$\mathbf{L}^p$ continuity of projectors of weighted harmonic Bergman spaces

Oscar Blasco*and Salvador Pérez-Esteva

Abstract

In this paper we study spaces $A^p(w)$ consisting of harmonic functions in B^n the unit ball in $\mathbb{R}^n$ and belonging to $L^p(w)$, where $dw(x) = w(1-|x|)dx$ and $w : (0, 1] \to \mathbb{R}^+$ will denote a continuous integrable function. For weights satisfying certain Dini type conditions we construct families of projections of $L^p(w)$ onto $A^p(w)$. We use this to get for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, a duality $A^p(w)^* = A^{p'}(w')$, where w' depends on p and w.

1. Introduction and preliminaries. Let us denote by $A^p(w)$ the spaces consisting of harmonic functions in B^n the unit ball in $\mathbb{R}^n$ and belonging to $L^p(w) = L^p(w(1-|x|)dx)$, where $w : (0, 1] \to \mathbb{R}^+$ will denote a continuous integrable function. The case $w(t) = t^\alpha$ was studied in [5, 8, 10, 12, 13] for $\alpha > -1$. Projections of $L^p((1-|x|)^\beta dx)$ onto $A^p(t^\beta)$ for different values $\beta > -1$, were defined in [5, 8, 12]. In particular, in [8], families of continuous projections are constructed for every $\beta > -1$ and $p \geq 1$.

In this paper we use the integral operators P_α with kernels $b_\alpha(x, y)$ related to the measure $(1-|x|)^\alpha dx$, $\alpha > -1$, defined originally in [5] for $\alpha \in \mathbb{N}$ and we give conditions (Dini type) on the weight function w related to α and $p > 1$, that make P_α a continuous projection of $L^p(w)$ onto $A^p(w)$. We use this to get for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, a duality $A^p(w)^* = A^{p'}(w')$, where w' depends on p and w.

We shall be using the following notation: for $x, y \in B^n$, we will write $x = Rx'$, $y = ry'$, with $R = |x'|$ and $r = |y'|$.

We denote by $P(x, y)$ the Poisson kernel in B^n,

*Partially supported by Proyecto DGICYT PB95-0261

$$
\begin{aligned}
P(x,y) &= c_n \frac{1-(rR)^2}{(1-2rRx'\cdot y'+r^2R^2)^{n/2}} \\
&= \sum_{k,j} (Rr)^k Y_j^k(x') Y_j^k(y'),
\end{aligned}
$$

being $\left\{Y_j^k\right\}_j$ the real orthonormal basis on S^{n-1} of spherical harmonics of degree k. Hence, any harmonic function on B^n can be written as

$$
f = \sum_{k,j} a_{k,j} Y_j^k
$$

where $a_{k,j} \in \mathbb{C}$ and the convergence is uniform on compacts in B^n.

We define for $\alpha > 0$ and $x, y \in B^n$

$$
b_\alpha(x,y) = \sum_{k,j} \frac{\Gamma(2k+n+\alpha)}{\Gamma(\alpha)\Gamma(2k+n)} (Rr)^k Y_j^k(x') Y_j^k(y') \tag{1}
$$

and the corresponding integral operator

$$
P_\alpha f(x) = \int_{B^n} (1-|y|)^{\alpha-1} b_\alpha(x,y) f(y) dy,
$$

which is well-defined for functions in $L^1(t^{\alpha-1})$.

We shall be using the operators of fractional differentiation $\mathcal{D}^\alpha$ defined by

$$
\mathcal{D}^\alpha \phi(t) = \int_0^t \frac{\phi^{(m+1)}(\tau)}{(t-\tau)^\beta} d\tau,
$$

if $\alpha = m+\beta, m \in \mathbb{N} \cup \{0\}$ and $0<\beta<1$; and $\mathcal{D}^\alpha \phi(t) = \phi^{(\alpha)}(t)$ if $\alpha \in \mathbb{N}$.

Throughout this paper we shall denote $\epsilon(x) = (1-|x|)$, for any $x \in \mathbb{R}^n$ or $x \in \mathbb{R}$ and we adopt the convention that C will denote a generic positive constant that may change in each occurrence.

2. Reproducing kernels and Projectors.

Definition 1 *Let $\gamma, q > 0$, and w a weight.function.*

1. We shall say that w is a d_γ-weight ($w \in d_\gamma$) if

$$\int_0^t \frac{w(s)}{s^\gamma} ds \leq C \frac{w(t)}{t^{\gamma-1}}.$$

2. w is said to be a b_q-weight ($w \in b_q$), provided

$$\int_t^1 \frac{w(s)}{s^q} ds \leq C \frac{w(t)}{t^{q-1}}.$$

3. We say that $w \in \mathcal{C}$, if $t^a w(t)$ is non-decreasing, for some $a \geq 0$.

The reader is referred to [1, 3, 4, 11] for some properties and uses of these type of weights functions.

Let us first mention some procedures to get examples of such weights whose elementary proofs are left to the reader.

Proposition 1 *Let $\gamma, q > 0$, and w be a weight function.*

(i) *If $t^b w(t)$ is non-increasing for some $b > q - 1$ then $w \in b_q$.*

(ii) *If $t^a w(t)$ is non-decreasing for some $a < 1 - \gamma$ then $w \in d_\gamma$.*

(iii) *If $w(ts) \leq C w(t) w(s)$ and $\frac{w(t)}{t^\gamma} \in L^1[0,1]$ then $w \in d_\gamma$.*

Next we give some basic but useful properties of the weights above.

Proposition 2 *Let $\gamma, q > 0$, and w be a weight.function.*

(i) *If $w \in b_q$, then for some $C > 0$*

$$w(t) \geq C t^{q-1} log \frac{1}{t}, \quad t \in (0, 1]. \tag{2}$$

(ii) *If $w \in d_\gamma \cap b_q$ then $q > \gamma$.*

(iii) *Let $w \in \mathcal{C} \cap\, b_q$ for some $q > 0$. Then there exists $C > 0$ such that*

$$w(2t) \leq C w(t). \tag{3}$$

provided $0 < t < \frac{1}{2}$.

Proof. (i):

By the continuity of w, there exists $C_1 > 0$, such that $w(t) \geq C_1 t^{q-1}$ for $t \in [\frac{1}{2}, 1]$. Also, if $t \leq 1/2$,

$$\int_{1/2}^{1} w(s)ds \leq \int_{t}^{1} \frac{w(s)}{s^q} ds \leq C \frac{w(t)}{t^{q-1}}.$$

Thus, $\frac{w(t)}{t^q} \geq \frac{C}{t}$, and (i) follows after integrating this expression.

(ii): Is an easy consequence of (i).

(iii) Since $w \in b_q$ implies that $w \in b_p$ for any $p > q$, we can assume that $t^a w(t)$ is non-decreasing, for some $a > 0$ such that $a + q > 1$. Now, if $t \leq 1/2$, then $\frac{1}{t^{a+q-1}} - 1 \geq \frac{C}{t^{a+q-1}}$, hence

$$\begin{aligned} \frac{w(t)}{t^{q-1}} &= \frac{t^a w(t)}{t^{a+q-1}} \leq C t^a w(t) \int_t^1 \frac{1}{s^{q+a}} ds \\ &\leq C \int_t^1 \frac{w(s)}{s^q} ds \leq C \frac{w(t)}{t^{q-1}}, \end{aligned}$$

that is,

$$\frac{w(t)}{t^{q-1}} \sim \int_t^1 \frac{w(s)}{s^q} ds.$$

Then, for $t \in (0, 1/4)$,

$$w(2t) \leq C t^{q-1} \int_{2t}^1 \frac{w(s)}{s^q} ds \leq C t^{q-1} \int_t^1 \frac{w(s)}{s^q} ds \leq C w(t).$$

If $t \in [1/4, 1/2]$,

$$w(2t) \leq \frac{max_{1/2 \leq s \leq 1} w(s)}{min_{1/4 \leq s \leq 1/2} w(s)} w(t).$$

■

Lemma 1 a) *Let* $0 < \beta < 1$ *and* $w \in \mathcal{C} \cap d_\gamma$ *for some* $\gamma > 0$. *Then there exists* $C > 0$ *such that*

$$\int_0^t \frac{w(s)}{s^\gamma (t-s)^\beta} dt \leq C \frac{w(t)}{t^{\gamma+\beta-1}}, \qquad t \in (0, 1]. \tag{4}$$

b) *Let $w \in d_\gamma \cap b_q$ for $q > \gamma$.*

If $0 < q_1 < \gamma$ and $q_1 + q_2 \geq q$ then

$$\int_0^1 \frac{w(t)}{t^{q_1}(s+t)^{q_2}} dt \leq C \frac{w(s)}{s^{q_1+q_2-1}}.$$

c) *For $\gamma > -1$ and for any $x' \in S^{n-1}$, $\int_{S^{n-1}} \frac{dy'}{(|x'-y'|+A)^{n+\gamma}} \leq c(\gamma, n) \frac{1}{A^{\gamma+1}}$.*

Proof. *a)* Since $w \in \mathcal{C}$ implies that $w(t/2) \leq 2^a w(t)$, for some $a \geq 0$,

$$\int_0^{t/2} \frac{w(s)}{s^\gamma (t-s)^\beta} ds \leq \frac{2^\beta}{t^\beta} \int_0^{t/2} \frac{w(s)}{s^\gamma} ds \leq C \frac{w(t)}{t^{\gamma+\beta-1}}.$$

On the other hand,

$$\int_{t/2}^t \frac{w(s)}{s^\gamma (t-s)^\beta} ds \leq C w(t) t^a \int_{t/2}^t \frac{ds}{(t-s)^\beta s^{\gamma+a}} \leq C \frac{w(t)}{t^{\gamma+\beta-1}} \int_{1/2}^1 \frac{dt}{(1-s)^\beta s^{\gamma+a}}.$$

b) The proof follows from the estimates

$$\int_0^s \frac{w(t)}{t^{q_1}(s+t)^{q_2}} dt \leq \frac{1}{s^{q_2}} \int_0^s \frac{w(t)}{t^\gamma} t^{\gamma-q_1} dt \leq C \frac{w(s)}{s^{q_1+q_2-1}},$$

$$\int_s^1 \frac{w(t)}{t^{q_1}(s+t)^{q_2}} dt \leq \int_s^1 \frac{w(t)}{t^{q_1+q_2}} dt \leq \frac{1}{s^{q_1+q_2-q}} \int_s^1 \frac{w(t)}{t^q} dt \leq C \frac{w(s)}{s^{q_1+q_2-1}}.$$

c) The proof is immersed in [5], we include it here for completeness: using spherical coordinates, we parametrize S^{n-1} on a cube $Q \subset \mathbb{R}^{n-1}$. Then

$$\int_{S^{n-1}} \frac{dy'}{(|x'-y'|+A)^{n+\gamma}} \leq C \int_Q \frac{d\xi}{(|y'(\xi_0)-y'(\xi)|+A)^{n+\gamma}}$$

where $y'(\xi_0) = x'$. Since $|y'(\xi_0) - y'(\xi)|_{\mathbf{R}^n} \sim |\xi_0 - \xi|_{\mathbf{R}^{n-1}}$, then

$$\int_Q \frac{d\xi}{(|y'(\xi_0)-y'(\xi)|+A)^{n+\gamma}} \leq C \int_Q \frac{d\xi}{(|\xi_0-\xi|+A)^{n+\gamma}} \leq C \int_{\mathbb{R}^{n-1}} \frac{d\xi}{(|\xi|+A)^{n+\gamma}}$$

$$= C \int_0^\infty \frac{r^{n-2}}{(r+A)^{n+\gamma}} dr = C \left(\frac{1}{A^{\gamma+1}} \right).$$

■

Lemma 2 *Let $\alpha = m + \beta > 0$, with $m \in \mathbb{N} \cup \{0\}$ and $0 \leq \beta < 1$, then*

$$b_\alpha(x, \rho^2 y') = C(\alpha)\rho^{1-n}\mathcal{D}^\alpha \left[\rho^{n-1+\alpha}P(x, \rho^2 y')\right].$$

Proof. An easy calculation shows that

$$\mathcal{D}^\alpha r^\gamma = \frac{\Gamma(\gamma+1)\,\Gamma(1-\beta)}{\Gamma(\gamma-\alpha+1)} r^{\gamma-\alpha}. \tag{5}$$

By (5), $\mathcal{D}^\alpha \rho^{n-1+\alpha+2k} = \Gamma(1-\beta)\frac{\Gamma(n+\alpha+2k)}{\Gamma(n+2k)}\rho^{n-1+2k}$. Then the proof follows after expanding $P(x, \rho^2 y')$. ■

Theorem 1 *Let $\alpha > 0$ and $w \in \mathcal{C} \cap d_\gamma \cap b_q$.*
If $\gamma + \alpha \geq q$, then

$$\int_{B^n} \frac{|b_\alpha(x,y)|}{(1-|y|)^\gamma} w(1-|y|)dy \leq C \frac{w(1-|x|)}{(1-|x|)^{\gamma+\alpha-1}}.$$

Proof. Write $\alpha = m + \beta$, with $m \in \mathbb{N} \cup \{0\}$ and $0 \leq \beta < 1$. Assume that $\beta > 0$ (the case $\beta = 0$ is similar).

$$\int_{B^n} \frac{|b_\alpha(x,y)|}{(1-|y|)^\gamma} w(1-|y|)dy = \int_{\frac{1}{4}B^n} + \int_{B^n \setminus \frac{1}{4}B^n} = I_1 + I_2.$$

Notice that Proposition 2 (i) implies that $\frac{w(t)}{t^{\gamma+\alpha-1}}$ is bounded below by a positive constant. Then, since $b_\alpha(x,y)$ is uniformly bounded in $B^n \times \frac{1}{4}B^n$, it is enough to estimate I_2. Let $a > 0$ such that $t^a w(t)$ is non-decreasing, then Proposition 2 (iii) clearly gives $w(1-r^2) \leq Cw(1-r)$, and therefore

$$\begin{aligned}
I_2 &= C\int_{1/4}^1 \int_{S^{n-1}} \frac{|b_\alpha(x, sy')|}{(1-s)^\gamma} w(1-s)s^{n-1}dy'ds \\
&\leq C\int_{1/2}^1 \int_{S^{n-1}} \frac{|b_\alpha(x, r^2y')|}{(1-r^2)^\gamma} w(1-r^2)dy'dr \\
&\leq C\int_{1/2}^1 \int_{S^{n-1}} \frac{|b_\alpha(x, r^2y')|}{(1-r)^\gamma} w\left(1-r\right)dy'dr
\end{aligned}$$

By Lemma 2, this integral equals

$$C\int_{S^{n-1}}\left(\int_{1/2}^{1}\frac{w(1-r)}{(1-r)^{\gamma}}\left|r^{1-n}\int_0^r\left(\frac{\partial^{m+1}}{\partial u^{m+1}}u^{n-1+\alpha}P(x,u^2y')\right)\frac{du}{(r-u)^{\beta}}\right|dr\right)\,dy'.$$

As in the begining of the proof, it suffices to estimate

$$C\int_{S^{n-1}}\int_{1/2}^{1}\frac{w(1-r)}{(1-r)^{\gamma}}\left|\int_{1/2}^r\left(\frac{\partial^{m+1}}{\partial u^{m+1}}u^{n-1+\alpha}P(x,u^2y')\right)\frac{du}{(r-u)^{\beta}}\right|dr\,dy'. \quad (6)$$

If we let $\widetilde{y}=\frac{y'}{|y|^2}$ for $y\neq 0$, we have the following estimate (see [5]):

$$\left|\frac{\partial^{m+1}}{\partial r^{m+1}}P(x,ry')\right|\leq C\left|x-\widetilde{y}\right|^{-n-m}, \quad (7)$$

for $x\in B^n$ and $r>1/2$. Then by (7),

$$\left|\frac{\partial^{m+1}}{\partial u^{m+1}}u^{n-1+\alpha}P(x,u^2y')\right|\leq C\left|x-\widetilde{u^2y'}\right|^{-n-m}.$$

Now, using elementary geometry we have that $|x-\widetilde{y}|\sim|x'-y'|+\epsilon(x)+\epsilon(y)$, uniformly for $x\in B^n$ and $y\in B^n\backslash\frac{1}{2}B^n$, hence (6) is bounded by a constant multiple of

$$\int_{S^{n-1}}\left(\int_{1/2}^{1}\frac{w(1-r)}{(1-r)^{\gamma}}\left(\int_0^r\left(|x'-y'|+\epsilon(x)+\epsilon(u)\right)^{-n-m}\frac{du}{(r-u)^{\beta}}\right)dr\right)dy'$$

(Lemma 1 (c))

$$\begin{aligned}
&\leq C\int_0^1\frac{w(1-r)}{(1-r)^{\gamma}}\left(\int_0^r\left(\epsilon(x)+\epsilon(u)\right)^{-1-m}\frac{du}{(r-u)^{\beta}}\right)dr\\
&= C\int_0^1\left(\int_u^1\frac{w(1-r)}{(1-r)^{\gamma}(r-u)^{\beta}}dr\right)\left(\epsilon(x)+\epsilon(u)\right)^{-1-m}du\\
&= C\int_0^1\left(\int_0^{1-u}\frac{w(s)}{s^{\gamma}(\epsilon(u)-s)^{\beta}}ds\right)\left(\epsilon(x)+\epsilon(u)\right)^{-1-m}du
\end{aligned}$$

(Lemma 1 (a))

$$\leq C\int_0^1\frac{w(1-u)}{\epsilon(u)^{\gamma+\beta-1}\left(\epsilon(x)+\epsilon(u)\right)^{m+1}}du$$

(Lemma 1 (b))

$$\leq C\frac{w(1-|x|)}{(1-|x|)^{\gamma+\alpha-1}}.$$

■

Corollary 1 *If* $0<\rho<\alpha$, *then*

$$\int_{B_n}|b_\alpha(x,y)|(1-|y|)^{-\rho+\alpha-1}dy \leq C(1-|x|)^{-\rho}.$$

Proof. Take $w(t)=t^{\alpha-1}$. Then $w\in d_\beta$ for every $\beta<\alpha$ and $w\in b_q$ if $q>\alpha$. Hence $w\in d_\rho$ and the proof follows from Theorem 1 taking $q=\alpha+\rho/2$.

■

Lemma 3 *(see [12, Proposition 2.1]). Let* $\alpha>0$, *and let* f *be a bounded harmonic function in* B^n. *Then*

$$P_\alpha f(x)=f(x).$$

Theorem 2 *Let* $1<p<\infty$, $0<\gamma<\alpha$ *and* $w\in\mathcal{C}\cap d_\gamma\cap b_q$, *with* $q>\gamma$. *If* $\alpha>\max\{\frac{p'}{p}\gamma, q-\gamma\}$ *then* P_α *can be extended as a continuous projection from* $L^p(w)$ *onto* $A^p(w)$.

Proof. Using Hölder's inequality with the measure $\epsilon(y)^{\alpha-1}dy$,

$$\begin{aligned}
& \int_{B^n}|P_\alpha f(x)|^p\, w(1-|x|)dx \\
\leq & \int_{B^n}\left(\int_{B^n}|b_\alpha(x,y)|\,|f(y)|\,\epsilon(y)^{\alpha-1}dy\right)^p w(1-|x|)dx
\end{aligned}$$

$$\leq \int_{B^n}\left(\int_{B^n}|b_\alpha(x,y)|\,\epsilon(y)^{-\frac{\gamma p'}{p}+\alpha-1}dy\right)^{p/p'}$$

$$\times\left(\int_{B^n}|b_\alpha(x,y)|\,|f(y)|^p\,\epsilon(y)^{\gamma+\alpha-1}dy\right)w(1-|x|)dx.$$

(let $\rho = \frac{\gamma p'}{p}$ in Corollary 1)

$$\begin{aligned} &\leq C\int_{B^n}\left(\int_{B^n}|b_\alpha(x,y)|\,|f(y)|^p\,\epsilon(y)^{-\gamma+\alpha-1}dy\right)\epsilon(x)^{-\gamma}w(1-|x|)dx \\ &= C\int_{B^n}|f(y)|^p\,\epsilon(y)^{-\gamma+\alpha-1}\left(\int_{B^n}|b_\alpha(x,y)|\,\epsilon(x)^{-\gamma}w(1-|x|)dx\right)dy. \end{aligned}$$

Since $w \in \mathcal{C}\cap d_\gamma \cap b_q$ and $\alpha > q-\gamma$, Theorem 1 implies that the last estimate is bounded by

$$C\int_{B^n}|f(y)|^p\,w(1-|y|)dy.$$

■

Next theorem extends the results of continuity in [5, 12], (compare with [8, Theorem 7.3]):

Theorem 3 *For $\beta > 0$, $p > 1$ and $\delta \geq 0$ denote $w_{\beta,\delta}(t) = t^{\beta p-1}\left(\log(\frac{e}{t})\right)^\delta$. If $\beta < \alpha$ then P_α can be extended as a continuous projection from $L^p(w_{\beta,\delta})$ onto $A^p(w_{\beta,\delta})$.*

Proof. An easy calculation shows that $w_{\beta,\delta} \in d_\gamma$ if $\gamma < \beta p$ and $w_{\beta,\delta} \in b_q$ if $q > \beta p$. In particular, if we let $0 < \varepsilon < \min\{\frac{\alpha-\beta}{2}, \frac{\beta p}{p'}\}$, $\gamma = \frac{\beta p}{p'} - \varepsilon$ and $q = \beta p + \varepsilon$, the proof follows from Theorem 2.

■

Given $\alpha > 0$ and a positive weight function w we can represent the dual of $L^p(w)$ as $L^{p'}(w')$, by the pairing $< f, g >= \int_{B^n} f(x)g(x)(1-|x|)^{\alpha-1}dx$, where $w'(t) = \frac{t^{p'(\alpha-1)}}{w(t)^{p'-1}}$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Corollary 2 *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $w \in \mathcal{C}\cap d_\gamma \cap b_q$, with $q > \gamma$. Also let $\alpha > \max\{\gamma, \frac{p'}{p}\gamma, q-\gamma\}$ and $w'(t) = t^{p'(\alpha-1)}/w(t)^{p'-1}$. Then*

$$(A^p(w))^* = A^{p'}(w') \text{ (with equivalent norms)}$$

under the pairing $< f, g >= \int_{B^n} f(x)g(x)(1-|x|)^{\alpha-1}dx$.

Proof.

It is clear that if $f \in A^{p'}(w')$ then $\Phi(g) = \int_{B^n} f(x)g(x)(1-|x|)^{\alpha-1}dx$ defines a functional in $(A^p(w))^*$ and $\|\Phi\| \leq \|f\|_{L^{p'}(w')}$.

Conversely, let $\Phi \in (A^p(w))^*$. Theorem 2 implies that $\Phi_1 = \Phi \circ P_\alpha \in (L^p(w))^*$. Hence there exists a function $f_1 \in L^{p'}(w)$ for which

$$\Phi_1(g) = \int_{B^n} f_1(x)g(x)w(1-|x|)dx$$

for all $g \in L^p(w)$.

Since we clearly have $\Phi_1(g) = \Phi_1(P_\alpha(g))$, then

$$\Phi_1(g) = \int_{B^n} f_1(x)\left(\int_{B^n} b_\alpha(x,y)g(y)(1-|y|)^{\alpha-1}dy\right)w(1-|x|)dx$$

for all $g \in L^p(w)$.

In particular, if $g \in C_c(B^n)$ we have

$$\begin{aligned}\Phi_1(g) &= \int_{B^n}\left(\int_{B^n} b_\alpha(x,y)f_1(x)w(1-|x|)dx\right)g(y)(1-|y|)^{\alpha-1}dy \\ &= \int_{B^n} f(y)g(y)(1-|y|)^{\alpha-1}dy,\end{aligned}$$

where $f(x) = \int_{B^n} b_\alpha(x,y)f_1(y)w(1-|y|)dy$ is a well-defined harmonic function, since $f_1 \in L^1(w)$.

Moreover, since $C_c(B^n)$ is dense in $L^p(w)$, and $\Phi_1 \in (L^p(w))^*$ then $\frac{f(y)(1-|y|)^{\alpha-1}}{w(1-|y|)} \in L^{p'}(w)$, or equivalently $f \in L^{p'}(w')$. Therefore it follows that $f \in A^{p'}(w')$ and represents Φ.

Finally, we need to prove that the correspondence $f \to \Phi$ is one to one:

Observe first that, from Theorem 2 and duality, one easily gets that P_α is also a projection from $L^{p'}(w')$ into $A^{p'}(w')$.

Now assume that $\Phi = 0$ is represented by $f \in A^{p'}(w')$, then for any $g \in C_c(B^n)$,

$$0 = \int_{B^n} f(y)P_\alpha g(y)(1-|y|)^{\alpha-1}dy = \int_{B^n} f(y)g(y)(1-|y|)^{\alpha-1}dy.$$

Once again the density of $C_c(B^n)$ in $L^p(w)$ implies that $f = 0$.

Finally we use the open mapping theorem to obtain that the norms $\|\Phi\|$ and $\|f\|_{L^{p'}(w')}$ are equivalent.

■

References

[1] J.L. Ansorena, O. Blasco, *Characterization of weighted Besov spaces*, Math. Nach. 171(1995), 5-17.

[2] S. Bell, *A duality theorem for harmonic functions*, Michigan Math. J. 29(1982), 123-128.

[3] O. Blasco, *Operators on weighted Bergman spaces and applications*, Duke Math. J. 66 , 1992, 443-467.

[4] O. Blasco, *Multipliers on weighted Besov spaces of analytic functions*, Contemporary. Math. 144 , 1993, 23-33.

[5] R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L^p*, Astérisque 77 (1980), 11-66.

[6] Ch. V. Coffman and J. Cohen, *The duals of harmonic Bergman spaces*, Proc. Amer. Math. Soc., Vol 110, No3, (1990),697-704.

[7] J. Diestel, J.J. Uhl *Vector measures,* Amer. Math. Soc., 1977.

[8] A.E. Djrbashian, F.A. Shamoian, *Topics in the Theory of A^p_α Spaces,* Teubner-Texte zur Mathematik, 1988.

[9] N. Dinculeanu, *Vector measures,* Pergamon Press, New York, 1967.

[10] T.M. Flett, *On the rate of growth of mean values of holomorphic and harmonic functions*, Proc. London Math. Soc., Vol 20, No3, (1976), 749-768.

[11] S. Janson, *Generalization of Lipschitz and application to Hardy spaces and bounded mean oscillation*, Duke Math. J., 47, (1980), 959-982.

[12] S. Pérez-Esteva, *Duality on vector-valued weighted harmonic Bergman spaces*, Studia Math. 118 (1) (1996), 37-47.

[13] A.L. Shields, D.L. Williams *Bounded projections, duality and multipliers in spaces of harmonic functions*, J. Reine Angew. Math., Vol 299/300, (1978), 256-279.

Departamento de Matemáticas
Universidad de Valencia
46100 Burjassot
Valencia
Spain
oblasco@uv.es

Instituto de Matemáticas
Universidad Nacional Autónoma
de México
Unidad Cuernavaca
A.P. 273-3 ADMON. 3
Cuernavaca, Morelos
62251 México
salvador@matcuer.unam.mx