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Interpolation between $H^1_{B_0}$ and $L^p_{B_1}$

by

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Abstract. It is proved that $[H^1_{B_0}, L^p_{B_1}]_\theta = L^q_{[B_0,B_1]_\theta}$ for $1/q = 1-\theta+\theta/p$.

§ **0. Introduction.** In this paper we are concerned with interpolation between Hardy spaces and L^p-spaces of vector-valued functions. Following the notation in [1] we write $[A_0, A_1]_\theta$ and $(A_0, A_1)_{\theta,q}$ for the interpolation spaces by the complex method [2] and the real method [13] respectively. Throughout this paper $(B, \|\cdot\|_B)$ stands for a Banach space and B_0, B_1 will be an interpolation pair of Banach spaces.

The Hardy space we shall deal with will be the following [7]:

$$H^1_B = \{f \in L^1_B(\boldsymbol{R}^n): \int \sup_{t>0} \|P_t * f(x)\|_B \, dx < +\infty\},$$

P_t being the Poisson kernel on $\boldsymbol{R}^n$, and the main result of the paper is:

$$[H^1_{B_0}, L^p_{B_1}]_\theta = L^q_{[B_0,B_1]_\theta},$$

where $0<\theta<1$, $1<p<\infty$, and $1/q = 1-\theta+\theta/p$.

For the case $B_0 = B_1 = \boldsymbol{R}$, this is the classical result of Fefferman and Stein [7]. They proved it using the duality $(H^1)^* = \mathrm{BMO}$, and considering the "sharp" maximal function. Their technique works also in the case $B_0 = B_1 = B$, but for the general case we shall use a different approach based on the atomic decomposition of functions in H^1_B. The ideas we shall use later have been considered by different authors (see [9], [8], [11]).

Recently several authors have extended Fefferman–Stein's complex interpolation result in the sense of replacing L^p on the right side by L^∞ and BMO, i.e. for $1/q = 1-\theta$,

$$[H^1, L^\infty]_\theta = [L^1, \mathrm{BMO}]_\theta = [H^1, \mathrm{BMO}]_\theta = L^q.$$

The reader is referred to [9], [12], [15] for different approaches to this result.

There are also interpolation results for H^p for $0<p<1$ (see [3], [6], [9]) but we restrict ourselves to the case $p=1$.

We shall denote by $(L^p_B, \|\cdot\|_{p,B})$ the space $L^p_B(\boldsymbol{R}^n)$ with its usual norm, for $1<p\leqslant\infty$, and since we are not going to use L^1_B let us denote by $\|\cdot\|_{1,B}$ the

norm in H_B^1. As we have already said we shall consider H_B^1 defined in terms of atoms (see [10], [4], [5]). The reader can realize that the classical proofs also work for vector-valued functions on merely replacing the absolute value by the norm, so for each f in H_B^1 we write

$$\|f\|_{1,B} = \inf\{\sum_k |\lambda_k| : f = \sum_k \lambda_k a_k,\ a_k \text{ are } B\text{-atoms}\}.$$

As usual, C will denote a constant but not necessarily the same at each occurrence.

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§ 1. The theorem and its corollaries. Let us formulate here a lemma which is essentially based on the Calderón–Zygmund decomposition and some arguments involved in Coifman's proof [4] for the atomic decomposition. The details are left to the reader.

LEMMA. *Given a B-valued simple function f, there exist a family of cubes $\{Q_j^k\}$ and a family of simple functions $\{a_j^k\}$ such that each a_j^k is supported in Q_j^k and*

$$\int a_j^k(x)\,dx = 0, \tag{1}$$

$$f = \sum_{k,j} a_j^k, \tag{2}$$

$$\|a_j^k(x)\|_B \leqslant C_0 2^k \chi_{Q_j^k}(x) \quad \text{for all } j, \tag{3}$$

$$\bigcup_j Q_j^k = \Omega_k = \{x\colon Mf(x) > 2^k\}, \tag{4}$$

where Mf stands for the Hardy–Littlewood maximal function of f.

THEOREM A. *Let $1 < p < \infty$, $0 < \theta < 1$ and $1/q = 1-\theta+\theta/p$. Then*

$$[H_{B_0}^1, L_{B_1}^p]_\theta = L_{B_\theta}^q, \quad \text{where} \quad B_\theta = [B_0, B_1]_\theta. \tag{5}$$

Proof. Since $H_{B_0}^1 \subseteq L_{B_0}^1$ the classical results about interpolation obviously imply

$$[H_{B_0}^1, L_{B_1}^p]_\theta \subseteq [L_{B_0}^1, L_{B_1}^p]_\theta = L_{B_\theta}^q.$$

Consider now a B_θ-valued simple function f. Using the lemma write $f = \sum a_j^k$, where a_j^k is also a B_θ-valued simple function which can be expressed as

$$a_j^k = \sum_{m=1}^{n(j,k)} x_m^{j,k} \chi_{E_m^{j,k}},$$

the $x_m^{j,k}$ being elements in B_θ and $\bigcup_m E_m^{j,k} = Q_j^k$.

Let $\Omega = \{z \in \boldsymbol{C}\colon\ 0 < \operatorname{Re} z < 1\}$. Given $\varepsilon > 0$ we choose continuous functions $f^{j,k}_m\colon \bar{\Omega} \to B_0 + B_1$, holomorphic in Ω and satisfying

$$f^{j,k}_m(\theta) = x^{j,k}_m, \qquad \|f^{j,k}_m(it)\|_{B_0} \leqslant (1+\varepsilon)\|x^{j,k}_m\|_{B_\theta},$$
$$\|f^{j,k}_m(1+it)\|_{B_1} \leqslant (1+\varepsilon)\|x^{j,k}_m\|_{B_\theta} \quad \text{for all } t \in \boldsymbol{R}.$$

Defining

$$F^k_j(z) = \sum_{m=1}^{n(j,k)} f^{j,k}_m(z)\,\chi_{E^{j,k}_m}$$

we get continuous functions $F^k_j\colon \bar{\Omega} \to L^\infty_{B_0}(Q^k_j) + L^\infty_{B_1}(Q^k_j)$ which are holomorphic in Ω and satisfy

$$F^k_j(\theta) = a^k_j, \tag{6}$$

$$\|F^k_j(it)(x)\|_{B_0} \leqslant (1+\varepsilon)\|a^k_j(x)\|_{B_\theta} \quad \text{for all } x \in Q^k_j,\ t \in \boldsymbol{R}, \tag{7}$$

$$\|F^k_j(1+it)(x)\|_{B_1} \leqslant (1+\varepsilon)\|a^k_j(x)\|_{B_\theta} \quad \text{for all } x \in Q^k_j,\ t \in \boldsymbol{R}. \tag{8}$$

Let us consider

$$G^k_j(z) = F^k_j(z) - \Big(|Q^k_j|^{-1} \int_{Q^k_j} F^k_j(z)(x)\,dx\Big)\chi_{Q^k_j}. \tag{9}$$

Setting $r(z) = q(1 - z + z/p) - 1$ we define

$$F(z) = \sum_{k,j} (2^k)^{r(z)}\, G^k_j(z). \tag{10}$$

From (2) and (6) we clearly have $F(\theta) = f$. Now we want to prove that

$$\sup_t \{\|F(it)\|_{1,B_0},\ \|F(1+it)\|_{p,B_1}\} \leqslant C\|f\|_{q,B_\theta}.$$

To check the norm $\|F(1+it)\|_{p,B_1}$ we first observe that

$$\|F(1+it)(x)\|_{B_1} \leqslant C \sum_{k,j} 2^{k(q/p-1)}\|G^k_j(1+it)(x)\|_{B_1}$$

and according to (8) and (3) we can write

$$\|F(1+it)(x)\|_{B_1} \leqslant C(1+\varepsilon)\sum_{k,j} 2^{k(q/p-1)}\big(\|a^k_j(x)\|_{B_\theta} + \|a^k_j\|_{\infty,B_\theta}\chi_{Q^k_j}(x)\big)$$
$$\leqslant C(1+\varepsilon)\sum_{k,j} 2^{kq/p}\chi_{Q^k_j}(x).$$

Hence we get

$$\|F(1+it)\|_{p,B_1} \leqslant C(1+\varepsilon)\sum_{k,j} 2^{kq}|Q^k_j| \leqslant C(1+\varepsilon)\sum_k 2^{kq}|\Omega_k|$$
$$\leqslant C(1+\varepsilon)\|Mf\|_q \leqslant C(1+\varepsilon)\|f\|_{q,B_\theta}.$$

To compute $\|F(it)\|_{1,B_0}$ let us write $\lambda_j^k = C_0(1+\varepsilon)2^{k+1}|Q_j^k|$ and $b_k^j = (\lambda_j^k)^{-1} G_j^k(it)$. From (9), (7), and (3) the b_j^k are B_0-atoms and we have

$$F(it) = \sum_{k,j} 2^{kr(it)} \lambda_j^k b_j^k,$$

therefore

$$\|F(it)\|_{1,B_0} \leqslant C \sum_{k,j} 2^{k(q-1)} |\lambda_j^k| = C(1+\varepsilon) \sum_{k,j} 2^{kq} |Q_j^k|$$

and the above computation shows that $\|F(it)\|_{1,B_0} \leqslant C(1+\varepsilon)\|f\|_{q,B_\theta}$.

Since ε can be chosen arbitrarily small we have just proved that for any simple function $\|f\|_\theta \leqslant C\|f\|_{q,B_\theta}$, and the proof is completed by a simple density argument.

Now we want to deduce some interpolation result for BMO_B, and some minor conditions have to be imposed on the Banach spaces B_0 and B_1 in order to be able to apply duality interpolation results [1]:

(*) $B_0 \cap B_1$ is dense in both B_0 and B_1,

(**) $B_0^* \cap B_1^*$ is dense in both B_0^* and B_1^*.

Corollary 1. *Suppose B_0 and B_1 satisfy (*) and (**) and let $0 < \theta < 1$, $1 < p < \infty$ and $1/q = (1-\theta)/p$. Then*

$$[L_{B_0}^p, \mathrm{BMO}_{B_1}]_\theta = L_{[B_0,B_1]_\theta}^q. \tag{11}$$

Proof. Since $L_{0,B_1}^\infty \subseteq \mathrm{BMO}_{B_1}$, where L_{0,B_1}^∞ is the closure of the simple functions in $L_{B_1}^\infty$, we already have

$$L_{B_\theta}^q = [L_{B_0}^p, L_{0,B_1}^\infty]_\theta \subseteq [L_{B_0}^p, \mathrm{BMO}_{B_1}]_\theta.$$

Recall now the dualities $L_{B_0}^p \subseteq (L_{B_0^*}^{p'})^*$, $1/p + 1/p' = 1$, and $\mathrm{BMO}_{B_1} \subseteq (H_{B_1^*}^1)^*$. Applying Theorem A and the duality interpolation theorem we can write

$$\begin{aligned}[L_{B_0}^p, \mathrm{BMO}_{B_1}]_\theta &\subseteq [(L_{B_0^*}^{p'})^*, (H_{B_1^*}^1)^*]_\theta = [(H_{B_1^*}^1)^*, (L_{B_0^*}^{p'})^*]_{1-\theta} \\ &= [H_{B_1^*}^1, L_{B_0^*}^{p'}]_{1-\theta}^* = (L_{[B_1^*,B_0^*]_{1-\theta}}^r)^* \\ &= (L_{[B_0,B_1]_\theta^*}^r)^*,\end{aligned}$$

where $1/r = 1-\theta+\theta/p'$, i.e. $r = q'$.

To finish the proof it suffices to realize that if a function f in L_B^1 belongs to $(L_{B^*}^{q'})^*$ then f has to belong to L_B^q.

Our next corollary will use Wolff's reiteration theorem [15]; let us recall it for the sake of clarity:

Theorem B ([15]). *Let A_1, A_2, A_3, A_4 be Banach spaces such that $A_1 \cap A_4$ is dense in both A_2 and A_3. Let $0 < \theta_1, \theta_2 < 1$ and $[A_2, A_4]_{\theta_1} = A_3$*

and $[A_1, A_3]_{\theta_2} = A_2$. *Then*

$$[A_1, A_4]_\eta = A_2, \quad \text{where} \quad \eta = \frac{\theta_1\,\theta_2}{1-\theta_2+\theta_1\,\theta_2}. \tag{12}$$

With this result and denoting by $L^\infty_{0,B}$ the closure of the simple functions in L^∞_B, we have the following corollary:

COROLLARY 2. *Let* $0 < \theta < 1$ *and* $1/p = 1-\theta$. *Then*

$$[H^1_{B_0}, L^\infty_{0,B_1}]_\theta = L^p_{[B_0,B_1]_\theta}. \tag{13}$$

If B_0 *and* B_1 *satisfy* (*) *and* (**) *we also have*

$$[L^1_{B_0}, \mathrm{BMO}_{B_1}]_\theta = [H^1_{B_0}, \mathrm{BMO}_{B_1}]_\theta = L^p_{[B_0,B_1]_\theta}. \tag{14}$$

Proof. Here we only present the proof of (13), leaving (14) as an exercise. Consider $p_0 = p+\sqrt{p(p-1)}$. This value is chosen to satisfy

$$(p_0-1)\,p = p_0(p_0-p). \tag{15}$$

Take $\theta_1 = 1-p/p_0$, $\theta_2 = p_0'/p'$. Then it is easy to show that (15) implies that η in Theorem B coincides with θ. Choosing

$$A_1 = H^1_{B_0}, \quad A_2 = L^p_{B_0}, \quad A_3 = L^{p_0}_{B_\gamma}, \quad A_4 = L^\infty_{0,B_1},$$

where $\gamma = (1-\theta_1)+\theta_2$, we can easily check all of the assumptions of Theorem B and then we get (13).

It is very well known that once the complex interpolation is obtained then the real interpolation can also be got by using the following theorem:

THEOREM C ([1]). *Let* $0 < \theta_1 < \theta_2 < 1$, $0 < \eta < 1$, $0 < p \leqslant \infty$. *Then for* $\theta = (1-\eta)\,\theta_1+\eta\theta_2$ *we get*

$$(A_0, A_1)_{\theta,p} = ([A_0, A_1]_{\theta_1}, [A_0, A_1]_{\theta_2})_{\eta,p}. \tag{16}$$

From this last theorem and the above results it is an easy exercise to derive the following corollary:

COROLLARY 3. *Let* $0 < \theta < 1$, $1 < p \leqslant \infty$, $1/p = 1-\theta+\theta/p$. *Then*

$$(H^1_{B_0}, L^p_{B_1})_{\theta,q} = L^q_{(B_0,B_1)_{\theta,q}}. \tag{17}$$

If B_0 *and* B_1 *satisfy* (*) *and* (**), *and* $0 < \theta < 1$, $1 \leqslant p < \infty$, *then*

$$(L^p_{B_0}, \mathrm{BMO}_{B_1})_{\theta,q} = L^q_{(B_0,B_1)_{\theta,q}}, \quad 1/q = (1-\theta)/p, \tag{18}$$

$$(H^1_{B_0}, \mathrm{BMO}_{B_1})_{\theta,q} = L^q_{(B_0,B_1)_{\theta,q}}, \quad 1/q = 1-\theta. \tag{19}$$

Remark. Finally, we would like to mention that in the case $B_0 = B_1 = B$ we can do real interpolation not only for a fixed value of q, as in the above corollary, but for all values $0 < r \leqslant \infty$, and it can be shown, either by using similar arguments to Corollary 3 or with an analogous proof to that given in [14], that for $1 < p \leqslant \infty$, $0 < r \leqslant \infty$, and $1/q = 1-\theta+\theta/p$,

$$(H_B^1, L_B^p)_{\theta,r} = L_B^{qr},$$

where L^{qr} stands for a Lorentz space.

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