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Boundary values of vector-valued functions in Orlicz-Hardy classes

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Introduction. The aim of this paper is to give a characterization of the "boundary values" of functions belonging to Orlicz-Hardy classes of harmonic and holomorphic functions on the disc, $\mathrm{Har}^{\Phi}(X)$ and $\mathrm{Hol}^{\Phi}(X)$ respectively, in terms of X-valued measures, being X a Banach space. We shall find the above spaces to be isometric to the spaces V_X^{Φ} and $V_{+,X}^{\Phi}$ respectively (see definitions below).

Some questions related to this problem have been considered in [6] when the Banach space X is a Hilbert space H or $\mathscr{L}(H)$. Several results for the case $\Phi(t) = t^p$ when $p > 1$ have been obtained in [1] and [2].

On the other hand, the classical theorems about boundary values remain valid in the vector-valued setting depending on the geometry of the Banach space X. In fact the Radon-Nikodym property and analytic Radon-Nikodym property [2] are the corresponding ones to guarantee the existence of boundary limits almost everywhere for functions in $\mathrm{Har}^{\Phi}(X)$ and $\mathrm{Hol}^{\Phi}(X)$ respectively.

Through this paper Φ will denote a Young function with Δ_2-condition and Ψ its complementary function (see [5] for definitions), X will be a complex Banach space and $(\mathbb{T}, \mathscr{B}, m)$ the Lebesgue measure space on the circle with $m(\mathbb{T}) = 1$.

Definitions and previous lemmas. Let us recall the definition of Orlicz-spaces of functions

$$(1) \qquad L_X^{\Phi} = \left\{ f\colon \mathbb{T} \to X \text{ measurable functions such that } \beta(f, \Phi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(\| f(t) \|)\, dt < +\infty \right\}.$$

Due to the assumptions on Φ, L_X^{Φ} is a vector space and becomes a Banach space with the following norm (see [5])

$$(1') \qquad | f |_{\Phi} = \inf \{ k > 0\colon \beta(f/k, \Phi) \leqq 1 \}.$$

In [7] J. J. Uhl studied a certain generalization of this space in terms of vector-valued measures. He did that for a general measure space and with more general assumptions on Φ. Here let us give the following definition more adequate for our purposes.

(2) $$V_X^\Phi = \left\{G: \mathscr{B} \to X \text{ finitely additive measures with } \beta(G, \Phi) = \sup_\pi \left\{\sum_{E \in \pi} \Phi\left(\frac{\|G(E)\|}{m(E)}\right) m(E)\right\} < +\infty\right\}$$

(where the supremum is taken over all finite partitions π of $\mathbb{T}$ in measurable sets of positive measure).

V_X^Φ becomes a Banach space endowed with the norm

(2′) $$|G|_\Phi = \inf\{k > 0: \beta(G/k, \Phi) \le 1\}.$$

Remark 1. If G belongs to V_X^Φ then G is m-continuous, that is $\lim_{m(E) \to 0} G(E) = 0$, and it has bounded variation.

Both facts follow easily from writing $\|G(E)\| = \int \frac{\|G(E)\|}{m(E)} \chi_E$ and using the scalar-valued result $\int |u(t) v(t)|\, dt \le |u|_\Phi |v|_\Psi$.

Remark 2 (see [7]). If $f \in L_X^\Phi$ then $G(E) = \int_E f(t)\, dt$ is a measure in V_X^Φ and $|G|_\Phi = |f|_\Phi$.

Let us modify a little bit the definition in (2) and consider

$$\beta'(G, \Phi) = \sup_\pi \sum_{E \in \pi} \Phi\left(\frac{|G|(E)}{m(E)}\right) m(E)$$

where $|G|(E)$ represents the variation of E.

It is clear that $\beta(G, \Phi) \le \beta'(G, \Phi)$, but actually we have the following

Lemma 1. $\beta(G, \Phi) = \beta'(G, \Phi)$.

Proof. Let us take a partition π_0 of sets of positive measure and consider E to be one of these sets.

$$\frac{|G|(E)}{m(E)} = \sup_{\pi_E} \frac{1}{m(E)} \sum_{A \in \pi_E} \|G(A)\| = \sup_{\pi_E} \sum_{A \in \pi_E} \frac{m(A)}{m(E)} \frac{\|G(A)\|}{m(A)}$$

where π_E denotes a finite partition of E in sets with positive measure.

By convexity and continuity of Φ we can write

$$\Phi\left(\frac{|G|(E)}{m(E)}\right) \le \sup_{\pi_E} \sum_{A \in \pi_E} \frac{m(A)}{m(E)} \Phi\left(\frac{\|G(A)\|}{m(A)}\right).$$

Therefore

$$\sum_{E \in \pi_0} \Phi\left(\frac{|G|(E)}{m(E)}\right) m(E) \le \sum_{E \in \pi_0} \sup_{\pi_E} \sum_{A \in \pi_E} \Phi\left(\frac{\|G(A)\|}{m(A)}\right) m(A)$$
$$\le \sup_\pi \sum_{B \in \pi} \Phi\left(\frac{\|G(B)\|}{m(B)}\right) m(B) = \beta(G, \Phi).$$

Taking supremum over all partitions we get the result. □

Lemma 2. *If G belongs to V_X^Φ then there exists a function $g \geqq 0$ in L^Φ such that*

$$|G|(E) = \int_E g(t)\,dt \quad \text{for all} \quad E \in \mathscr{B} \tag{3}$$

$$|G|_\Phi = |g|_\Phi. \tag{4}$$

Proof. By Remark 1 we can say that $|G|$ is a positive finite measure which is m-continuous and therefore by using the Radon-Nikodym theorem we find a positive function g in L^1 verifying (3). Now (4) follows from Lemma 1 and Remark 2. □

The main theorems. Let us recall the following definitions for Orlicz-Hardy classes:

$$\operatorname{Har}^\Phi(X) = \left\{F: D \to X \quad \text{harmonic such that} \quad \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \Phi(\|F_r(t)\|)\,dt < +\infty\right\} \tag{5}$$

where $F_r(t) = F(r e^{it})$, and D is the unit disc.

We give the following norm in it

$$|F|_\Phi = \sup_{0<r<1} |F_r|_\Phi. \tag{5'}$$

We shall denote $\operatorname{Hol}^\Phi(X)$ the subspace of it formed only by holomorphic functions.

As usual if we are given a X-valued measure G with bounded variation we can consider the Poisson integral of it as follows:

$$F(r e^{i\theta}) = P(G)(r e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\,dG(t) \tag{6}$$

where P_r stands for the Poisson kernel on the circle $\mathbb{T}$.

Theorem 1. *V_X^Φ is isometric (via Poisson integral) to $\operatorname{Har}^\Phi(X)$.*

Proof. If G is a measure with bounded variation the $F = P(G)$ is a harmonic function and it verifies

$$\|F_r(\theta)\| = \left\|\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\,dG(t)\right\| \leqq \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\,d|G|(t).$$

Hence, according to Lemma 2, we can write

$$\|F_r(\theta)\| \leqq P_r * g(\theta) \quad \text{for some} \quad g \text{ in } L^\Phi. \tag{7}$$

Since $\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\,dt = 1$ for all θ then Jensen's inequality and (9) allow us to do the following computation:

$$\begin{aligned}
\beta(\|F_r\|, \Phi) &\leqq \int_0^{2\pi} \Phi\left(\int_0^{2\pi} P_r(\theta - t)\,g(t)\frac{dt}{2\pi}\right)\frac{d\theta}{2\pi} \\
&\leqq \int_0^{2\pi}\left(\int_0^{2\pi} P_r(\theta - t)\,\Phi(g(t))\frac{dt}{2\pi}\right)\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi}\left(\int_0^{2\pi} P_r(\theta - t)\frac{d\theta}{2\pi}\right)\Phi(g(t))\frac{dt}{2\pi} = \beta(g, \Phi).
\end{aligned}$$

Therefore $|F|_\Phi \leqq |g|_\Phi = |G|_\Phi$.

Conversely, let us take F in $\operatorname{Har}^\Phi(X)$ and let us consider $\{F_r\}$ as a net uniformly bounded in L_X^Φ. Now we look at L_X^Φ as a subspace of a dual space in the following way: $L_X^\Phi \subseteqq L_{X^{**}}^\Phi \subseteqq V_{X^{**}}^\Phi = (L_{X^*}^\Psi)^*$ (See [7] for duality).

Therefore there exist a sequence r_n and a measure G in $V_{X^{**}}^\Phi$ such that F_{r_n} converges to G in the w^*-topology.

Now let us take ξ in X^* with $\|\xi\|_{X^*} = 1$, $0 < s < 1$ and θ in $\mathbb{T}$, and consider the element $\xi P_s(\theta - t) = \eta(t)$ belonging to $L_{X^*}^\Psi$, we can write

$$\int_0^{2\pi} \langle F_{r_n}(t), \xi\rangle P_s(\theta - t)\frac{dt}{2\pi} \xrightarrow[n\to\infty]{} \left\langle \frac{1}{2\pi}\int_0^{2\pi} P_s(\theta - t)\,dG(t), \xi\right\rangle.$$

From this it follows that $F = P(G)$. To show now that the range of G is actually in X, let us observe the following fact.

$$\begin{aligned}
G(E) &= \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \chi_E * P_r(t)\,dG(t) \\
&= \lim_{r\to 1}\int_0^{2\pi}\left(\int_E P_r(\theta - t)\frac{d\theta}{2\pi}\right)dG(t) = \lim_{r\to 1}\int_E\left(\int_0^{2\pi} P_r(\theta - t)\,dG(t)\right)\frac{d\theta}{2\pi}.
\end{aligned}$$

All these limits are a priori in X^{**}, and to justify the use of Fubini's theorem we can apply both members to elements in X^* and use the scalar-valued version. Now notice that the last term is "$\lim \int_E F_r(\theta)\,d\theta$" and therefore $G(E)$ is a limit in X^{**} but of elements in X, so $G(E) \in X$. Finally it is easy to see that $|G|_\Phi \leqq \sup |F_r|_\Phi = |F|_\Phi$. □

Denoting by

$$V_{+,X}^\Phi = \{G \in V_X^\Phi : \hat{G}(n) = 0 \text{ for } n < 0\},$$

where $\hat{G}(n)$ stands for $\frac{1}{2\pi}\int_0^{2\pi} e^{-int}\,dG(t)$, we can establish the following

Corollary 1. $\operatorname{Hol}^\Phi(X) = V_{+,X}^\Phi$ *(Via Poisson integral).*

So far we have found a space to identify the "boundary values" of functions in $\mathrm{Har}^{\Phi}(X)$ and $\mathrm{Hol}^{\Phi}(X)$ without any property on the Banach space X. A different point of view would be to look for conditions on X to make the classical result remains valid, that is any function F from the disc into X has limits at the boundary a.e.

This was studied by Bukhvalov and Danilevich [2] in the particular case $\Phi(t) = t^p$. Here we shall extend their results and we shall use a different approach.

Theorem 2. *X has the Radon-Nikodym property if and only if the Poisson integral is an isometry between* $\mathrm{Har}^{\Phi}(X)$ *and* L^{Φ}_X.

Proof. By Theorem 1 we shall prove that the RNP is equivalent to the fact that any measure in V^{Φ}_X is representable by a function in L^{Φ}_X. Let us suppose X has the RNP and take G in V^{Φ}_X, then from Remark 1 there exists a function f such that $G(E) = \int_E f(t)\,dt$. Moreover the function g in Lemma 2 is actually $\|f(t)\|$ what implies that $f \in L^{\Phi}_X$.

Conversely let us take an operator $T: L^1 \to X$, and according to the formulation of RNP in terms of operators (see [3]) we have to show that T is representable by a function in L^1_X. Consider now $G(E) = T(\chi_E)$. It is immediate that G belongs to V^{Φ}_X and then G is representable and so T is also representable. □

Due to a result like this, the following property was introduced in [2] for holomorphic functions:

Definition. A complex Banach space X is said to have the *analytic Radon-Nikodym property* (ARNP) if every bounded holomorphic function from the disc D into X has limits at the boundary a.e.

They proved that this is equivalent to saying that the Poisson integral is an isometry between $\mathrm{Hol}^p(X)$ and $\{f \in L^p_X : \hat{f}(n) = 0 \text{ for } n < 0\}$ for any $1 \leqq p \leqq \infty$.

Obviously this can be extended to Orlicz spaces.

Corollary 2. *X has the* ARNP *if and only if* $\mathrm{Hol}^{\Phi}(X) = \{f \in L^{\Phi}_X : \hat{f}(n) = 0, n < 0\}$.

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