Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS Vol. 147, No. 2, April 1990
All Rights Reserved by Academic Press, New York and London Printed in Belgium

Spaces of Analytic Functions on the Disc where the Growth of $M_p(F, r)$ Depends on a Weight

OSCAR BLASCO*

Departamento de Matematicas, Universidad de Zaragoza, Zaragoza – 50009 Spain

AND

GERALDO SOARES DE SOUZA

Division of Mathematics, Auburn University, Auburn, Alabama 36849-5310

Submitted by Bruce C. Berndt

Received October 14, 1988

DEDICATED TO THE MEMORY OF J. L. RUBIO DE FRANCIA

We consider spaces of analytic functions depending on a weight $\rho(t) \geqslant 0$, $t \in [0, 1)$, defined by certain conditions, namely

(1) $M_p(F', r) = O(\rho(1-r)/(1-r))$,

(2) $M_p(F'', r) = O(\rho(1-r)/(1-r)^2)$,

(3) $\int_0^1 (\rho(1-r)/(1-r))\, M_p(F, r)\, dr < +\infty$.

We study boundary value problems and duality for these spaces depending on the properties of the weight function

INTRODUCTION

In this paper we shall deal with spaces of analytic functions F closely related to H^p spaces. We shall look at those functions F, where the growth of the L^p-norm of F_r (restriction of F to $|z| = r$) depends on a certain weight function ρ. We connect these spaces to weighted Besov–Lipschitz classes and prove several duality results depending on the properties of ρ.

* Partially supported by the grant C.A.I.C.Y.T. PB-85-0338.

0022-247X/90 $3.00

Following ideas from Hardy and Littlewood [H–L] and Zygmund [Z1], we relate functions satisfying

$$M_p(F', r) = O\left(\frac{\rho(1-r)}{1-r}\right) \tag{0.1}$$

and

$$M_p(F'', r) = O\left(\frac{\rho(1-r)}{(1-r)^2}\right) \tag{0.2}$$

with the behavior of their boundary values. We find conditions on ρ to get results analogous to those proved in [H-L] for $\rho(t) = t^\alpha (0 < \alpha < 1)$ in (0.1) and [Z1] for $\rho(t) = t$ in (0.2). Under certain assumptions we show that the boundary value functions must satisfy respectively

$$\|\Delta_t f\|_p = O(\rho(t)) \qquad (t \to O^+), \tag{0.1$'$}$$

$$\|\Delta_t^2 f\|_p = O(\rho(t)) \qquad (t \to O^+), \tag{0.2$'$}$$

where $\Delta_t f(\theta) = f(\theta + t) - f(\theta)$ and $\Delta_t^2 f(\theta) = f(\theta + t) + f(\theta - t) - 2f(\theta)$.

The study of the previous spaces leads in a natural way to a dual condition

$$\int_0^1 \frac{\rho(1-r)}{1-r} M_p(F, r)\, dr < +\infty. \tag{0.3}$$

Special cases of this condition have already been considered in [D-R-S], [S], and [S-W].

We find the equivalent formulation for the boundary values of functions verifying (0.3), reaching certain Besov–Lipschitz classes. These results extend to more general weights for some theorems in [T] and [F] proved for $\rho(t) = t^\alpha$.

The last section is devoted to the study of duality for these spaces. It is inspired by some results in [D-R-S], [A-C-P], and [S-W], when very special cases are shown. We extend them to values of p, $1 < p < \infty$, finding conditions on the weight ρ to get analogous results.

The reader is referred to [J] and [S-W] to see some results on general weighted spaces, and to [B-S1] and [B-S2] where the second named author and S. Bloom have recently proved some results of the same type for the special case $p = \infty$.

Throughout the paper $M_p(F, r)$ will mean $(1/2\pi \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta)^{1/p}$, $F_r(e^{i\theta}) = F(re^{i\theta})$, by H^p we mean the set of all analytic functions on the disc D, such that $\sup_{0<r<1} M_p(F, r) < \infty$, and C will denote a constant not necessarily the same at each occurrence.

1. PREVIOUS DEFINITI

We shall write $\rho(t)$ for a n
on [0, 1). The following pro
[B-S1] and [B-S2].

DEFINITION 1.1. ρ is said
constant C such that

$$\int_0^t \frac{\rho(s)}{s}\, ds \leqslant$$

ρ is said to be a b_n-weight, $\rho \in$

$$\int_t^1 \frac{\rho(s)}{s^{n+1}}\, ds \leqslant$$

The reader is referred to [B-
some characterizations, and e
Let us mention here some e

PROPOSITION 1.1.

(i) *Let ρ be Dini and ρ(*

$$\rho(t) \log \frac{1}{t} \leqslant$$

(ii) *If $\rho \in b_n$ then*

Proof. (i) Since $\rho(0) = 0$
function we have a positive m
write

$$\int_0^t \frac{\rho(s}{s}$$

From (1.5), Fubbini and Dini

$$C\rho(t) \geqslant \int_0^t \log(t/u$$

$$\geqslant \int_0^{t^\alpha} \log(t/u$$

1. Previous Definitions and Basic Lemmas on Weights

We shall write $\rho(t)$ for a non-negative non-decreasing function defined on $[0, 1)$. The following properties on a weight ρ were introduced in [B-S1] and [B-S2].

Definition 1.1. ρ is said to be *Dini* if $\rho(t)/t \in L^1(0, 1)$ and there is a constant C such that

$$\int_0^t \frac{\rho(s)}{s}\, ds \leqslant C\rho(t) \qquad \text{for all} \quad 0<t<1. \tag{1.1}$$

ρ is said to be a b_n-weight, $\rho \in b_n$, $(n \geqslant 1)$ if there is a constant C such that

$$\int_t^1 \frac{\rho(s)}{s^{n+1}}\, ds \leqslant C\frac{\rho(t)}{t^n} \qquad \text{for all} \quad 0<t<1. \tag{1.2}$$

The reader is referred to [B-S1] to see the motivation for the definitions, some characterizations, and examples of weights with these properties.

Let us mention here some easy properties of ρ that we shall use later on.

Proposition 1.1.

(i) *Let ρ be Dini and $\rho(0)=0$. If $1<\alpha<\infty$ then*

$$\rho(t) \log\frac{1}{t} \leqslant C_\alpha \rho(t^{1/\alpha}) \qquad (0<t<1) \tag{1.3}$$

(ii) *If $\rho \in b_n$ then*

$$\log\frac{1}{t} \leqslant C\frac{\rho(t)}{t^n}. \tag{1.4}$$

Proof. (i) Since $\rho(0)=0$ and ρ is a non-negative non-decreasing function we have a positive measure $d\rho(u)$ associated with it. Let us then write

$$\int_0^t \frac{\rho(s)}{s}\, ds = \int_0^t \int_0^s \frac{d\rho(u)}{s}\, ds. \tag{1.5}$$

From (1.5), Fubbini and Dini condition we have

$$\begin{aligned} C\rho(t) &\geqslant \int_0^t \log(t/u)\, d\rho(u) \\ &\geqslant \int_0^{t^\alpha} \log(t/u)\, d\rho(u) \geqslant (\alpha-1)\log\left(\frac{1}{t}\right)\rho(t^\alpha). \end{aligned}$$

(ii) It is obvious that $\rho(s)/s^n \geqslant C$ if $\rho \in b_n$. Then

$$\log \frac{1}{t} = \int_t^1 \frac{ds}{s} \leqslant \frac{1}{C} \int_t^1 \frac{\rho(s)}{s^{n+1}}\, ds \leqslant C \frac{\rho(t)}{t^n}.$$

We remark that the converse of (1.4) is false. To see that, take $\rho(t) = t^2 \log 1/t$ which does not belong to b_2.

We now establish two elementary but useful lemmas. We include only the proof of the second one. The proof of the first one is similar but easier.

Lemma 1.1. *Let $\rho \in b_n$. Then*

$$\int_0^1 \frac{\rho(s)}{(s^2 + ct^2)^{(n+1)/2}}\, ds \leqslant C \frac{\rho(t)}{t^n} \tag{1.6}$$

$$\int_0^1 \frac{\rho(1-s)}{(1-rs)^{n+1}}\, ds \leqslant C \frac{\rho(1-r)}{(1-r)^n}. \tag{1.7}$$

Lemma 1.2. *Let $\varepsilon \in b_n$ and Dini. Then*

$$\int_0^1 \frac{\rho(s)}{s(s^2 + ct^2)^{n/2}}\, ds \leqslant C \frac{\rho(t)}{t^n} \tag{1.8}$$

$$\int_0^1 \frac{\rho(1-s)}{(1-s)(1-rs)^n}\, ds \leqslant C \frac{\rho(1-r)}{(1-r)^n}. \tag{1.9}$$

Proof.

$$\int_0^1 \frac{\rho(s)}{s(s^2+ct^2)^{n/2}}\, ds = \int_0^t \frac{\rho(s)}{s(s^2+ct^2)^{n/2}}\, ds + \int_t^1 \frac{\rho(s)}{s(s^2+ct^2)^{n/2}}\, ds$$
$$\leqslant \frac{C}{t^n} \int_0^t \frac{\rho(s)}{s}\, ds + \int_t^1 \frac{\rho(s)}{s^{n+1}}\, ds \leqslant C \frac{\rho(t)}{t^n},$$

where Dini and b_n conditions are used in the last inequality.

$$\int_0^1 \frac{\rho(1-s)}{(1-s)(1-rs)^n}\, ds = \int_0^r \frac{\rho(1-s)}{(1-s)(1-rs)^n}\, ds + \int_r^1 \frac{\rho(1-r)}{(1-s)(1-rs)^n}\, ds$$
$$\leqslant \int_0^r \frac{\rho(1-s)}{(1-s)^{n+1}}\, ds + \frac{1}{(1-r)^n} \int_r^1 \frac{\rho(1-s)}{1-s}\, ds$$
$$\leqslant \int_{1-r}^1 \frac{\rho(u)}{u^{n+1}}\, du + \frac{1}{(1-r)^n} \int_0^{1-r} \frac{\rho(u)}{u}\, du \leqslant C \frac{\rho(1-r)}{(1-r)^n},$$

where again Dini and b_n are used at the end.

Next we introduce the spaces of analytic functions whose definitions were inspired by conditions introduced by Hardy and Littlewood [H-L], Zygmund [Z1], and the spaces defined in [D-R-S] and [S], called weighted Bergman spaces.

DEFINITION 1.2. *Let* $\rho \geqslant 0$ non-decreasing and $1 \leqslant p \leqslant \infty$.

$$HL_\rho^p = \left\{ F: D \to C \text{ analytic}; \, M_p(F', r) = O\left(\frac{\rho(1-r)}{1-r}\right) \right\}$$

$$Z_\rho^p = \left\{ F: D \to C \text{ analytic}; \, M_p(F'', r) = O\left(\frac{\rho(1-r)}{(1-r)^2}\right) \right\}$$

$$B_\rho^p = \left\{ F: D \to C \text{ analytic}; \, \int_0^1 \frac{\rho(1-r)}{1-r} M_p(F, r)\, dr < +\infty \right\}.$$

The obvious norms in the spaces are given by

$$\|F\|_{HL,p,\rho} = |F(0)| + \inf\left\{ C: M_p(F', r) \leqslant C \frac{\rho(1-r)}{1-r}, \, 0 < r < 1 \right\} \tag{1.10}$$

$$\|F\|_{Z,p,\rho} = |F(0)| + |F'(0)| + \inf\left\{ C: M_p(F'', r) \leqslant C \frac{\rho(1-r)}{(1-r)^2}, \, 0 < r < 1 \right\} \tag{1.11}$$

$$\|F\|_{B,p,\rho} = \int_0^1 \frac{\rho(1-r)}{1-r} M_p(F, r)\, dr. \tag{1.12}$$

There are two conditions on ρ we must assume if we want the analytic polynomials to belong to these spaces.

$$\frac{\rho(t)}{t} \geqslant C, \qquad 0 < t < 1 \tag{*}$$

$$\rho(t)/t \in L^1(0, 1). \tag{**}$$

From Cauchy's formula we have that

$$M_p(F', r) \leqslant C \frac{M_p(F, r)}{1-r}$$

which implies $HL_\rho^p \subseteq Z_\rho^p$.

On the other hand (**) obviously implies that $H^p \subseteq B_\rho^p$. Let us show now that (**) also implies $Z_\rho^p \subseteq H^p$. Indeed, let us take F in Z_ρ^p.

$$F(re^{i\theta}) = \int_0^r \int_0^s F''(te^{i\theta})\, e^{2i\theta}\, dt\, ds + re^{i\theta} F'(0) + F(0).$$

Hence

$$M_p(F, r) \leqslant \int_0^r \int_0^s M_p(F'', t)\, dt\, ds + |F'(0)| + |F(0)|$$

$$\leqslant \int_0^1 \int_0^s M_p(F'', t)\, dt\, ds + |F'(0)| + |F(0)|$$

$$\leqslant \int_0^1 (1-t)\, M_p(F'', t)\, dt + |F'(0)| + |F(0)|$$

$$\leqslant C \int_0^1 \frac{\rho(1-t)}{1-t}\, dt + C \leqslant C.$$

LEMMA 1.3. *Let* $1 \leqslant p \leqslant \infty$.

If $\rho \in b_1$ *then* $HL^p_\rho = Z^p_\rho$ (1.13)

If ρ *is Dini then*

$$F \in B^p_\rho \text{ if and only if } \int_0^1 \rho(1-r)\, M_p(F', r)\, dr < +\infty. \qquad (1.14)$$

Proof. We know that $F'(re^{i\theta}) = \int_0^r F''(se^{i\theta})\, e^{i\theta}\, ds + F'(0)$ so

$$M_p(F', r) \leqslant \int_0^r M_p(F'', s)\, ds + C.$$

Assume that $F \in Z^p_\rho$ and $\rho \in b_1$, therefore

$$M_p(F', r) \leqslant C \int_0^r \frac{\rho(1-s)}{(1-s)^2}\, ds + C \leqslant C \int_{1-r}^1 \frac{\rho(t)}{t^2}\, dt + C \leqslant C\, \frac{\rho(1-r)}{1-r}.$$

To show (1.14) notice that Cauchy's formula implies that if $F \in B^p_\rho$ then $\int_0^1 \rho(1-r)\, M_p(F', r)\, dr < +\infty$.

On the other hand, using Dini condition, we can write

$$\int_0^1 \frac{\rho(1-r)}{1-r} M_p(F, r)\, dr \leqslant \int_0^1 \frac{\rho(1-r)}{1-r} \int_0^r M_p(F', s)\, ds\, dr + C \int_0^1 \frac{\rho(s)}{s}\, ds$$

$$\leqslant \int_0^1 \left(\int_s^1 \frac{\rho(1-r)}{1-r}\, dr \right) M_p(F', s)\, ds + C$$

$$\leqslant \int_0^1 \left(\int_0^{1-s} \frac{\rho(u)}{u}\, du \right) M_p(F', s)\, ds + C$$

$$\leqslant C \int_0^1 \rho(1-s)\, M_p(F', s)\, ds + C$$

2.

It is well known fr
[Z2], [T]) that the gro
the behavior of the first
Our aim in this section i
allows us to get the kno
for f defined on T,

$\Delta_t f(\theta) = f(\theta + t) - f(\theta$

THEOREM 2.1. *Let* ρ

(i) *F belongs to H*

$\|\Delta_{t}$

where f is the boundary l

(ii) *F' belongs to B*

where f is the boundary l

Proof. We shall prov
boundary values for funct
as we showed in the pr
$M_p(F', r) \in L^1(0, 1)$ and

$\|F_r - F_{r'}\|_p \leqslant \int_{r'}^r$

This implies that $\{F_r\}$ is
and then there is f in L^p

F

Let us use now an argu
page 78]) and write for 0

$F(r_2 e^{i(}$

$'(0)| + |F(0)|$

$'(0)| + |F(0)|$

$|F'(0)| + |F(0)|$

(1.13)

$F', r)\, dr < +\infty.$ (1.14)

$^{i\theta}\, ds + F'(0)$ so

$s + C.$

$\frac{(t)}{^2}\, dt + C \leqslant C \frac{\rho(1-r)}{1-r}.$

implies that if $F \in B^p_\rho$ then

can write

$', s)\, ds\, dr + C \int_0^1 \frac{\rho(s)}{s}\, ds$

$M_p(F', s)\, ds + C$

$M_p(F', s)\, ds + C$

$', s)\, ds + C$

2. Boundary Value Problems

It is well known from the effort of several authors (see [H-L], [Z2], [T]) that the growth of the first and second derivatives depends on the behavior of the first and second differences of the boundary function. Our aim in this section is to exhibit this relation for certain weights which allows us to get the known results as corollaries. Let us denote, as usual, for f defined on T,

$$\Delta_t f(\theta) = f(\theta + t) - f(\theta) \qquad \text{and} \qquad \Delta_t^2 f(\theta) = f(\theta + t) + f(\theta - t) - 2f(\theta).$$

THEOREM 2.1. *Let ρ be Dini and b_1, $1 \leqslant p \leqslant \infty$ and F analytic.*

(i) *F belongs to HL^p_ρ if and only if F belongs to H^p and*

$$\|\Delta_t f\|_p = O(\rho(t)) \qquad as \quad t \to 0^+, \tag{2.1}$$

where f is the boundary limit of F.

(ii) *F' belongs to B^p_ρ if and only if F belongs to H^p and*

$$\int_0^1 \frac{\rho(t)}{t^2} \|\Delta_t f\|_p\, dt < +\infty, \tag{2.2}$$

where f is the boundary limit of F.

Proof. We shall prove (i) and (ii) in a parallel way. The existence of boundary values for functions in HL^p_ρ follows from the fact that $HL^p_\rho \subseteq H^p$, as we showed in the previous section. Assume now that $\rho(1-r)/(1-r)\, M_p(F', r) \in L^1(0, 1)$ and use $\rho(t)/t > C$ to write

$$\|F_r - F_{r'}\|_p \leqslant \int_{r'}^r M_p(F', s)\, ds \leqslant C \int_{r'}^r \frac{\rho(1-s)}{1-s} M_p(F', s)\, ds.$$

This implies that $\{F_r\}$ is a Cauchy net in L^p and therefore F belongs to H^p and then there is f in L^p such that

$$F(z) = \frac{1}{2} \int_{-1}^1 \frac{f(e^{i\pi t})}{(e^{i\pi t} - z)} e^{i\pi t}\, dt. \tag{2.3}$$

Let us use now an argument due to Hardy and Littlewood (see [D, page 78]) and write for $0 < r_1 < r_2 < 1$ and $0 < t$, $\theta < 1$.

$$F(r_2 e^{i(\theta + t)\pi}) - F(r_2 e^{i\theta\pi}) = \int_\Gamma F'(\xi)\, d\xi,$$

where the contour Γ goes radially from $r_2 e^{i\pi\theta}$ to $r_1 e^{i\pi\theta}$ along $|z| = r_1$ to $r_1 e^{i(\theta+t)\pi}$ and then radially again to $r_2 e^{i(\theta+t)\pi}$. Therefore

$$|F(r_2 e^{i(\theta+t)\pi}) - F(r_2 e^{i\theta\pi})|$$
$$\leqslant \int_{r_1}^{r_2} (|F'(se^{i\pi\theta})| + |F'(se^{i(\theta+t)\pi})|)\, ds + \int_0^t |F'(r_1 e^{i(\theta+u)\pi}|\, du.$$

Hence, taking L^p-norms, $r_1 = 1 - t$, and limit as $r_2 \to 1$ we get that for all $0 < t < 1$,

$$\|\Delta_t f\|_p \leqslant 2 \int_{1-t}^{1} M_p(F', s)\, ds + t M_p(F', 1-t). \tag{2.4}$$

We apply (2.4) to situations (i) and (ii).

If F belongs to HL^p_ρ then

$$\|\Delta_t f\|_p \leqslant C \int_{1-t}^{1} \frac{\rho(1-s)}{1-s}\, ds + C\rho(t)$$
$$\leqslant C \int_0^t \frac{\rho(u)}{u}\, du + C\rho(t).$$

Hence (2.1) follows from Dini condition.

If F' belongs to B^p_ρ then

$$\int_0^1 \frac{\rho(t)}{t^2} \|\Delta_t f\|_p\, dt \leqslant 2 \int_0^1 \int_{1-t}^1 \frac{\rho(t)}{t^2} M_p(F', s)\, ds\, dt$$
$$+ \int_0^1 \frac{\rho(t)}{t} M_p(F', 1-t)\, dt$$
$$= 2 \int_0^1 \left(\int_{1-s}^1 \frac{\rho(t)}{t^2}\, dt \right) M_p(F', s)\, ds$$
$$+ \int_0^1 \frac{\rho(1-s)}{1-s} M_p(F', s)\, ds.$$

Now using that ρ is b_1 we get

$$\int_0^1 \frac{\rho(t)}{t^2} \|\Delta_t f\|_p\, dt \leqslant C \int_0^1 \frac{\rho(1-s)}{1-s} M_p(F', s)\, ds < +\infty.$$

Let us start now with a function F in H^p and a representation given by (2.3). Then we have

$$F'(re^{i\theta}) = \int_{-1}^{1} \frac{f(e^{i\pi(\theta+t)}) - f(e^{i\pi\theta})}{(e^{i\pi t} - r)^2} e^{i\pi(t-\theta)}\, dt.$$

Notice that $\|\Delta_t f\|_p = \|\Delta$

M

Let us recall the estim
implies

$M_p($

which gives that F belong
the assumption (2.1).
If we apply (2.5) and F

$\int_0^1 \frac{\rho(1-r)}{1-r} M$

$\leqslant C \int_0^1$

Assumption (2.2) togeth
belongs to B^p_ρ. The proof

Note that $\rho(t) = t^\alpha$ (0
[H-L] or [D, page 78]
for Theorem 2.1(ii) for th

THEOREM 2.2. *Let ρ b*

(i) *F belongs to Z*

$\|\Delta$

(ii) *F'' belongs to B*

(*as before f is the bounda*

Proof. As in the prev
(i) and (ii).
It was proved in Sectio
then $F \in H^p$.

to $r_1 e^{i\pi\theta}$ along $|z| = r_1$ to
herefore

$\int_0^t |F'(r_1 e^{i(\theta+u)\pi}|\, du.$

s $r_2 \to 1$ we get that for all

$M_p(F', 1-t).$ (2.4)

$+ C\rho(t)$

$t).$

$M_p(F', s)\, ds\, dt$

$, 1-t)\, dt$

$t\Big)\, M_p(F', s)\, ds$

$M_p(F', s)\, ds.$

$F', s)\, ds < +\infty.$

d a representation given by

$e^{i\pi(t-\theta)}\, dt.$

Notice that $\|\Delta_t f\|_p = \|\Delta_{-t} f\|_p$, so

$$M_p(F', r) \leqslant 2 \int_0^1 \frac{\|\Delta_t f\|_p}{|e^{i\pi t} - r|^2}\, dt.$$

Let us recall the estimate $|e^{i\pi t} - r|^2 \geqslant (1-r)^2 + ct^2$ for $0 < t < 1$ which implies

$$M_p(F', r) \leqslant C \int_0^1 \frac{\|\Delta_t f\|_p}{(1-r)^2 + ct^2}\, dt \tag{2.5}$$

which gives that F belongs to HL^p_ρ using (1.6) in Lemma 1.1 together with the assumption (2.1).

If we apply (2.5) and Fubini we can write

$$\int_0^1 \frac{\rho(1-r)}{1-r} M_p(F', r)\, dr$$
$$\leqslant C \int_0^1 \|\Delta_t f\|_p \left(\int_0^1 \frac{\rho(1-r)\, dr}{(1-r)((1-r)^2 + ct^2)} \right) dt.$$

Assumption (2.2) together with (1.8) in Lemma 1.2 shows now that F' belongs to B^p_ρ. The proof is completed.

Note that $\rho(t) = t^\alpha$ $(0 < \alpha < 1)$ is Dini and b_1. The reader is referred to [H-L] or [D, page 78] for Theorem 2.1(i) and [T] or [F, Theorem 10] for Theorem 2.1(ii) for the special case $\rho(t) = t^\alpha$, $0 < \alpha < 1$.

THEOREM 2.2. *Let ρ be Dini and b_2, $1 \leqslant p \leqslant \infty$ and F analytic*

(i) *F belongs to Z^p_ρ if and only if F belongs to H^p and*

$$\|\Delta_t^2 f\|_p = O(\rho(t)) \qquad (t \to 0^+) \tag{2.6}$$

(ii) *F'' belongs to B^p_ρ if and only if F belongs to H^p and*

$$\int_0^1 \frac{\rho(t)}{t^3} \|\Delta_t^2 f\|_p\, dt < +\infty \tag{2.7}$$

(as before f is the boundary value function of F).

Proof. As in the previous theorem we shall proceed simultaneously for (i) and (ii).

It was proved in Section 1 that $Z^p_\rho \subseteq H^p$. Let us show now that if $F'' \in B^p_\rho$ then $F \in H^p$.

As before

$$M_p(F, r) \leqslant \int_0^r \int_0^s M_p(F'', u)\, du\, ds + |F'(0)| + |F(0)|.$$

All we have to estimate is the first term on the right hand side. In fact,

$$\begin{aligned}
&\int_0^r \int_0^s M_p(F'', u)\, du\, ds \\
&\leqslant \int_0^1 \int_0^s M_p(F'', u)\, du\, ds \\
&= \int_0^1 \int_u^1 M_p(F'', u)\, ds\, du \\
&= \int_0^1 (1-u)\, M_p(F'', u)\, du \leqslant C \int_0^1 \frac{\rho(1-u)}{1-u} M_p(F'', u)\, du.
\end{aligned}$$

In the last inequality we used the fact that $C \leqslant \rho(1-u)/(1-u)^2$ since $\rho \in b_2$. Therefore we conclude

$$\begin{aligned}
&\int_0^r \int_0^s M_p(F', u)\, du\, ds \\
&\leqslant C \int_0^1 \frac{\rho(1-u)}{1-u} M_p(F'', u)\, du < +\infty, \text{ since } F'' \in B_\rho^p.
\end{aligned}$$

Hence in both cases F is the Poisson integral of its boundary limit f. That is

$$F(re^{i\theta}) = \frac{1}{2} \int_{-1}^1 P(r, \theta - t)\, f(e^{i\pi t})\, dt, \tag{2.8}$$

where

$$P(r, t) = \frac{1 - r^2}{1 + r^2 - 2r \cos \pi t};$$

To estimate $\|\Delta_t^2 f\|_p$ let us use an argument due to Zygmund (see [D, p. 77]).

Given $0 < r_1 < r_2 < 1$ and $0 < t < 1$ we write

$$\Delta_t^2 F_{r_2} = \Delta_t^2 (F_{r_2} - F_{r_1}) + \Delta_t^2 F_{r_1}$$

$$\begin{aligned}
\Delta_t^2 F_{r_1}(\theta) = {} & i r_1 \int_0^t e^{i\pi(\theta+u)} \left(\int_{-u}^u F''(r_1 e^{i\pi(\theta+v)})\, e^{i\pi(\theta+v)}\, dv \right) du \\
& + i r_1 \int_0^t (e^{i\pi(\theta+u)} - e^{i\pi(\theta-u)})\, F'(r_1 e^{i\pi(\theta-u)})\, du.
\end{aligned}$$

Therefore

$$\|\Delta_t^2 F_{r_1}\|_p \leqslant Ct^2(M_p(F'', r_1) + M_p(F', r_1)). \tag{2.9}$$

On the other hand

$$F(r_2 e^{i\pi\theta}) - F(r_1 e^{i\pi\theta}) = e^{2\pi i\theta} \int_{r_1}^{r_2} (r_2 - s)\, F''(se^{i\pi\theta})\, ds + (r_2 - r_1)\, e^{i\pi\theta} F'(r_1 e^{i\pi\theta})$$

which implies

$$\|\Delta_t^2(F_{r_2} - F_{r_1})\|_p \leqslant 4 \int_{r_1}^{1} (1-s)\, M_p(F'', s)\, ds + (1 - r_1)\, \|\Delta_t^2(e^{i\pi\theta} F'(r_1 e^{i\pi\theta}))\|_p.$$

Note that

$$\Delta_t(e^{i\pi\theta} F'(r_1 e^{i\pi\theta})) = \Delta_t(e^{i\pi\theta})\, F'(r_1 e^{i\pi(\theta + t)}) + e^{i\pi\theta}\, \Delta_t F'(r_1 e^{i\pi\theta}).$$

Hence

$$\begin{aligned} \|\Delta_t^2\, (e^{i\pi\theta} F'(r_1 e^{i\pi\theta}))\|_p &\leqslant 2\, \|\Delta_t(e^{i\pi\theta} F'(r_1 e^{i\pi\theta}))\|_p \\ &\leqslant Ct(M_p(F', r_1) + M_p(F'', r_1)). \end{aligned}$$

Thus

$$\begin{aligned} \|\Delta_t^2(F_{r_2} - F_{r_1})\|_p \leqslant{}& C(1 - r_1)\, t(M_p(F', r_1) \\ &+ M_p(F'', r_1)) + C \int_{r_1}^{1} (1-s)\, M_p(F'', s)\, ds. \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10), writing $r_1 = 1 - t$ and taking limit as $r_2 \to 1$ we get

$$\begin{aligned} \|\Delta_t^2 f\|_p \leqslant{}& Ct^2(M_p(F', 1-t) + M_p(F'', 1-t)) \\ &+ C \int_{1-t}^{1} (1-s)\, M_p(F'', s)\, ds. \end{aligned} \tag{2.11}$$

Let us consider (2.11) for F in Z_ρ^p and recall that

$$M_p(F', r) \leqslant M_p(F'', r) + C \qquad \text{and} \qquad C \leqslant \frac{\rho(t)}{t^2} \qquad \text{since} \quad \rho \in b_2$$

$$\begin{aligned} \|\Delta_t^2 f\|_p &\leqslant Ct^2(M_p(F'', 1-t) + C) + C \int_{1-t}^{1} (1-s)\, M_p(F'', s)\, ds \\ &\leqslant C\rho(t) + C \int_{1-t}^{1} \frac{\rho(1-s)}{1-s}\, ds = C\rho(t) + C \int_0^t \frac{\rho(u)}{u}\, du. \end{aligned}$$

Using the Dini condition we get (2.6).

Using (2.11) for F'' in B_ρ^p we have

$$\int_0^1 \frac{\rho(t)}{t^3} \|\Delta_t^2 f\|_p \, dt \leqslant C \int_0^1 \frac{\rho(t)}{t} M_p(F'', 1-t)\, dt + C \int_0^1 \frac{\rho(t)}{t}\, dt$$

$$+ C \int_0^1 \frac{\rho(t)}{t^3} \int_{1-t}^1 (1-s)\, M_p(F'', s)\, ds\, dt$$

$$\leqslant C \int_0^1 \frac{\rho(1-s)}{1-s} M_p(F'', s)\, ds + C$$

$$+ C \int_0^1 \left(\int_{1-s}^1 \frac{\rho(t)}{t^3}\, dt \right) (1-s)\, M_p(F'', s)\, ds.$$

Therefore the b_2 condition implies (2.7).

To see the converse of (i) and (ii) let us assume $F \in H^p$ and is represented by (2.8).

From (2.8) a standard argument shows that

$$F_{\theta\theta}(re^{i\theta}) = \int_0^1 P_{\theta\theta}(r, t)\, \Delta_t^2 f(\theta)\, dt.$$

Then

$$M_p(F_{\theta\theta}, r) \leqslant \int_0^1 |P_{\theta\theta}(r, t)|\, \|\Delta_t^2 f\|_p\, dt.$$

It is easy to see that

$$|P_{\theta\theta}(r, t)| \leqslant \frac{C}{((1-r)^2 + ct^2)^{3/2}} \qquad \text{for} \quad 0 < t < 1$$

which implies

$$M_p(F_{\theta\theta}, r) \leqslant C \int_0^1 \frac{\|\Delta_t^2 f\|_p}{((1-r)^2 + (t^2))^{3/2}}\, dt. \tag{2.12}$$

Note that $z^2 F''(z) = F_\theta(z) - F_{\theta\theta}(z)$ and $F_{\theta r}(z) = |z|\, F_{\theta\theta}(z)$ then it follows that for $r > \frac{1}{2}$, $M_p(F'', r) \leqslant M_p(F_\theta, r) + M_p(F_{\theta\theta}, r) \leqslant M_p(F_{\theta\theta}, r) + C$. Hence

$$M_p(F'', r) \leqslant C \int_0^1 \frac{\|\Delta^2 f\|_p\, dt}{((1-r)^2 + ct^2)^{3/2}} + C. \tag{2.13}$$

If we assume (2.6) then (2.13) says

$$M_p(F'', r) \leqslant C \int_0^1 \frac{\rho(t)\, dt}{((1-r)^2 + ct^2)^{3/2}} + C$$

and (1.6) in Lemma 1.1, together with $C \leqslant \rho(t)/t^2$ gives that F must belong to Z_ρ^p.

Assume now (2.7). Using (2.13) again we have

$$\int_0^1 \frac{\rho(1-r)}{1-r} M_p(F'', r)\, dr \leqslant C + \int_{1/2}^1 \frac{\rho(1-r)}{1-r} M_p(F'', r)\, dr$$

$$\leqslant C + \int_{1/2}^1 \frac{\rho(1-r)}{1-r} \left(\int_0^1 \frac{\|\Delta_t^2 f\|_p}{((1-r)^2 + ct^2)^{3/2}} \right) dr$$

$$\leqslant C + \int_0^1 \left(\int_0^1 \frac{\rho(1-r)\, dt}{(1-r)((1-r)^2 + ct^2)^{3/2}} \right) \|\Delta_t^2 f\|_p\, dr.$$

Observe now that if $\rho \in b_2$ then also $\rho \in b_3$, so applying (1.8) in Lemma 1.2 we get that

$$\int_0^1 \frac{\rho(1-r)}{1-r} M_p(F'', r)\, dr \leqslant C + \int_0^1 \frac{\rho(t)}{t^3} \|\Delta_t^2 f\|_p\, dt < +\infty.$$

The reader is referred to [Z1] and [T] to get special cases of Theorem 2.2(i) and (ii), respectively, and to [B-S2] to see a slightly different approach to part (i) for the case $p = \infty$.

3. Duality Results

Our next objective is to find the predual space of HL_ρ^q and Z_ρ^q depending on conditions for ρ.

The next theorems are inspired by ideas from several papers. In [D-R-S], among other things, the predual spaces of Λ_α $(0 < \alpha < 1)$ and Λ_* were characterized in terms of the following space

$$B_p = \left\{ F: D \to C \text{ analytic}: \int_0^1 \int_{-\pi}^{\pi} (1-r)^{1/p-2} |F(re^{i\theta})|\, d\theta\, dr < +\infty \right\},$$

where $p = 1/(1+\alpha)$ if $0 < \alpha < 1$ and $p = \frac{1}{2}$, respectively.

Flett, in [F], has got an extension to Λ_α^q for $1 < q < \infty$. Another interesting result of duality was achieved in [A-C-P] a few years later, namely it was shown that the predual of the space of Bloch functions can be identified as

$$J = \left\{ F: D \to C \text{ analytic}: \int_0^1 \int_{-\pi}^{\pi} |F'(re^{i\theta})|\, d\theta\, dr < +\infty \right\}.$$

Here we shall present two theorems which cover all these cases and also get some extensions of those. The reader is referred to [B-S1] and [B-S2] for some other duality results using block decompositions for the special case $q = \infty$.

THEOREM 3.1. *Let ρ be b_1, $1 < q \leqslant \infty$ and $1/p + 1/q = 1$. The predual space of HL^q_ρ is isomorphic to*

$$J^p_\rho = \left\{ F: D \to C \text{ analytic}: \int_0^1 \rho(1-r)\, M_p(F', r)\, dr < +\infty \right\}$$

endowed with the norm $\|F\|_{J,p,\rho} = |F(0)| + \int_0^1 \rho(1-r)\, M_p(F', r)\, dr$.

Proof. Let us take $G(z) = \sum_{n=0}^{\infty} a_n z^n$ in HL^q_ρ and $F(z) = \sum_{n=0}^{\infty} b_n z^n$ in J^p_ρ. Then define

$$\phi(r) = \sum_{n=1}^{\infty} a_n b_n r^{n-1} \qquad \text{for} \quad 0 < r < 1. \tag{3.1}$$

(The reader can easily show that $|a_n| = O(1)$ and $|b_n| = O(n)$ which gives sense to (3.1) for $0 < r < 1$.)

We shall show that $\{\phi(r)\}_{0<r<1}$ is a Cauchy net. Let us rephrase (3.1) using the equality

$$2(n+1)n \int_0^1 (1-s^2)\, s^{2n-1}\, ds = 1 \qquad n \geqslant 1.$$

Therefore

$$\phi(r) = 2 \int_0^1 (1-s^2) \left(\sum_{n=1}^{\infty} n b_n (rs)^{n-1} (n+1)\, a_n s^n \right) ds$$

which implies

$$\phi(r) = (1/\pi) \int_0^1 (1-s^2) \int_{-\pi}^{\pi} F'(rse^{i\theta})\, G_1'(se^{-i\theta})\, e^{-i\theta}\, d\theta\, ds, \tag{3.2}$$

where $G_1(z) = z[G(z) - G(0)]$.

Notice that $G_1'(z) = zG'(z) + \int_0^r G'(se^{i\theta})\, e^{i\theta}\, ds$ for $z = re^{i\theta}$ which implies that $M_q(G_1', r) \leqslant 2r M_q(G', r)$ and then $G_1 \in HL^q_\rho$ and also

$$\|G_1\|_{HL,q,\rho} \leqslant C \|G\|_{HL,q,\rho}.$$

Using (3.2) we have

$$\phi(r) - \phi(r') = (1/\pi) \int_0^1 \int_{-\pi}^{\pi} (1-s^2)[F'(rse^{i\theta}) - F'(r'se^{i\theta})] \times G_1'(se^{-i\theta})\, e^{-i\theta}\, d\theta\, ds$$

d to [B-S1] and [B-S2] for
ɔositions for the special case

$d\ 1/p+1/q=1$. *The predual*

$M_p(F', r)\, dr < +\infty \Big\}$

$(1-r)\, M_p(F', r)\, dr.$

L^q_ρ and $F(z)=\sum_{n=0}^{\infty} b_n z^n$ in

$0<r<1.$ (3.1)

and $|b_n| = O(n)$ which gives

ıy net. Let us rephrase (3.1)

$1 \qquad n \geqslant 1.$

$(n+1)\, a_n s^n \Big)\, ds$

$F_1(se^{-i\theta})\, e^{-i\theta}\, d\theta\, ds,$ (3.2)

s for $z=re^{i\theta}$ which implies
L^q_ρ and also

$_{,q,\rho}.$

$(rse^{i\theta}) - F'(r'se^{i\theta})]$

and then applying Holder's inequality, we get

$$|\phi(r)-\phi(r')| \leqslant (2/\pi) \int_0^1 (1-s)\, M_q(G_1, s)\, M_p(F'_r - F'_{r'}, s)\, ds$$

$$\leqslant C \int_0^1 \rho(1-s)\, M_p(F'_r - F'_{r'}, s)\, ds.$$

A simple application of the Lebesgue convergence theorem shows that $|\phi(r)-\phi(r')| \to 0$ as $r, r' \to 1$. This allows us to define the linear functional

$$\Phi(F) = \lim_{r\to 1} \phi(r).$$

Its boundedness follows also from (3.2)

$$|\Phi(F)| \leqslant \sup_{0<r\leqslant 1} |\phi(r)| \leqslant \sup_{0<r\leqslant 1} \int_0^1 (1-s)\, M_q(G_1, s)\, M_p(F', rs)\, ds$$

$$\leqslant C \int_0^1 \rho(1-s)\, M_p(F', s)\, ds \leqslant C \|F\|_{F,p,\rho}.$$

Conversely let us take ψ in $(J^p_\rho)^*$ and consider $a_n = \psi(u_n)$, where $u_n(z) = z^n$ $(n \geqslant 0)$.

Since $\|u_n\|_{J,p,\rho} \leqslant C \int_0^1 \rho(1-s)\, s^{n-1}\, ds \leqslant Cn$, we may define the analytic function

$$G(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{3.3}$$

Now we estimate $M_q(G', r)$ as follows. Fix $0<r<1$,

$$M_q(G', r) = \left| (1/2\pi) \int_{-\pi}^{\pi} G'(re^{i\theta})\, f(e^{-i\theta})\, d\theta \right|$$

for some $f \in L^p$ with $\|f\|_p = 1$.

Using Fourier expansion we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} G'(re^{i\theta})\, f(e^{-i\theta})\, d\theta = \sum_{n=1}^{\infty} n\psi(u_n)\, \hat{f}(n-1)\, r^{n-1}.$$

Notice that $\sum_{n=1}^{\infty} n\hat{f}(n-1)\, r^{n-1} u_n = F_{(r)}$ converges absolutely in J^p_ρ which allows us to write

$$M_q(G', r) = |\psi(F_{(r)})| \leqslant \|\psi\| \cdot \|F_{(r)}\|_{J,p,\rho}. \tag{3.4}$$

It is elementary to show that

$$F_{(r)}(z)=z^2F'(rz)+zF(rz) \text{ and } F(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(e^{it})}{e^{it}-z}\,e^{it}\,dt$$

which gives

$$M_p(F'_{(r)},s)\leqslant CM_p(F'',rs)+C.$$

Notice that $\|f\|_p=1$ and

$$F''(re^{i\theta})=\frac{3}{\pi}\int_{-\pi}^{\pi}\frac{f(e^{i(\theta+t)})}{(e^{it}-r)^3}\,e^{i(t-2\theta)}\,dt$$

then

$$M_p(F'',rs)\leqslant C\int_{-\pi}^{\pi}\frac{dt}{|e^{it}-rs|^3}\leqslant C\left(\frac{1}{(1-rs)^2}\right).$$

(The reader is referred to [D, p. 65] for the last inequality.) Hence

$$\int_0^1 \rho(1-s)\,M_p(F'_{(r)},s)\,ds\leqslant C+C\int_0^1\rho(1-s)\,M_p(F'',rs)\,ds$$

$$\leqslant C+C\int_0^1\frac{\rho(1-s)}{(1-rs)^2}\,ds\leqslant C\,\frac{\rho(1-r)}{1-r},$$

where the last inequality follows from the fact $\rho\in b_1$ and (1.7) in Lemma 1.1. Finally $G\in H^q_\rho$ from this inequality and (3.4).

THEOREM 3.2. *Let ρ be Dini and b_2, $1<q\leqslant\infty$ and $1/p+1/q=1$. Then*

$$(B^p_\rho)^*=Z^q_\rho.$$

Proof. We follow a similar argument as in the previous theorem , but interpreting things slightly differently. Take

$$G(z)=\sum_{n=0}^{\infty}a_nz^n \quad \text{in } Z^q_\rho \quad \text{and} \quad F(z)=\sum_{n=0}^{\infty}b_nz^n \quad \text{in } B^p_\rho.$$

Now we define

$$\phi(r)=\sum_{n=0}^{\infty}a_nb_nr^n \tag{3.5}$$

and rewrite it as

$$\phi(r) = a_0 b_0 + 2\int_0^1 (1-s^2)\left(\sum_{n=1}^{\infty} (n+1)\, n a_n s^{n-1} \cdot a_n (sr)^n\right) ds.$$

Hence

$$\phi(r) = a_0 b_0 + \frac{1}{\pi}\int_0^1 (1-s^2)\left(\int_{-\pi}^{\pi} F(rse^{i\theta})\, G_2''(se^{-i\theta})\, e^{i\theta}\, d\theta\right) ds, \tag{3.6}$$

where $G_2(z) = zG(z)$.

Note that $G_2''(z) = zG''(z) + 2G'(z)$ which gives

$$M_q(G_2'', r) \leqslant CM_q(G'', r) + 2G'(0).$$

Using condition b_2 (in particular $C \leqslant \rho(t)/t^2$) we get that G_2 belongs to Z_ρ^q and $\|G_2\|_{z,q,\rho} \leqslant C\|G\|_{Z,q,\rho}$.

Using (3.6) we can write

$$\phi(r) - \phi(r') = (1/\pi)\int_0^1 \int_{-\pi}^{\pi} (1-s^2)[F(rse^{i\theta}) - F(r'se^{i\theta})] \times G_2''(se^{-i\theta})\, e^{i\theta}\, d\theta\, ds.$$

Therefore Holder's inequality and $G_2 \in Z_\rho^q$ imply

$$|\phi(r) - \phi(r')| \leqslant C\int_0^1 \frac{\rho(1-s)}{1-s} M_p(F_r - F_{r'}, s)\, ds.$$

To finish the direct implication we repeat the argument in the previous theorem and we take into account that

$$|F(0)| \leqslant C\int_0^1 |F(0)|\, \frac{\rho(1-s)}{1-s}\, ds \leqslant C\int_0^1 \frac{\rho(1-s)}{1-s} M_p(F, s)\, ds$$

which allows us to prove that $|\Phi(F) \leqslant C\|F\|_{B,p,\rho}$.

To do the converse we have at our disposal an extra fact to use. That is, H^p is dense in B_ρ^p (since $\rho(t)/t \in L^1$) then if $\psi \in (B_\rho^p)^*$ there is a G in $H^q = (H^p)^*$, $1 < p < \infty$, or in BMOA for $p = 1$ with boundary values g such that

$$\psi(F) = (1/2\pi)\int_{-\pi}^{\pi} g(e^{i\theta})\, f(e^{-i\theta})\, d\theta \tag{3.7}$$

for all F in H^p with boundary limit f.

Again we can write

$$M_q(G'', r) = \left| (1/2\pi) \int_{-\pi}^{\pi} G''(re^{i\theta}) f(e^{i\theta})\, d\theta \right|$$

for some f in the unit ball of L^p.

It is easy to write now

$$M_q(G'', r) = \left| (1/2\pi) \int_{-\pi}^{\pi} g(e^{i\theta}) F_{(r)}(e^{-i\theta})\, d\theta \right| = |\psi(F_{(r)}|, \tag{3.8}$$

where $F_{(r)}(z) = \sum_{n=0}^{\infty} (n+2)(n+1) \hat{f}(n) r^n z^{n+2}$ that is

$$F_{(r)}(z) = \int_{-\pi}^{\pi} \frac{f(e^{-it}) H(r, e^{it}, z)}{(1 - re^{it}z)^3}\, dt \tag{3.9}$$

and $|H(r, e^{it}, z)| \leqslant C$.

The same estimate as before gives

$$M_p(F_{(r)}, s) \leqslant C \|f\|_p \int_{-\pi}^{\pi} \frac{dt}{|1 - rse^{it}|^3} \leqslant \frac{C}{(1 - rs)^2}.$$

Thus

$$\|F_{(r)}\|_{B,p,\rho} = \int_0^1 \frac{\rho(1-s)}{1-s} M_p(F_{(r)}, s)\, ds$$

$$\leqslant C \int_0^1 \frac{\rho(1-s)}{(1-s)} \frac{ds}{(1-rs)^2}.$$

Applying (1.9) in Lemma 1.2 together with (3.8) gives

$$M_q(G'', r) \leqslant \|\psi\| \cdot \|F_{n,r}\|_{J,p,\rho} \leqslant C \frac{\rho(1-r)}{(1-r)^2}.$$

References

[A-C-P] J. M. Anderson, J. Clunie, and Ch. Pommerenke, On Bloch functions and normal functions, *J. Reine Angew. Math.* **270** (1974), 12–37.

[B-S1] S. Bloom and G. S. de Souza, Weighted Lipschitz spaces and their analytic characterization, preprint.

[B-S2] S. Bloom and G. S. de Souza, Atomic decomposition of generalize Lipschitz spaces, *Illinois J. Math.* **33**, No. 2 (1989), 181–209.

[D] P. L. Duren, "Theory of H^p spaces." Academic Press, New York, 1970.

[D-R-S] P. L. Duren, B. W. Romberg, and A. L. Shields, Linear functionals on H^p-spaces $0 < p < 1$, *J. Reine Angew. Math.* **258** (1969), 32–60.

[F] T. M. Flett, Lipschitz s *Appl.* **39** (1972), 125–15

[H-L] C. H. Hardy and J. E *Math. Z.* **34** (1932), 403

[J] S. Janson, Generalizati and bounded mean osci

[S] H. Shapiro, Mackey to and Bergman spaces, *D*

[S-W] A. L. Shields and D. in spaces of analytic fun

[T] M. Taibleson, On the *n*-space, I, II, III, *J. Ma* 973–981.

[Z1] A. Zygmund, Smooth f

[Z2] A. Zygmund, "Trigonor 1959.

Printed by Cat

$\left.{}^{\theta}) f(e^{i\theta})\, d\theta\right|$

$\left.{}^{-i\theta})\, d\theta\right| = |\psi(F_{(r)}|,$ (3.8)

that is

$\frac{{}^{it}, z)}{{}^{3}}\, dt$ (3.9)

$\frac{}{{}^{3}} \leq \frac{C}{(1-rs)^2}.$

$F_{(r)}, s)\, ds$

$\frac{ds}{-rs)^2}.$

8) gives

$C\frac{\rho(1-r)}{(1-r)^2}.$

KE, On Bloch functions and normal
-37.
ipschitz spaces and their analytic

mposition of generalize Lipschitz
09.
Press, New York, 1970.
os, Linear functionals on H^p-spaces
-60.

[F] T. M. FLETT, Lipschitz spaces of functions on the circle and the disc, *J. Math. Anal. Appl.* **39** (1972), 125–158.

[H-L] C. H. HARDY AND J. E. LITTLEWOOD, Some properties of fractional integrals II, *Math. Z.* **34** (1932), 403–439.

[J] S. JANSON, Generalization on Lipschitz spaces and applications to Hardy spaces and bounded mean oscillation, *Duke Math. J.* **47** (1980), 959–982.

[S] H. SHAPIRO, Mackey topologies, reproducing kernels and diagonal maps on Hardy and Bergman spaces, *Duke Math. J.* **43** (1976), 187–202.

[S-W] A. L. SHIELDS AND D. L. WILLIAMS, Bounded projections, duality and multipliers in spaces of analytic functions, *Trans. Amer. Math. Soc.* **162** (1971), 287–302.

[T] M. TAIBLESON, On the theory of Lipschitz spaces and distributions on Euclidean *n*-space, I, II, III, *J. Math. Mech.* **13** (1964), 407–479, **14** (1965), 821–839, **15** (1966), 973–981.

[Z1] A. ZYGMUND, Smooth functions, *Duke Math. J.* **12** (1945), 47–76.

[Z2] A. ZYGMUND, "Trigonometric Series," Cambridge Univ. Press, London/New York, 1959.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium