac{1}{(1-\overline{a}\Phi(z))^2} dA(z)\right)^{1/p'}
\]
\[
\leq C\left(\int_0^1 \frac{1}{(1-\overline{a}\Phi(z))^2} dA(z)\right)^{1/p'}
\]
\[
\leq C(1-|a|^2)^{1-1/p-1/p'} \leq C.
\]

References


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