International Journal of Modern Physics D
Vol. 17, Nos. 3 & 4 (2008) 489–494

THE ROLE OF THE PLANCK SCALE IN BLACK HOLE RADIANCE*

IVÁN AGULLÓ† and JOSÉ NAVARRO-SALAS‡

Departamento de Física Teórica and IFIC,
Universidad de Valencia–CSIC, Facultad de Física,
Burjassot-46100, Valencia, Spain
†*ivan.agullo@uv.es*
‡*jnavarro@ific.uv.es*

GONZALO J. OLMO§ and LEONARD PARKER¶

Physics Department, University of Wisconsin–Milwaukee,
PO Box 413, Milwaukee, WI 53201, USA
§*olmoalba@uwm.edu*
¶*leonard@uwm.edu*

Received 4 December 2007
Communicated by D. V. Ahluwalia-Khalilova

Lorentz invariance plays a pivotal role in the derivation of the Hawking effect, which crucially requires an integration in arbitrarily small distances or, equivalently, in unbounded energies. New physics at the Planck scale could, therefore, potentially modify the emission spectrum. We argue, however, that the kinematic invariance can be deformed in such a way that the thermal spectrum remains insensitive to trans-Planckian physics.

Keywords: Hawking radiation; Planck scale.

The combination of gravity, relativity and quantum mechanics offers us natural scales for energy and length — $E_P = \sqrt{\hbar c^5/G}$ and $l_P = \sqrt{\hbar G/c^3}$, respectively — at which the standard description of space–time is expected to break down, opening an exciting window for new physics. However, this same combination leads to the phenomenon of black hole radiance,[1–4] whose derivation requires the assumption of exact local Lorentz invariance. In fact, any emitted Hawking quanta will have an unbounded, exponentially increasing energy when propagated backward in time and measured by a freely falling observer near the horizon. In a sense, the horizon acts as an unbounded boost machine. Any cutoff for the energy or the wavelength of

*This essay received an "honorable mention" in the 2007 Essay Competition of the Gravity Research Foundation.

the quanta created by the horizon that breaks Lorentz invariance will also destroy the steady thermal radiation.[5–7]

How can this tension between black hole radiance and the existence of new energy/length scales be overcome? Obviously one could give up (semiclassical) black hole radiance. But this is not for free, because of the deep connection between the Hawking effect and black hole thermodynamics,[8] including the generalized second law.[9,10] A less dramatic way out is to modify Lorentz invariance in such a way that the bulk of the Hawking effect remains unaltered by Planck scale effects. The simplest approach would be to deform the dispersion relation $E^2 = m^2 + \mathbf{p}^2$ of special relativity, introducing a Planck length parameter $\alpha \sim l_P$ and letting $E^2 = m^2 + \mathbf{p}^2 + \alpha F(\mathbf{p}^2)$, where the function F specifies the deformation. This approach has been exploited in the last decade in high energy physics and astrophysics,[11,12] and also in condensed matter analogs for gravity.[13] In this article we will follow an alternative route. We study the phenomenon of black hole radiance by means of the correlation functions of the matter fields and explore the effects that the existence of the Planck length might have on the radiation.

The number of particles of a scalar field in the mode $u_i^{\text{out}}(x)$ measured by a congruence of "out" observers in the vacuum state $|\text{in}\rangle$ of a congruence of "in" observers can in general be expressed, in curved or flat space–time, as the integral

$$\langle \text{in}|N_i^{\text{out}}|\text{in}\rangle = \frac{1}{\hbar}\int_{\Sigma} d\Sigma_1^{\mu} d\Sigma_2^{\nu}[u_i^{\text{out}}(x_1)\overleftrightarrow{\partial}_{\mu}][u_i^{\text{out}*}(x_2)\overleftrightarrow{\partial}_{\nu}]$$
$$\times [\langle \text{in}|\phi(x_1)\phi(x_2)|\text{in}\rangle - \langle \text{out}|\phi(x_1)\phi(x_2)|\text{out}\rangle], \quad (1)$$

where Σ is an initial value hypersurface. A glance at this expression shows that in Minkowski space–time the "out" observers will detect no particles if they are related to the "in" observers by Lorentz transformation. If their relation is not inertial, then $\langle \text{in}|N_i^{\text{out}}|\text{in}\rangle \neq 0$ in general. Note that the regularity of (1) is guaranteed by the universality of the short-distance singularity of the two-point function (the so-called Hadamard condition[14]). In fact, for any physical state $|\psi\rangle$, we have $\langle\psi|\phi(x_1)\phi(x_2)|\psi\rangle \sim \frac{\hbar}{4\pi^2\sigma(x_1,x_2)}$ for any two nearby points x_1 and x_2, where $\sigma(x_1, x_2)$ is the squared geodesic distance.

From (1) it is straightforward to derive the Hawking effect. In the simplest case of a Schwarzschild black hole and a massless scalar field, the expectation values $\langle \text{in}|N_{wlm}^{\text{out}}|\text{in}\rangle$, where w, l, m are frequency and angular momentum quantum numbers, can be worked out as[15,16]

$$\langle \text{in}|N_{wlm}^{\text{out}}|\text{in}\rangle = \frac{4}{\hbar}\int_{I^-} dv_1 r_1^2 d\Omega_1 \int_{I^-} dv_2 r_2^2 d\Omega_2 u_{wlm}^{\text{out}}(x_1) u_{wlm}^{\text{out}*}(x_2)$$
$$\times [\langle \text{in}|\partial_{v_1}\phi(x_1)\partial_{v_2}\phi(x_2)|\text{in}\rangle - \langle \text{out}|\partial_{v_1}\phi(x_1)\partial_{v_2}\phi(x_2)|\text{out}\rangle]. \quad (2)$$

The propagated-backward "out" modes can be approximated by $u_{wlm}^{\text{out}} \approx t_l(w)(4\pi w)^{-1/2} r^{-1} e^{-iwu(v)} Y_{lm}(\Omega)\Theta(v_H - v)$, where v_H represents the location of the null ray that will form the event horizon and $t_l(w)$ are the transmission coefficients. The relation between null inertial coordinates u at I^+ and v at I^- is given by the well-known expression $u(v) \approx \text{constant} - \kappa^{-1} \ln \kappa(v_H - v)$. The relevant derivatives of the two-point functions in (2) can be also expanded in spherical harmonics:

$$\langle \text{in}|\partial_{v_1}\phi(x_1)\partial_{v_2}\phi(x_2)|\text{in}\rangle = \frac{1}{r_1 r_2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\Omega_1)Y^*_{lm}(\Omega_2)G_l^{\text{in}}(v_1, r_1; v_2, r_2). \quad (3)$$

A similar expansion holds for $\langle \text{out}|\partial_{u_1}\phi(x_1)\partial_{u_2}\phi(x_2)|\text{out}\rangle$. In the asymptotic regions I^- and I^+, we have $G_l^{\text{in}}|_{I^-} = -\frac{\hbar}{4\pi}\frac{1}{(v_1-v_2)^2}$ and $G_l^{\text{out}}|_{I^+} = -\frac{\hbar}{4\pi}\frac{1}{(u_1-u_2)^2}$. The latter, when propagated backward to I^-, becomes $G_l^{\text{out}}|_{I^-} = -\frac{\hbar}{4\pi}\frac{\frac{du}{dv}(v_1)\frac{du}{dv}(v_2)}{(u(v_1)-u(v_2))^2}$. Inserting the above expressions in (2), and performing the angular integrations, we end up with

$$\langle \text{in}|N^{\text{out}}_{wlm}|\text{in}\rangle = -\frac{|t_l(w)|^2}{4\pi^2 w}\int_{-\infty}^{v_H} dv_1 dv_2 e^{-iw[u(v_1)-u(v_2)]} \times \left[\frac{1}{(v_1-v_2)^2} - \frac{\frac{du}{dv}(v_1)\frac{du}{dv}(v_2)}{[u(v_1)-u(v_2)]^2}\right], \quad (4)$$

which, for $u(v) \approx \text{constant} - \kappa^{-1} \ln \kappa(v_H - v)$, leads to the thermal emission rate (per unit frequency and unit time)

$$\langle \text{in}|N^{\text{out}}_{wlm}|\text{in}\rangle = \frac{|t_l(w)|^2}{e^{2\pi w/\kappa} - 1}. \quad (5)$$

Note that this result crucially requires that the short-distance singularity of the "in" state be the same as that of the "out" state. Note also that the above integral displays an apparent sensitivity to ultrashort distances due to the highly oscillatory behavior of the modes in the region close to v_H.

Let us now assume that the two-point functions are *deformed* by unknown quantum gravity effects and focus on the derivatives of the two-point functions relevant to the calculation for black hole particle production. The simplest deformation in the asymptotically flat regions I^- and I^+ can be obtained by replacing $G_l^{\text{in}}|_{I^-}$ and $G_l^{\text{out}}|_{I^+}$ with (from now on we omit the subindex l)

$$G_\alpha^{\text{in}}|_{I^-} = -\frac{\hbar}{4\pi}\frac{1}{(v_1-v_2)^2+\alpha^2}, \quad G_\alpha^{\text{out}}|_{I^+} = -\frac{\hbar}{4\pi}\frac{1}{(u_1-u_2)^2+\alpha^2}, \quad (6)$$

where $\alpha \sim l_P$ is the deformation parameter. To ensure the invariance of the above correlation functions under radial boosts with rapidity ξ: $(u, v) \to (\bar{u}, \bar{v}) = (e^{\xi}u, e^{-\xi}v)$ at both asymptotic regions, the action of Lorentz transformations should also be deformed as

$$\bar{G}_\alpha^{\text{out}}|_{I^+} = -\frac{\hbar}{4\pi}\frac{\frac{d\bar{u}_1}{du_1}\frac{d\bar{u}_2}{du_2}}{(\bar{u}_1-\bar{u}_2)^2+\alpha^2\frac{d\bar{u}_1}{du_1}\frac{d\bar{u}_2}{du_2}}, \quad (7)$$

and an analogous expression for $\bar{G}^{\rm in}_\alpha$ at I^-. When $\alpha \to 0$ we recover the transformation law of the unmodified theory. An important consequence of the above deformation is the absence of particle creation under Lorentz boosts, as required on physical grounds if all inertial observers are regarded in flat space–time as physically equivalent.

Our interest now is to compute the particle production rate with the above deformed two-point functions. Due to our heuristic modification of the theory, we face two problems for the computation of the particle production rate: (1) how to evolve the "out" two-point function $G^{\rm out}_\alpha|_{I^+}$ from I^+ to I^-, and (2) how to evolve the "out" modes from I^+ to I^-. Since the propagation to I^- implies a strong blueshift, the $u^{\rm out}_w$ modes and $G^{\rm out}_\alpha$ might manifest some dependence on the particular details of the modified theory, which are unknown to us. Thus, we see no simple way to estimate the form of the $u^{\rm out}_w$ modes. For this reason, it is preferable to evaluate the particle production as an integral on I^+. In this region we can use the standard form of the "out" modes if we consider emission frequencies much lower than the Planck frequency $w_P \sim 1/l_P$. We thus find that

$$\langle {\rm in}|N^{\rm out}_{wlm}|{\rm in}\rangle = \frac{|t_l(w)|^2}{\hbar\pi w}\int_{I^+} du_1 du_2 e^{-iw(u_1-u_2)} \times \left[G^{\rm in}_\alpha|_{I^+} + \frac{\hbar}{4\pi}\frac{1}{(u_1-u_2)^2+\alpha^2}\right]. \tag{8}$$

The problem is then reduced to simply unraveling the form of $G^{\rm in}_\alpha|_{I^+}$. The modified short-distance physics near the horizon could have dramatic effects on the evolution of the two-point function, so that its form at I^+ could be rather different from that at I^-. However, the deformed action of boosts (7) implies that, under the "black hole boost" $v \to u = u(v) \approx \text{constant} - \kappa^{-1}\ln\kappa(v_H - v)$, we should have

$$G^{\rm in}_\alpha|_{I^+} = -\frac{\hbar}{4\pi}\frac{\dfrac{dv_1}{du_1}\dfrac{dv_2}{du_2}}{(v_1-v_2)^2+\alpha^2\dfrac{dv_1}{du_1}\dfrac{dv_2}{du_2}}. \tag{9}$$

This, in turn, implies that the short-distance behavior of $G^{\rm in}_\alpha$ at I^+ is identical to that of the two-point function for the "out" state: $\lim_{u_1\to u_2} G^{\rm in}_\alpha(u_1,u_2)|_{I^+} \sim \lim_{u_1\to u_2} G^{\rm out}_\alpha(u_1,u_2)|_{I^+}$. This condition, which by itself requires (9), can be seen as a natural generalization of the Hadamard condition, i.e. universality of the short-distance behavior for all quantum states. Using (9) in (8) and performing the integration in the complex plane, the particle production rate becomes [we omit the overall factor $|t_l(w)|^2$ for simplicity]

$$\langle {\rm in}|N^{\rm out}_{wlm}|{\rm in}\rangle = \frac{e^{\pi w/\kappa}}{e^{2\pi w/\kappa}-1}\frac{\sinh\left[\dfrac{w}{\kappa}(\theta-\pi)\right]}{\dfrac{w}{\kappa}\sin\theta} + \frac{e^{-w\alpha}}{2w\alpha}, \tag{10}$$

where $\theta \equiv 2\arcsin\left(\frac{\alpha\kappa}{2}\right)$. The thermal Planckian spectrum is smoothly recovered in the limit $\theta \approx \alpha\kappa \ll 1$ as follows:

$$\langle \mathrm{in}|N_w^{\mathrm{out}}|\mathrm{in}\rangle \approx \frac{1}{e^{2\pi w/\kappa}-1} - \frac{\alpha\kappa}{16w/\kappa}. \tag{11}$$

Since $\alpha \sim l_P \approx 1.6\times 10^{-33}\,\mathrm{cm}$, the departure from thermality is negligible in general. In fact, for a solar mass black hole, $\alpha\kappa \sim 10^{-40}$, the ratio of $\frac{\alpha\kappa}{16w/\kappa}$ by $\frac{1}{e^{2\pi w/\kappa}-1}$ in (11) gives a correction at $w_{\mathrm{typical}} \equiv \kappa/2\pi$ of order 10^{-40}. For primordial black holes, $M \sim 10^{15}\,\mathrm{g}$, the correction is of order 10^{-21}. Moreover, in the solar mass case, we need to look at the high frequency region, $w/w_{\mathrm{typical}} \approx 100$, to find a ratio of order unity. For primordial black holes this happens at $w/w_{\mathrm{typical}} \approx 50$. Note that for mini–black holes in TeV gravity scenarios[17,18] the deviations from thermality might therefore be nontrivial, even before reaching the end stages of the evaporation.

Our calculations indicate that only at high emission frequencies could an underlying quantum theory of gravity, with the usual Planck scale, predict significant deviations from the thermal spectrum. Therefore, the fact that string theory exactly reproduces the semiclassical result for some near-extremal black holes and for low emission frequencies[19–21] should not come as a big surprise. Furthermore, since string theory respects Lorentz invariance, it seems that this kinematical symmetry group, with a suitable deformation of its action on the fields, should be able to maintain the essential properties of the Hawking effect.

We have seen that the generalized Hadamard condition is enough to preserve the conformal (Möbius) symmetry, and particularly the Lorentz symmetry, present in the theory with $\alpha = 0$. When a Planck scale parameter α is introduced, that condition implies the existence of a universal quantity, namely $\lim_{x_1\to x_2} G_\alpha^{\mathrm{in}} = \lim_{x_1\to x_2} G_\alpha^{\mathrm{out}} = -\frac{\hbar}{4\pi\alpha^2}$. This invariant length scale, α, modifies the divergent behavior when $x_1 \to x_2$ and becomes the new universal label of the theory. The main lesson of this note is that the way one keeps Hawking radiation as a low energy effect has deep implications for physics and symmetries at very high energy.

References

1. S. W. Hawking, *Nature* **248** (1974) 30.
2. S. W. Hawking, *Commun. Math. Phys.* **43** (1975) 199.
3. L. Parker, *Phys. Rev. D* **12** (1975) 1519.
4. R. M. Wald, *Commun. Math. Phys.* **45** (1975) 9.
5. G. 't Hooft, *Nucl. Phys. B* **335** (1990) 138.
6. T. Jacobson, *Phys. Rev. D* **44** (1991) 1731.
7. T. Jacobson, *Phys. Rev. D* **48** (1993) 728.
8. J. M. Bardeen, B. Carter and S. W. Hawking, *Commun. Math. Phys.* **31** (1973) 181.
9. J. D. Bekenstein, *Phys. Rev. D* **7** (1973) 2333.
10. J. D. Bekenstein, *Phys. Rev. D* **9** (1974) 3292.
11. G. Amelino-Camelia, J. R. Ellis, N. E. Mavrovatos, D. V. Nanopoulos and S. Sarkar, *Nature* **393** (1998) 763.

12. D. Mattingly, *Living Rev. Relativ.* **8** (2005) 5.
13. W. G. Unruh, *Phys. Rev. D* **51** (1995) 2827.
14. R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (Chicago University Press, Chicago, 1994).
15. I. Agullo, J. Navarro-Salas and G. J. Olmo, *Phys. Rev. Lett.* **97** (2006) 041302.
16. I. Agullo, J. Navarro-Salas, G. J. Olmo and L. Parker, *Phys. Rev. D* **76** (2007) 044018.
17. S. B. Giddings, *Gen. Relativ. Gravit.* **34** (2002) 1775.
18. S. Dimopoulos and G. Landsberg, *Phys. Rev. Lett.* **87** (2001) 161602.
19. C. Callan and J. Maldacena, *Nucl. Phys. B* **475** (1996) 645.
20. A. Dhar, G. Mandal and S. R. Wadia, *Phys. Lett. B* **388** (1996) 51.
21. S. Das and S. Mathur, *Nucl. Phys. B* **478** (1996) 561.