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Boundary Values of Functions in Vector-Valued Hardy Spaces and Geometry on Banach Spaces*

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The spaces of boundary values of vector-valued functions in Hardy spaces defined by either holomorphic functions on the disk or harmonic functions with maximal function in L^p are characterized in terms of vector-valued measures of bounded p-variation. We extend to the case $p=1$ a characterization of the Radon–Nikodym property based on the existence of limits at the boundary for harmonic functions with maximal function in L^1. In the case $0<p\leqslant 1$ we find the UMD property as the necessary and sufficient condition to make the spaces defined by maximal function and by conjugate Poisson kernel coincide.

INTRODUCTION

In this paper we are concerned with Hardy spaces of vector-valued functions on the disk. Our main objectives are: To extend several definitions of these spaces to the vector-valued setting, to study their relationships depending on the geometry of the Banach space, and to find representations for the boundary values of functions in these different Hardy spaces when we do not require any condition on the Banach space. The paper is divided into three sections. In the first one we show that the Hardy space of B-valued holomorphic functions $H^p_B(D)$ is isometric (via Poisson integral) to certain space of B-valued measures, the so-called measures of bounded p-variation. With this result we can regard the analytic Radon–Nikodym property in an equivalent way, which allows us to give another formulation of this property. The second section is devoted to solving the same question but for B-valued harmonic functions whose maximal function belongs to L^p $(1\leqslant p\leqslant\infty)$. We find now the Radon–Nikodym property as the right condition to make the classical result remain

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valid in the B-valued setting. This formulation allows us to extend a Bukhvalov–Danilevich result to the case $p=1$. In the last section we deal with the case $0<p<1$. Several definitions for the space H_B^p as space of B-valued distributions are considered. The main result in this section consists of characterizing the class of Banach spaces B such that the B-valued Hardy spaces defined in terms of maximal functions and by means of the conjugate Poisson kernel coincide. They are the called UMD spaces.

Throughout this paper $(\mathbb{T}, \mathscr{B}, m)$ denotes the Lebesgue measure space on the circle $\mathbb{T}$ with $m(\mathbb{T})=1$, D will be the unit disk, and we shall write either $\int f(t)\,dm(t)$ or $(1/2\pi)\int_0^{2\pi} f(e^{it})\,dt$.

1. Spaces of Holomorphic Functions

Through this section $(B, \|\ \|)$ denotes a *complex* Banach space. Given $0<p<\infty$ we shall denote by $H_B^p(D)$ the space of holomorphic functions $F\colon D\to B$ such that

$$|F|_p = \sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi} \|F(re^{it})\|^p\,dt\right)^{1/p} < +\infty. \tag{1.1}$$

$H_B^\infty(D)$ will be the space of B-valued bounded holomorphic functions and its norm is given by

$$|F|_\infty = \sup_{z\in D} \|F(z)\|. \tag{1.2}$$

Recall that for $1\leqslant p\leqslant\infty$ we have a simple way to build functions in $H_B^p(D)$. This consists of taking a function f in $L_B^p(\mathbb{T})$ whose negative Fourier coefficient vanishes and considering its Poisson integral

$$F(re^{i\theta}) = \int P_r(\theta-t)\,f(t)\,dm(t) = P_r * f(\theta), \tag{1.3}$$

where P_r stands for the Poisson kernel on $\mathbb{T}$. For $1\leqslant p\leqslant\infty$, let us consider the space

$$H_B^p(\mathbb{T}) = \{f\in L_B^p(\mathbb{T})\colon \hat{f}(n)=0\ \forall n<0\}, \tag{1.4}$$

where $\hat{f}(n)=\int f(t)\,e^{-int}\,dm(t)$ is a Bochner integral. It is not difficult to see that if f belongs to $H_B^p(\mathbb{T})$ then its Poisson integral F belongs to $H_B^p(D)$ and $|F|_p=\|f\|_p$, where $\|f\|_p$ denotes the norm of f in $L_B^p(\mathbb{T})$.

The following example shows us that we cannot expect, in general, that every function in $H_B^p(D)$ is the Poisson integral of a function in $H_B^p(\mathbb{T})$.

Example 1.1. Take $B=c_0(\mathbb{N})$ and $F(z)=(z^n)_{n\in\mathbb{N}}$. Denote by $F_r(t)=F(re^{it})$ and assume that $F_r=P_r*f$ for some f in $L_{c_0}^1(\mathbb{T})$. Clearly $\hat{F}_r(k)=\hat{P}_r(k)\cdot\hat{f}(k)$ for all $k\in\mathbb{Z}$, which implies that $\{\hat{f}(k)\}$ is the canonic basis in c_0 and this contradicts the Riemann–Lebesgue lemma since $\|\hat{f}(k)\|_{c_0}=1$ does not go to zero as $k\to\infty$.

From this example two natural questions arise:

(a) To find a larger space containing $L_B^p(\mathbb{T})$ whose elements can be interpreted as "boundary values" of functions in $H_B^p(D)$, and

(b) To characterize the class of Banach spaces for which the Poisson integral is an isometry between $H_B^p(\mathbb{T})$ and $H_B^p(D)$.

The second question was studied by A. V. Bukhvalov and A. A. Danilevich [6]. They called the analytic Radon–Nikodym property the condition on B to satisfy (b). Since then several characterizations of this property have been obtained (see [15, 16]). The answer to the first question will be given in terms of B-valued measures. The reader is referred to [11, 12] for a general treatment of vector measures, but we shall recall here several concepts and results we shall use later on. Let G be a B-valued finitely additive measure on $(\mathbb{T}, \mathscr{B})$ with bounded variation and let ϕ be a continuous function on $\mathbb{T}$, then we consider

$$\int \phi(t)\,dG(t) = \lim \sum_{i=1}^{n} \phi(s_i)\,G([t_{i-1}, t_i]), \tag{1.5}$$

where $0=t_0<t_1<\cdots<t_n=2\pi$, $t_{i-1}<s_i\leqslant t_i$, and the limit is taken as $\max|t_i-t_{i-1}|$ goes to zero. Notice that such a measure G defines an operator T_G from $\mathbb{C}(\mathbb{T})$ into B by setting $T_G(\phi)=\int\phi(t)\,dG(t)$. Let us denote by M_B the space of B-valued regular measures with bounded variation. According to (1.5) we can give sense to the Poisson integral and the Fourier coefficients for measures G in M_B:

$$P(G)(z) = \int P_z(t)\,dG(t), \tag{1.6}$$

where $P_z(t)=P_r(\theta-t)$ being $z=re^{i\theta}$,

$$\hat{G}(n) = \int e^{-int}\,dG(t) \qquad \text{for} \quad n\in\mathbb{Z}. \tag{1.7}$$

Given $1<p<\infty$ and a finitely additive measure G we define the p-variation of G as

$$|G|_p = \sup\left(\sum_{E\in\pi} \frac{\|G(E)\|^p}{m(E)^{p-1}}\right)^{1/p}, \tag{1.8}$$

where the supremum is taken over all finite partitions π of $\mathbb{T}$, where we use the convention $\lambda/0$ equals 0 or ∞ provided $\lambda=0$ or $\lambda>0$.

For $p=\infty$, we define

$$|G|_\infty = \inf\{C: \|G(E)\| \leqslant Cm(E) \text{ for all } E\in\mathscr{B}\}. \tag{1.9}$$

We denote by V_B^p the spaces of measures with bounded p-variation for $1<p\leqslant\infty$.

For measures G in V_B^p we can give sense to $\int\phi(t)\,dG(t)$ not only for continuous functions ϕ but for functions in $L^{p'}(\mathbb{T})$, where $1/p+1/p'=1$. To see that, let us take a simple function $s=\sum_{i=1}^n \lambda_i\kappa_{E_i}$ and define

$$T_G(s)=\int s(t)\,dG(t)=\sum_{i=1}^n \lambda_i G(E_i). \tag{1.10}$$

Notice that

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i G(E_i)\right\| &= \left\|\sum_{i=1}^n \lambda_i m(E_i)^{1/p'} m(E_i)^{-1/p'} G(E_i)\right\| \\ &\leqslant \left(\sum_{i=1}^n |\lambda_i|^{p'} m(E_i)\right)^{1/p'} \left(\sum_{i=1}^n \frac{\|G(E_i)\|^p}{m(E_i)^{p-1}}\right)^{1/p} \\ &\leqslant \|s\|_{p'}\cdot|G|_p.\end{aligned}$$

This simple computation allows us to extend T_G as a bounded operator to $L^{p'}(\mathbb{T})$. For a general study of V^p spaces the reader is referred to [19, 12, 13]. Some of the good properties that these spaces have are reflected in the following:

Remark 1.1. Every measure in V_B^p for $1<p\leqslant\infty$ is countably additive, m-continuous, and with bounded variation.

This allows us to look at $|G|$, the variation of G, as a positive finite m-continuous measure and therefore by the Radon–Nikodym theorem to represent $|G|$ by a positive function g in $L^1(\mathbb{T})$. After this observation it is easy to conclude the following result (see [1]).

PROPOSITION 1.1. *Let $1<p\leqslant\infty$ and let G be a finitely additive measure. G belongs to V_B^p if and only if there exists a positive function g in $L^p(\mathbb{T})$ such that $\|g\|_p=|G|_p$ and it verifies that for all ϕ in $L^{p'}(\mathbb{T})$*

$$\left\|\int\phi(t)\,dG(t)\right\| \leqslant \int g(t)\,|\phi(t)|\,dm(t). \tag{1.11}$$

Before we state the main result of this section, let us formulate a lemma

which is proved in either [5] or [2] for $p=1$. The proof in [2] can be extended for all values of p.

LEMMA 1.1. *Let $0<p<\infty$, and let F be a function in $H_B^p(D)$. Then $F^*(t)=\sup_{0<r<1}\|F(re^{it})\|$ belongs to $L^p(\mathbb{T})$, and*

$$\|F^*\|_p \leqslant C\,|F|_p. \tag{1.12}$$

THEOREM 1.1. *Let $1<p\leqslant\infty$. Then*

$$H_B^1(D)=\{G\in M_B: \hat{G}(n)=0 \text{ for } n<0\}$$

and

$$H_B^p(D)=\{G\in V_B^p: \hat{G}(n)=0 \text{ for } n<0\},$$

where both identifications are by means of the Poisson integral.

Proof. We shall do both parts in a parallel way. Let G belong to M_B (resp. V_B^p), and assume that $\hat{G}(n)=0$ for $n<0$. We define a holomorphic function on the unit disk by

$$F(z)=\sum_{n=0}^\infty \hat{G}(n)\,z^n. \tag{1.13}$$

For any $z=re^{i\theta}$ in the disk, we have

$$P_z(t)=\lim_{n,m\to\infty}\sum_{k=-m}^n r^{|k|}e^{ik(\theta-t)}$$

being the convergence in $\mathbb{C}(\mathbb{T})$ (resp. $L^{p'}(\mathbb{T})$). Therefore from (1.6) we easily deduce that $P(G)=F$. For $p=1$ we can write

$$\|F(re^{i\theta})\| \leqslant \int P_r(\theta-t)\,d\,|G|(t)=P_r*|G|(\theta)$$

and since $|G|$ is a finite measure we get $|F|_1\leqslant|G|(\mathbb{T})$. For $p>1$, use Proposition 1.1 to find g in $L^p(\mathbb{T})$ such that $\|F(re^{i\theta})\|\leqslant P_r*g(\theta)$ and $|G|_p=\|g\|_p$. From this it is clear that $|F|_p\leqslant|G|_p$.

Conversely, let us take any p, $1\leqslant p\leqslant\infty$, and consider a function F in $H_B^p(D)$. For any ξ in B^* the function $\langle\xi, F(z)\rangle$ belongs to $H^p(D)$ and therefore, by the classical result, there exists a function f_ξ in $L^p(\mathbb{T})$ with $\hat{f}(n)=0$ for $n<0$ satisfying

$$\langle\xi, F(re^{i\theta})\rangle=\int P_r(\theta-t)\,f_\xi(t)\,dm(t). \tag{1.14}$$

Now for each measurable set E we can define the following linear map from B^* into $\mathbb{C}$,

$$\langle G(E), \xi \rangle = \int_E f_\xi(t)\, dm(t), \tag{1.15}$$

and since $f_\xi(t) = \lim_{r\to 1}\langle \xi, F(re^{it})\rangle$ a.e. then Fatou's lemma implies

$$|\langle G(E), \xi \rangle| \leq \|\xi\|_{B^*} \cdot \lim_{r\to 1} \int_E \|F(re^{it})\|\, dm(t).$$

Therefore, using the notation $F^*(t)$ as in Lemma 1.1,

$$\|G(E)\|_{B^{**}} \leq \int_E F^*(t)\, dm(t). \tag{1.16}$$

Now from (1.16) it clearly follows that G belongs to $M_{B^{**}}$. Besides if $p>1$ it is easy to see that G belongs to $V^p_{B^{**}}$. To see that the range of G is actually in B, let us prove that for any ϕ in either $C(\mathbb{T})$ or $L^{p'}(\mathbb{T})$ provided $p=1$ or $p>1$ we have that $\int \phi(t)\, dG(t)$ belongs to B. By using the fact that $\phi * P_r$ converges to ϕ as $r\uparrow 1$ in either $\mathbb{C}(\mathbb{T})$ or $L^{p'}(\mathbb{T})$ $(1\leq p'<\infty)$, together with Fubini's theorem, we can write

$$\int \phi(t)\, dG(t) = \lim_{r\uparrow 1} \int (P_r * \phi(t))\, dG(t)$$

$$= \lim_{r\uparrow 1} \int \left(\int P_r(\theta - t)\, dG(t) \right) \phi(\theta)\, d\theta.$$

Therefore if we show that $\int P_r(\theta - t)\, dG(t)$ belongs to B for all $0<r<1$ and θ in $\mathbb{T}$ we shall have the range of B contained in B. But on the other hand we have

$$\int P_r(\theta - t)\, dG(t) = F(re^{i\theta})$$

as the following computation shows: From (1.14) and (1.15) we have that for every ξ in B,

$$\langle \xi, F(re^{i\theta})\rangle = \int P_r(\theta - t) f_\xi(t)\, dm(t)$$

$$= \left\langle \int P_r(\theta - t)\, dG(t), \xi \right\rangle.$$

An analogous argument shows that $\hat{G}(n)=0$ for $n<0$ and this completes the proof.

From the proof of Theorem 1.1, we can establish the following corollary

COROLLARY 1.1. *Let G be a measure in M_B. If $\hat{G}(n)=0$ for $n<0$ then G is m-continuous.*

Let us introduce the following property in a different way from that given in [6].

DEFINITION 1.1. A complex Banach space B is said to have the analytic Radon–Nikodym property (ARNP) if every measure G in M_B with $\hat{G}(n)=0$ for $n<0$ is representable by a function f in $L^1_B(\mathbb{T})$, that is, $G(E) = \int_E f(t)\, dm(t)$ for all E in $\mathscr{B}$.

Now Corollary 1.1 shows that the class of spaces with ARNP is larger than that with the RNP. An example of space with ARNP and without RNP is $L^1(\mathbb{T})$ (see [6, 11]).

From Definition 1.1 and Theorem 1.1 we can rewrite Bukhvalov and Danilevich's result as follows (see [6]).

COROLLARY 1.2. *The following statements are equivalent:*

(a) *B has ARNP.*

(b) *For all p $(1<p\leq\infty)$ every measure G in V^p_B with $\hat{G}(n)=0$ for $n<0$ is representable.*

(c) *For some p $(1<p\leq\infty)$ every measure in V^p_B with $\hat{G}(n)=0$ for $n<0$ is representable.*

(d) *Every measure in V^∞_B with $\hat{G}(n)=0$ for $n<0$ is representable.*

Remark 1.2. According to (1.10), the space V^∞_B can be interpreted as the space of bounded operators from $L^1(\mathbb{T})$ into B, to be denoted by $\mathscr{L}(L^1, B)$, and it is just a computation to show that $\{G\in V^\infty_B : \hat{G}(n)=0$ for $n<0\}$ corresponds to operators T in $\mathscr{L}(L^1, B)$ such that $T = S\cdot q$ being $S: L^1/H^1_0 \to B$ and $q: L^1 \to L^1/H^1_0$ the natural quotient operator (H^1_0 stands for $\{\phi \in L^1(\mathbb{T}) : \hat{\phi}(n)=0$ for $n\geq 0\}$).

From this remark and Corollary 1.2 we can recover the following result [15].

COROLLARY 1.3. *B has the ARNP if and only if $S\cdot q: L^1 \to B$ is representable for each bounded linear operator $S: L^1/H^1_0 \to B$.*

Our next goal is to connect Hardy spaces of vector-valued functions with Hardy spaces on the bi-disk (see [21] for definitions).

PROPOSITION 1.2. *Let $1 \leqslant p < \infty$ and $B = H^p(D)$. Then*

$$H^p_B(D) = H^p(D^2),$$

where $F(z_1, z_2)$ in $H^p(D^2)$ is identified to $z_1 \to F_{z_1}(z_2) = F(z_1, z_2)$ in $H^p_B(D)$.

Proof. Recall that $H^p(D^2)$ can be interpreted as a space at the boundary $\mathbb{T}^2$, that is, the space of functions ψ in $L^p(\mathbb{T}^2)$ such that $\hat{\psi}(n, m) = 0$ for $(n, m) \notin \mathbb{Z}^+ \times \mathbb{Z}^+$.

On the other hand, for $1 \leqslant p < \infty$, $H^p(D)$ has the RNP since it is a separable dual space. Therefore we can identify $H^p_B(D)$ with $H^p_B(\mathbb{T})$. Now according to the identification between L^p-functions with values in $L^p(\mathbb{T})$ and the space $L^p(\mathbb{T}^2)$ we only have to check that if f belongs to $L^p_B(\mathbb{T})$ (here we regard now B as $H^p(\mathbb{T})$) and ψ is the corresponding function in $L^p(\mathbb{T}^2)$ then

$$(\hat{f}(n))^\wedge(m) = \hat{\psi}(n, m). \tag{1.17}$$

Notice that (1.17) follows easily from Fubini's theorem.

We shall finish this section by mentioning another interesting space of B-valued holomorphic functions, $H^p(D) \hat{\otimes} B$, that is, the tensor product with the projective norm. It is a simple computation to see that $H^p(D) \hat{\otimes} B$ is embedded in $H^p_B(D)$. The following result gives a necessary condition for both spaces to coincide.

PROPOSITION 1.3. *Let $1 \leqslant p < \infty$. If $H^p(D) \hat{\otimes} B = H^p_B(D)$ (with equivalent norms) then B has the ARNP.*

Proof. Provided $H^p(D) \hat{\otimes} B = H^p_B(D)$ we have that any function in $H^p_B(D)$ can be approached by functions in $H^p_B(D) \otimes B$. Since the functions in $H^p_B(D) \otimes B$ are Poisson integrals of functions in $H^p(\mathbb{T}) \otimes B \subseteq H^p_B(\mathbb{T})$ we obtain that the Poisson integral is surjective. Therefore B has the ARNP.

Let us give some examples to guarantee that this is not a sufficient condition.

PROPOSITION 1.4. *Let $1 < p \leqslant 2$. Then $B = l^p$ has the ARNP but $H^p_B(D) \hat{\otimes} l^p$ is strictly contained in $H^p_B(D)$.*

Proof. Of course it is clear that l^p has ARNP, since in fact it has RNP. Now since $H^p_B(D) \hat{\otimes} B$ is dense in $H^p_B(D)$ then it suffices to show that $(H^p_B(D))^*$ is strictly contained in $(H^p(D) \hat{\otimes} B)^*$. The fact that any function F in $H^p_B(D)$ can be regarded as a sequence F_n of functions in $H^p(D)$ allows

us to identify $H^p_{l^p}(D)$ with $l^p_{H^p(D)}$. By using this identification and taking into account that $(l^p_B)^* = l^{p'}_{B^*}$ we can write

$$(H^p_{l^p}(D))^* = H^{p'}_{l^{p'}}(D).$$

On the other hand, $(H^p(D) \hat{\otimes} B)^*$ can be identified with $\mathscr{L}(H^p(D), B^*)$. Consequently it is sufficient to find an operator from $H^p_B(\mathbb{T})$ into $l^{p'}$ which cannot be represented by a function f in $H^{p'}_{l^{p'}}(\mathbb{T})$. Let us take $T: H^p(\mathbb{T}) \to l^{p'}$ defined by $T(\phi) = (\hat{\phi}(n))_{n \in \mathbb{Z}}$. A very well known result in Harmonic Analysis says that T is bounded [17], and the no representability of T can be again proved by a simple argument involving the Fourier coefficients as in Example 1.1.

Remark 1.3. The last example does not work for $p = 1$:

$$H^1(D) \hat{\otimes} l^1 = H^1_{l^1}(D). \tag{1.18}$$

The reason for this is that $l^1_B = l^1 \hat{\otimes} B$ for any Banach space, and we also have the identification $H^1_{l^1} = l^1_{H^1}$. A necessary condition for $H^1 \hat{\otimes} B = H^1_B$ is obtained by using duality (see [2]).

2. SPACES OF HARMONIC FUNCTIONS

One of the most important techniques for studying Hardy spaces from a "real point of view" was introduced by D. L. Burkholder, R. F. Grundy, and M. L. Silverstein [8] by considering maximal functions to check whether or not a harmonic function belongs to Re $H^p(D)$. In this section we shall deal with spaces defined by means of maximal functions in the vector-valued case.

Through this section $(B, \| \ \|)$ is allowed to be a real or complex Banach space. For any function F defined on the disk and with values in B we can define the radial maximal function by

$$F^*(t) = \sup_{0 < r < 1} \|F(re^{it})\|. \tag{2.1}$$

Let us mention here that the following development could be done by using non-tangential maximal function [8]. Let us denote by $H^p_{\max,B}(D)$, $0 < p \leqslant \infty$, the space of harmonic functions from D into B with maximal function F^* in $L^p(\mathbb{T})$. We set in this space the norm (p-norm for $p < 1$) given by

$$|F|_{\max,p} = \|F^*\|_p. \tag{2.2}$$

For $1 < p \leqslant \infty$ we can get functions in $H^p_{\max,B}(D)$ just by taking Poisson integrals of functions in $L^p_B(\mathbb{T})$. The reason for that is simply that the Poisson integral of a function f in $L^p_B(\mathbb{T})$ is harmonic and verifies

$$\|F(re^{it})\| \leqslant P_r * \|f\|(t).$$

From this, applying the classical result to $\|f\|$, we get that F belongs to $H^p_{\max,B}(D)$.

In the case $p=1$, not any function f in $L^1_B(\mathbb{T})$ leads us to an element in $H^p_{\max,B}(D)$ by taking its Poisson integral, we have to restrict ourselves to the space

$$H^1_{\max,B}(\mathbb{T}) = \{f \in L^1_B(\mathbb{T}): f^*(t) = \sup_{0<r<1} \|P_r * f(t)\| \in L^1(\mathbb{T})\}.$$

Let us present an example to see that the Poisson integral is not surjective.

EXAMPLE 2.1. Take $B = L^1(\mathbb{T})$ and $F(z) = P_z$. Clearly $F^*(t) = 1$ for all t. Hence F belongs to $H^p_{\max,B}(D)$ for all p. Assume F is the Poisson integral of some f with values in $L^1(\mathbb{T})$. Then we would have that $\hat{f}(n) = \phi_n$ being $\phi_n(t) = e^{int}$, which contradicts the Riemann–Lebesgue lemma.

From this example we can again ask ourselves the same questions as in Section 1: To give a characterization of the space of boundary values of functions belonging to $H^p_{\max,B}(D)$ when B is a general Banach space, and to find the class of spaces where the Poisson integral maps the function spaces $H^1_{\max,B}(\mathbb{T})$ or $L^p_B(\mathbb{T})$ onto $H^p_{\max,B}(D)$.

The following remark tells us that the second question was already answered in [6] for $1 < p \leqslant \infty$. Here we shall give a proof which extends to $p=1$.

Remark 2.1. For $1 < p \leqslant \infty$ the space $H^p_{\max,B}(D)$ coincides with $h^p_B(D)$, the space of B-valued harmonic functions such that

$$\sup_{0<r<1} \int \|F(re^{it})\|^p \, dt < +\infty.$$

Indeed, given F in $h^p_B(D)$ we can consider the subharmonic function $g(z) = \|F(z)\|$. Since $g_r(t) = \|F(re^{it})\|$ are uniformly bounded in $L^p(\mathbb{T})$, then there exists a positive function h in $L^p(\mathbb{T})$ such that $\|F(re^{it})\| \leqslant P_r * h(t)$. This implies that F^* belongs to $L^p(\mathbb{T})$.

The proof in Theorem 1.1 could be slightly modified to get the following result, but here we sketch a proof based on the w^*-compactness of the balls in dual spaces.

THEOREM 2.1. *Let $1 < p \leqslant \infty$. Then*

$$H^p_{\max,B}(D) = V^p_B$$

and

$$H^1_{\max,B}(D) = \left\{ G \in M_B : G^*(\theta) = \sup_{0<r<1} \left\| \int P_r(\theta - t) \, dG(t) \right\| \in L^1(\mathbb{T}) \right\},$$

where both identifications are by means of the Poisson integral and the norms are equivalent.

Proof. Let us begin with the case $1 < p \leqslant \infty$. Let us take a function F in $H^p_{\max,B}(D)$ and write $F_r(t) = F(re^{it})$. We regard F_r as a set of uniformly bounded functions $L^p_B(\mathbb{T})$ and we use the inclusions

$$L^p_B(\mathbb{T}) \subseteq L^p_{B^{**}}(\mathbb{T}) \subseteq V^p_{B^{**}}.$$

Since $V^p_{B^{**}}$ is the dual space of $L^{p'}_{B^*}$ (see [13]) then there exist a measure G in $V^p_{B^{**}}$ and a sequence r_n such that F_{r_n} converges to G in w^*-topology. Now arguments similar to those used in Theorem 1.1 lead us to conclude that $P(G) = F$ and the range of G in B. The case $p=1$ follows in a similar way by using Singer's duality theorem [22] $(C_{B^*}(\mathbb{T}))^* = M_{B^{**}}$.

The converse inclusion is a consequence of either the definition for $p=1$ or Proposition 1.1 for $p>1$.

LEMMA 2.1. *If G belongs to M_B and G^* belongs to $L^1(\mathbb{T})$ then G is m-continuous.*

Proof. Let us take a measurable set E in $\mathscr{B}$. Since G is regular we have that for every $n \in \mathbb{N}$ there exist a compact set K_n and an open set O_n such that

$$K_n \subseteq E \subseteq O_n \quad \text{and} \quad \|G(O_n \backslash K_n)\| < 1/n.$$

Let us consider a continuous function ϕ_n such that $0 < \phi_n \leqslant 1$, $\phi_n(t) = 1$ for $t \in K_n$ and $\phi_n(t) = 0$ for $t \notin O_n$, since $\phi_n * P_r$ converges to ϕ_n in $\mathbb{C}(\mathbb{T})$ as $r \uparrow 1$, then we can write

$$\int \phi_n(t) \, dG(t) = \lim_{r \uparrow 1} \int \phi_n * P_r(t) \, dG(t) = \lim_{r \uparrow 1} \int P(G)(re^{it}) \, \phi_n(t) \, dm(t).$$

From this we get $\|\int \phi_n(t) \, dG(t)\| \leqslant \int \phi_n(t) \, G^*(t) \, dm(t)$ for each $n \in \mathbb{N}$. Now make n go to infinity to obtain

$$\|G(E)\| \leqslant \int_E G^*(t) \, dm(t),$$

which clearly implies that G is m-continuous.

According to Remark 2.1, the next result is a different proof of a result of Bukhvalov and Danilevich [6] as well as an extension of it to the case $p=1$.

THEOREM 2.2. *The following statements are equivalent:*

(a) *B has the RNP.*

(b) *Every function F in $H^1_{\max,B}(D)$ is the Poisson integral of a function f in $H^1_{\max,B}(\mathbb{T})$.*

(c) *For all p, $1<p\leqslant\infty$, every function F in $H^p_{\max,B}(D)$ is the Poisson integral of a function $L^p_B(\mathbb{T})$.*

(d) *For some p, $1<p\leqslant\infty$, every function F in $H^p_{\max,B}(D)$ is the Poisson integral of a function f in $L^p_B(\mathbb{T})$.*

(e) *Every B-valued harmonic function which is bounded has limits at the boundary almost everywhere.*

Proof. The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) are either obvious or just using the inclusions between L^p-spaces and Fatou's lemma.

(a) $\Rightarrow$ (b) Let F be a function in $H^1_{\max,B}(D)$. According to Theorem 2.1 it is the Poisson integral of a measure G in M_B with G^* in $L^1(\mathbb{T})$. Now by Lemma 2.1 this measure is m-continuous and consequently the Radon–Nikodym property of B implies that there is a function f in $L^1_B(\mathbb{T})$ verifying that $G(E)=\int_E f(t)\,dm(t)$. Obviously f belongs to $H^1_{\max,B}(\mathbb{T})$ since $f^*=G^*$.

(e) $\Rightarrow$ (a) Using the characterization of the RNP in terms of operators (see [11, p. 63]) we have to prove that any operator $T: L^1(\mathbb{T})\to B$ is representable by a function. Assume we take such an operator and consider $F(z)=T(P_z)$. It is immediate that F is a bounded harmonic function on the disk. Therefore there exists a function $f(t)=\lim_{r\to 1}F(re^{it})$ t-a.e. Finally, it can be easily checked that F is the Poisson integral of f and that T is represented by f.

3. SPACES OF B-VALUED DISTRIBUTIONS: CASE $0<p<1$

Throughout this section $(B, \|\ \|)$ will denote a real Banach space, p will be a number $0<p<1$, and C will be a constant, not necessarily the same at each occurrence. We shall denote by $C^\infty(\mathbb{T})$ the space of functions in $C^\infty(\mathbb{R})$ with period 2π. Endowing $C^\infty(\mathbb{T})$ with its usual topology we shall call B-valued distributions to the continuous linear maps from $C^\infty(\mathbb{T})$ to B. $\mathscr{D}'_B$ will denote the space of such distributions.

LEMMA 3.1. *Let B_0 be a complex Banach space and $0<p<1$. If F belongs to $H^p_{B_0}(D)$ then there is a B_0-valued distribution Φ in $\mathscr{D}'_{B_0}$ such that $F(z)=\Phi(P_z)$.*

Proof. Suppose $F(z)=\sum_{n=0}^\infty a_n z^n$ being a_n in B_0. Since $\langle\xi, F(z)\rangle$ belongs to $H^p(D)$ for each ξ in B_0^* then we can write (see [14, p. 98])

$$|\langle\xi, a_n\rangle|\leqslant C\cdot|\langle\xi, F\rangle|_p\cdot n^{1/p-1},\qquad n\neq 0. \tag{3.1}$$

From (3.1) we have $\|a_n\|\leqslant C\cdot|F|_p\cdot n^{1/p-1}$. Now techniques analogous to those in the scalar-valued case [18] allow us to find Φ in $\mathscr{D}'_{B_0}$ such that $\Phi(\phi_n)=a_n$ for $n\geqslant 0$ and $\Phi(\phi_n)=0$ for $n<0$ being $\phi_n(t)=e^{-int}$. From this it easily follows that $F(z)=\Phi(P_z)$.

We shall consider three definitions for H^p_B in the case $0<p<1$ based on the corresponding ones from the scalar-valued case. Let us begin with a definition in terms of p-atoms (see [9, 10]). A function a in $L^\infty_B(\mathbb{T})$ is called (p, B)-atom if it is supported by an interval I and it verifies

$$\|a(t)\|\leqslant m(I)^{-1/p}\qquad\text{for all } t\in I \tag{3.2}$$

$$\int_I t^k a(t)\,dm(t)=0\qquad\text{for all integer } k,\ 0\leqslant k\leqslant[1/p]-1. \tag{3.3}$$

DEFINITION 3.1. We define $H^p_{\mathrm{at},B}$ as the space of B-valued distributions Φ which can be represented as $\sum_{k=1}^\infty \lambda_k a_k=\Phi$ in the sense of distributions being a_k (p, B)-atoms and $\sum|\lambda_k|^p<+\infty$. The "norm" in it is given by

$$|\Phi|_{\mathrm{at},p}=\inf\left\{\left(\sum|\lambda_k|^p\right)^{1/p}: \Phi=\sum\lambda_k a_k\right\}. \tag{3.4}$$

DEFINITION 3.2. We define the maximal function of a B-valued distribution Φ in $\mathscr{D}'_B$ as

$$\Phi^*(t)=\sup_{0<r<1}\|\Phi(P_{re^{it}})\| \tag{3.5}$$

and denote by $H^p_{\max,B}$ the space of distributions Φ in $\mathscr{D}'_B$ whose Φ^* belongs to $L^p(\mathbb{T})$. The norm in it is given by

$$|\Phi|_{\max,p}=\|\Phi^*\|_p. \tag{3.6}$$

DEFINITION 3.3. For $z=re^{it}$, let us write $Q_z(t)=Q_r(s-t)$, where Q_r stands for the conjugate Poisson kernel. Given a distribution Φ in $\mathscr{D}'_B$ we can consider two different harmonic functions $P(\Phi)$ and $Q(\Phi)$ defined by $P(\Phi)(z)=\Phi(P_z)$ and $Q(\Phi)(z)=\Phi(Q_z)$. We shall denote by H^p_B the space of distributions Φ in $\mathscr{D}'_B$ such that $P(\Phi)$ and $Q(\Phi)$ belong to $h^p_B(D)$. We set in it the "norm"

$$|\Phi|_p = \sup_{0<r<1} \left(\int (\|\Phi(P_{re^{it}})\|^p + \|\Phi(Q_{re^{it}})\|^p)\, dm(t) \right)^{1/p}. \tag{3.7}$$

The first result we want to mention is that the proof in [9] can be adapted for the Poisson kernel and in a Banach-valued setting to get

$$H^p_{\mathrm{at},B} \subseteq H^p_{\max,B} \quad \text{(with continuity).} \tag{3.8}$$

J. Garcia-Cuerva and the author have recently proved the following:

THEOREM 3.1 [3]. *The following statements are equivalent:*

(a) *B has RNP.*

(b) *$H^p_{\mathrm{at},B} = H^p_{\max,B}$ (with equivalent norms).*

Next we shall try to understand the relationship between H^p_B and $H^p_{\max,B}$.

PROPOSITION 3.1. *$H^p_B \subseteq H^p_{\max,B}$ (with continuity).*

Proof. Let us take Φ in H^p_B and consider $F(z) = \Phi(P_z) + i\Phi(Q_z)$. We can look at F as a holomorphic function on the disk with values in the complex space $B_0 = B + iB$, being $\|a + ib\|_{B_0} = \|a\| + \|b\|$. From (3.7) it follows that F belongs to $H^p_{B_0}(D)$. Moreover, from Lemma 1.1 we see that F belongs to $H^p_{\max,B_0}(D)$. Finally, since $\Phi^*(t) \leq \sup_{0<r<1} \|F(re^{it})\|_{B_0}$ then Φ^* belongs to $L^p(\mathbb{T})$ or, in other words, Φ belongs to $H^p_{\max,B}$ and $|\Phi|_{\max,p} \leq C\,|\Phi|_p$.

PROPOSITION 3.2. *If there exists a constant C such that $\sup_{0<r<1} \|Q_r * a\|_p \leq C$ for all (p, B)-atom then $H^p_{\mathrm{at},B} \subseteq H^p_B$ (with continuity).*

Proof. Suppose Φ belongs to $H^p_{\mathrm{at},B}$ and write Φ as $\sum \lambda_k a_k$. We see the action on P_z and Q_z as

$$\Phi(P_z) = \sum \lambda_k \int P_z(t)\, a_k(t)\, dt$$

and

$$\Phi(Q_z) = \sum \lambda_k \int Q_z(t)\, a_k(t)\, dt.$$

From this clearly we have that for $z = re^{it}$

$$\|\Phi(P_z)\|^p + \|\Phi(Q_z)\|^p \leq \sum |\lambda_k|^p (\|P_r * a_k(t)\|^p + \|Q_r * a_k(t)\|^p).$$

Integrating and taking supremums we get

$$|\Phi|^p_p \leq \sum |\lambda_k|^p \sup_{0<r<1} (\|P_r * a_k\|^p_p + \|Q_r * a_k\|^p_p).$$

From (3.8) we have that $\sup_{0<r<1} \|P_r * a_k\|_p \leq C\,\|a^*_k\|_p \leq C$. Now the assumptions allow us to say that $|\Phi|_p \leq C(\sum |\lambda_k|^p)^{1/p}$ for any decomposition, which implies the result.

To find the property on B to make the spaces H^p_B and $H^p_{\max,B}$ coincide will be our next goal. This property turns out to be the UMD property (see [7, 4]). We shall define it in a useful way for our purposes.

DEFINITION 3.4. A Banach space B is said to be a UMD space if there exists a constant C such that

$$\|\tilde{f}\|_2 \leq C\,\|f\|_2 \qquad \text{for all } f \text{ in } L^2_B(\mathbb{T}), \tag{3.9}$$

where $\tilde{f}$ stands for the conjugate function $\tilde{f}(t) = \lim Q_r * f(t)$ a.e.

To see the connection with martingales and the characterization in terms of the conjugate function the reader is referred to [4, 7]. Here we will write an equivalent and useful formulation (see [20, 23]).

For every q, $1 < q < \infty$, there exists a function $C_q(\lambda)$ converging to zero as $\lambda \to \infty$ such that for all f in $L^q_B(\mathbb{T})$ with $\|f\|_q \leq 1$ it verifies

$$m(\{t \in \mathbb{T} : \|\tilde{f}(t)\| > \lambda\}) \leq C_q(\lambda). \tag{3.10}$$

Now we are ready to formulate the main result of this section. The author proved an analogous result for $p = 1$ in [2].

THEOREM 3.2. *The following statements are equivalent:*

(a) *B is a UMD space.*

(b) *$H^p_B = H^p_{\max,B}$ (with equivalent norms).*

Proof. Let us assume B is a UMD space. Therefore B has the RNP and according to Propositions 3.1 and 3.2 and Theorem 1.1 we only have to prove the existence of a constant C such that

$$\sup_{0<r<1} \|Q_r * a\|_p \leq C \qquad \text{for all } (p, B)\text{-atom } a. \tag{3.11}$$

Identify now $\mathbb{T}$ with $(-\pi, \pi]$ and let us take a (p, B)-atom a supported by

$(-\delta, \delta)$ for some $\delta > 0$. By using Hölder's inequality twice and the UMD property we can write

$$\begin{aligned}\int_{|t| \leqslant 2\delta} \|Q_r * a(t)\|^p \, dt &= \int_{|t| \leqslant 2\delta} \|P_r * \tilde{a}(t)\|^p \, dt \\ &\leqslant C\delta^{1-p} \left(\int_{|t| \leqslant 2\delta} \|P_r * \tilde{a}(t)\| \, dt \right)^p \\ &\leqslant C\delta^{1-p/2} \left(\int \|P_r * \tilde{a}(t)\|^2 \, dt \right)^{p/2} \\ &\leqslant C\delta^{1-p/2} \left(\int \|\tilde{a}(t)\|^2 \, dt \right)^{p/2} \\ &\leqslant C\delta^{1-p/2} \left(\int_{|t| < \delta} \|a(t)\|^2 \, dt \right)^{p/2} \leqslant C.\end{aligned}$$

Next we shall prove

$$\sup_{0<r<1} \int_{2\delta}^{\pi - 2\delta} \|Q_r * a(t)\|^p \, dt \leqslant C. \tag{3.12}$$

The standard argument involving the cancellation properties of atoms lead us to the integral expression

$$Q_r * a(s) = \int_{-\delta}^{\delta} Q_r^{(N)}(\xi_{s,t}) \frac{t^N}{N!} a(t) \, dt, \tag{3.13}$$

where N denotes $[1/p]$ and we are taking $s \in (2\delta, \pi - 2\delta)$, $t \in (-\delta, \delta)$, and $\xi_{s,t}$ is an intermediate value between $s-t$ and s and consequently $\xi_{s,t} \in (s-\delta, s+\delta)$.

Writing $Q_r(t) = \operatorname{Im}((1+re^{it})/(1-re^{it}))$, it easily follows that

$$|Q_r^{(k)}(t)| \leqslant C_k \cdot ((1 - r\cos t)^2 + (r \sin t)^2)^{-(k+1)/2}$$

for all $k \in \mathbb{N}$.

On the other hand, we can estimate the N-derivative as follows: For all $t \in (-\delta, \delta)$ we have

$$\sup_{0<r\leqslant 1/2} |Q_r^{(N)}(t)| \leqslant \sup_{0<r\leqslant 1/2} C_N (1 - r\cos t)^{-(N+1)} \leqslant C'_N \tag{3.14}$$

$$\sup_{1/2<r<1} |Q_r^{(N)}(\xi_{s,t})| \leqslant C''_N \{ \sin(s-\delta)^{-(N+1)} \chi^{(s)}_{(2\delta;\pi/2-2\delta)} + \chi^{(s)}_{(\pi/2-2\delta,\pi/2+2\delta)} + \sin(s+\delta)^{-(N+1)} \chi^{(s)}_{(\pi/2+2\delta,\pi-2\delta)} \}. \tag{3.15}$$

From (3.14) we have

$$\|Q_r * a(s)\| \leqslant C \qquad \text{for} \quad 0 < r \leqslant \tfrac{1}{2},$$

which implies (3.12) for these values of r. To study the case $\frac{1}{2} < r < 1$, we split the integral in (3.12) into three parts, $\int_{2\delta}^{\pi/2-2\delta}$, $\int_{\pi/2-2\delta}^{\pi/2+2\delta}$, and $\int_{\pi/2+2\delta}^{\pi-2\delta}$. Let us compute only the first one, the others being similar or easier.

$$\begin{aligned}&\int_{2\delta}^{\pi/2-2\delta} \left(\int_{-\delta}^{\delta} \frac{|Q_r^{(N)}(\xi_{s,t})|}{N!} |t|^N \|a(t)\| \, dt \right)^p ds \\ &\quad\leqslant C \int_{2\delta}^{\pi/2-2\delta} \delta^{(N+1)p-1} \sup_{t \in (-\delta,\delta)} |Q_r^{(N)}(\xi_{s,t})|^p \, ds \\ &\quad\leqslant C\delta^{(N+1)p-1} \int_{2\delta}^{\pi/2-2\delta} (\sin(s-\delta))^{-(N+1)p} \, ds. \qquad (**)\end{aligned}$$

Now changing the variable $\operatorname{tag}(s-\delta) = y$ and using the facts that $1-(N+1)p < 0$ and $2-(N+1)p > 0$ we can write

$$\begin{aligned}&\int_{\operatorname{tag}\delta}^{\operatorname{tag}(\pi/2-3\delta)} y^{-(N+1)p} (1+y^2)^{((N+1)/2)p-1} \, dy \\ &\quad\leqslant C \int_{\operatorname{tag}\delta}^{\operatorname{tag}(\pi/2-3\delta)} y^{-(N+1)p} \, dy \\ &\quad\leqslant C \left((\operatorname{tag}\delta)^{1-(N+1)p} - \left(\operatorname{tag}\left(\frac{\pi}{2} - 3\delta \right) \right)^{1-(N+1)p} \right).\end{aligned}$$

When $\delta \to 0$ we only need to use the fact that $\delta/\operatorname{tag}\delta \leqslant C$ to show $(**) \leqslant C$, and this finishes the proof of (3.12). Similar arguments would compute the integral over $(-\pi, -\pi+2\delta)$, finishing the direct implication.

To see the converse let us consider the operators T_r from $H^p_{\max,B}$ to L^p_B defined by

$$T_r(\Phi)(t) = \Phi(Q_{re^{it}}). \tag{3.16}$$

The assumption now means that T_r are uniformly bounded, that is, $\|T_r\| \leqslant C$ for all $0 < r < 1$. We shall prove that B is UMD by showing (3.10). To do that let us take any q, $1 < q < \infty$, and consider $L^q_B(\mathbb{T})$ included into $H^p_{\max,B}$ in the natural way, that is, every function f in L^q_B defines the distribution $\Phi_f(\phi) = \int f(t)\,\phi(t)\,dt$. This allows us to consider T_r acting on L^q_B and according to Kolmogorov's inequality we can write

$$m(\{t \in \mathbb{T} : \|T_r(\Phi_f)\| > \lambda\}) \leqslant \lambda^{-p} \|T_r(\Phi_f)\|_p^p \leqslant C\lambda^{-p} \|\Phi_f\|^p_{\max,p} \leqslant C\lambda^{-p} \|f\|_q^p.$$

Let us assume now that f is a trigonometric polynomial. Then $\tilde{f}$ exists and $T_r(\Phi_f) \to \tilde{f}$ as $r \to 1$. Therefore the last inequality can be written as

$$m(\{t \in \mathbb{T} : \|\tilde{f}(t)\| > \lambda\}) \leqslant C \cdot \lambda^{-p} \|f\|_q^p$$

for any trigonometric polynomial. This fact is extended to any function in $L_B^q(\mathbb{T})$ by the density of trigonometric polynomials in $L_B^q(\mathbb{T})$ and (3.10) is proved.

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