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Insensitivity of Hawking radiation to an invariant Planck-scale cutoff

Iván Agulló,[1,2,*] José Navarro-Salas,[2,†] Gonzalo J. Olmo,[3,‡] and Leonard Parker[1,§]

[1]*Physics Department, University of Wisconsin-Milwaukee, P.O.Box 413, Milwaukee, Wisconsin 53201 USA*

[2]*Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC. Facultad de Física, Universidad de Valencia, Burjassot-46100, Valencia, Spain.*

[3]*Instituto de Estructura de la Materia, CSIC, Serrano 121, 28006 Madrid, Spain*

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A disturbing aspect of Hawking's derivation of black hole radiance is the need to invoke extreme conditions for the quantum field that originates the emitted quanta. It is widely argued that the derivation requires the validity of the conventional relativistic field theory to arbitrarily high, trans-Planckian scales. We stress in this note that this is not necessarily the case if the question is presented in a covariant way. We point out that Hawking radiation is immediately robust against an invariant Planck-scale cutoff. This important feature of Hawking radiation is relevant for a quantum gravity theory that preserves, in some way, the Lorentz symmetry.

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The Hawking effect [1] plays a pivotal role in the interplay between quantum mechanics and general relativity and, hence, it is of special relevance in any proposal for a quantum gravity theory. The original derivation of Hawking is based on the general framework of particle creation on curved spacetimes, first developed in a cosmological setting in [2] (see also [3,4]). The derivation considers the propagation of modes that represent particles in the asymptotically flat regions; the first at early times before a dust cloud has begun to collapse, and the second at late times long after it has collapsed to form a black hole as seen by a distant observer. In short, the expansion of a field in two different sets of modes, $u_j^{\rm in}(x)$ (that are positive frequency on past null infinity) and $u_j^{\rm out}(x)$ (that are positive frequency on future null infinity) leads to a relation for the corresponding creation and annihilation operators: $a_i^{\rm out} = \sum_j(\alpha_{ij}^* a_j^{\rm in} - \beta_{ij}^* a_j^{\rm in\dagger})$. When the coefficients β_{ij} do not vanish, the "in" and "out" vacuum states do not coincide and, therefore, the number of particles measured in the i^{th} mode by an out observer in the in vacuum state, is given by $\langle N_i\rangle = \sum_k|\beta_{ik}|^2$. For a Schwarzschild black hole, one obtains [1] for the average number of particles observed at late times in the state in which no particles are present at early times (we omit angular quantum numbers)

$$\langle N_w\rangle = \int_0^{+\infty} dw'|\beta_{ww'}|^2, \tag{1}$$

where the beta coefficients, up to a transmission amplitude factor and a trivial phase are given by

$$\beta_{w,w'} = \frac{1}{2\pi\kappa}\sqrt{\frac{w'}{w}}\frac{\Gamma(1+\kappa^{-1}wi)}{(-\kappa^{-1}w')^{1+\kappa^{-1}wi}}, \tag{2}$$

where κ is the surface gravity. Since these coefficients behave like $1/\sqrt{w'}$ for large w', the integral (1) diverges. This is naturally interpreted as the fact that the total number of created quanta is infinite, as corresponds to a finite steady rate of emission. The steady rate can be easily obtained from (1) and turns out to be thermal

$$\langle \dot{N}_w\rangle = \frac{1}{2\pi}\frac{1}{e^{2\pi\kappa^{-1}w} - 1}. \tag{3}$$

However, there is a disturbing point in this derivation. One needs to perform an unbounded integration in the frequencies w' to obtain the steady thermal rate of radiation [5–8]. Any outgoing Hawking quanta at infinity will have an exponentially increasing frequency as they are propagated backwards in time to reach the near-horizon region.

A cutoff in the frequencies w' of order of the Planck length (we take units with $c = 1$) would require that we consider only early-time frequencies satisfying

$$w' < \ell_P^{-1}, \tag{4}$$

where ℓ_P is the Planck length. This will change completely the Hawking effect. It will introduce a damping time-dependent factor in formula (3). The Hawking radiation is then converted into a transient phenomena (see, for instance, [9] and also [10]).

However, as first shown in [11], it is possible to rederive the Hawking radiation from a different perspective. In this derivation it is just the universal Hadamard short-distance behavior of the two-point function for all physically allowed states near horizon, namely

*ivan.agullo@uv.es
†jnavarro@ific.uv.es
‡olmo@iem.cfmac.csic.es
§leonard@uwm.edu

$$G(x_1, x_2) \approx \frac{\hbar}{4\pi^2\sigma}, \tag{5}$$

where σ is the squared geodesic distance between x_1 and x_2, that is responsible for the steady thermal emission. A somewhat related approach was developed in [9,12]. The mean number operator at late times can be expressed, in general, as [9,12]

$$\langle N_i\rangle = \hbar^{-1}\int_\Sigma d\Sigma_1^\mu d\Sigma_2^\nu [u_i^{\rm out}(x_1)\overleftrightarrow{\partial}_\mu] \times [u_i^{\rm out *}(x_2)\overleftrightarrow{\partial}_\nu] G(x_1, x_2). \tag{6}$$

After some algebra, one arrives at the expression

$$\langle N_w\rangle = -\frac{1}{4\pi^2 w}\int_{-\infty}^0 dU_1 dU_2 \frac{e^{-iw(u(U_1)-u(U_2))}}{(U_1 - U_2 - i\epsilon)^2}, \tag{7}$$

where U is the null Kruskal coordinate $U = -\kappa^{-1}e^{-\kappa u}$ and $u = t - r^*$ is the corresponding retarded time of a Schwarzschild black hole. The double integral above is divergent, but this divergence is expected due to the infinite number of quanta emitted in the infinite amount of time involved in the formula. Restricting the computation to the mean particle number per unit time one gets the finite thermal result

$$\langle \dot N_w\rangle = -\frac{1}{4\pi^2 w}\frac{d}{du}\int_{-\infty}^0 dU_1 dU_2 \frac{e^{-iw(u(U_1)-u(U_2))}}{(U_1 - U_2 - i\epsilon)^2} = \frac{1}{2\pi}\frac{1}{e^{2\pi\kappa^{-1}w} - 1}. \tag{8}$$

Again, the disturbing point in the above derivation is that a cutoff in distances requiring that

$$(U_1 - U_2)^2 > \ell_P^2, \tag{9}$$

turns the otherwise steady Hawking radiation into a transient phenomenon. One notices immediately that the common point in the cutoff (9) and that of (4) is that both are not Lorentz-invariant. Since we have put an upper limit, $w' \sim 1/\ell_P$, on the early-time frequencies, the in modes remaining after this amputation are not sufficient to generate the radiated out modes at late times. This produces the described decay of Hawking radiation with time as a consequence of breaking the principle of relativity by means of a noninvariant cutoff.

It is possible, however, to introduce a cutoff in an invariant way. On dimensional grounds, one can demand that the two-point function $G(x_1, x_2)$ that appears in our integrals does not exceed the inverse of Newton's constant

$$|G(x_1, x_2)| < \hbar\ell_P^{-2} \equiv G_N^{-1}. \tag{10}$$

It is not difficult to show, as we will see, that this condition translates into a restriction in the integration range of the U_1, U_2 coordinates in (8) given by

$$(U_1 - U_2)^2 > \ell_P^2\kappa^2 U_1 U_2/4\pi^2. \tag{11}$$

The factor $\kappa^2 U_1 U_2$ on the right-hand side of (11) is absent in Eq. (9). This factor is required to have an invariant cutoff for all locally inertial observers and immediately ensures the robustness of Hawking radiation.

An understanding of how (11) follows from (10) can be obtained in a simple way by considering the Unruh effect [13]. A detector held at constant r just outside the horizon behaves like a uniformly accelerated detector in Minkowski space (equivalence principle). The thermal radiation detected by the accelerated observer can be related to the Hawking emission. The detector will have some internal energy states $|E\rangle$ and it can interact with the field by absorbing or emitting quanta. The interaction can be modeled in the standard way by coupling the field $\phi(x)$ along the detector trajectory $x = x(\tau)$ (τ is the detector proper time) to some operator $m(\tau)$ acting on the internal detector eigenstates

$$g\int d\tau m(\tau)\Phi(x(\tau)), \tag{12}$$

where g is the strength of the coupling. The probability for the detector to make the transition from $|E_i\rangle$ to $|E_f\rangle$ is given by the expression $P(E_i \to E_f) = g^2|\langle E_f|m(0)|E_i\rangle|^2 F(\Delta E)$, where $F(\Delta E)$ is the so-called response function

$$F(\Delta E) = \int_{-\infty}^{+\infty} d\tau_1 d\tau_2 e^{-i\Delta E\Delta\tau/\hbar}\langle 0_M|\Phi(x(\tau_1)) \times \Phi(x(\tau_2))|0_M\rangle, \tag{13}$$

where $\Delta\tau = \tau_1 - \tau_2$. For a massless field the Wightman two-point function in (13), where $|0_M\rangle$ is the Minkowski vacuum, is given by

$$\langle 0_M|\Phi(x_1)\Phi(x_2)|0_M\rangle = -\frac{\hbar}{4\pi^2[(\Delta t - i\epsilon)^2 - (\Delta\vec{x})^2]}. \tag{14}$$

For trajectories having a proper-time translational symmetry under $\tau \to \tau + \tau_0$, it is natural to consider the constant transition probability per unit proper time and the corresponding response rate per unit proper time

$$\dot F(\Delta E) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-i\Delta E\Delta\tau/\hbar}\langle 0_M|\Phi(x(\tau_1))\Phi(x(\tau_2))|0_M\rangle. \tag{15}$$

Both the inertial detector and the uniformly accelerated detector possess proper-time translational symmetry. For an inertial detector trajectory, the response rate is given by

$$\dot F(\Delta E) = -\int_{-\infty}^{+\infty} d\Delta\tau e^{-i\Delta E\Delta\tau/\hbar}\left[\frac{\hbar}{4\pi^2(\Delta\tau - i\epsilon)^2}\right] = -\frac{\Delta E}{2\pi}\theta(-\Delta E), \tag{16}$$

in agreement with the principle of relativity. If the detector's initial state is the ground state $E_i = E_0$, then $\Delta E > 0$

and the probability for an inertial detector to be excited is exactly zero, irrespective of the velocity of the detector. (When $\Delta E < 0$ the result is nonvanishing and this leads to the expected nonzero probability for the spontaneous decay $E_i \rightarrow E_f < E_i$.)

For a uniformly accelerated trajectory in Minkowski spacetime

$$t = \frac{1}{a}\sinh a\tau, \qquad x = \frac{1}{a}\cosh a\tau, \tag{17}$$

where a is the acceleration, the response function is then

$$F(\Delta E) = \int_{-\infty}^{+\infty} d\tau_1 d\tau_2 e^{-i(\Delta E\Delta\tau/\hbar)} \frac{-\hbar(a/2)^2}{4\pi^2 \sinh^2[\frac{a}{2}(\Delta\tau - i\epsilon)]}. \tag{18}$$

The corresponding response rate function turns out to be $\dot{F}(\Delta E) = (\Delta E/2\pi)(e^{2\pi\Delta E/\hbar a} - 1)^{-1}$, which implies, via the detailed balance relation, $\dot{P}(\Delta E) = \dot{P}(-\Delta E)e^{-2\pi\Delta E/a\hbar}$, that a uniformly accelerated observer in Minkowski space feels himself immersed in a thermal bath at the temperature $k_B T = \frac{a\hbar}{2\pi}$.

Performing the change of variable

$$U \equiv t - x = -a^{-1}e^{-a\tau}, \tag{19}$$

one can rewrite the integral (18) in the form

$$F(\Delta E) = -\int_{-\infty}^{0} dU_1 dU_2 e^{-i\Delta E\Delta\tau/\hbar} \frac{\hbar}{4\pi^2(U_1 - U_2 - i\epsilon)^2}. \tag{20}$$

The time derivative of this expression is exactly the same (up to the factor $1/\hbar w$) as (8) obtained before in computing the expectation value of the number operator in the Hawking effect (identifying the acceleration a with the surface gravity κ and the coordinate U with the corresponding Kruskal coordinate). It is now easy to see that the invariant cutoff condition

$$\left| \frac{\hbar}{4\pi^2[(\Delta t)^2 - (\Delta\vec{x})^2]} \right| < G_N^{-1} \tag{21}$$

on the accelerated trajectory (17) becomes

$$\frac{\hbar(\frac{a}{2})^2}{4\pi^2 \sinh^2\frac{a}{2}\Delta\tau} < G_N^{-1}. \tag{22}$$

Expanding the denominator of (22) to lowest order in $\Delta\tau$ and using (19) to express $(\Delta\tau)^2$ in terms of $(\Delta U)^2 \equiv (U_1 - U_2)^2$, it is straightforward to show that this inequality is equivalent to (11). This confirms our statement that (10) implies (11).

The natural question now is to see if the invariant cutoff suffices to preserve the bulk of the Hawking effect. The answer is in the affirmative, but to see this requires an additional step [9,12]. Let us use again the Unruh effect to illustrate the argument. We want to take advantage of the fact that there is a state of the field, $|0_A\rangle$, for which the response function of the accelerated detector vanishes for $\Delta E > 0$

$$F_A(\Delta E > 0) = \int_{-\infty}^{+\infty} d\tau_1 d\tau_2 e^{-i\Delta E\Delta\tau} \times \langle 0_A|\Phi(x(\tau_1)) \times \Phi(x(\tau_2))|0_A\rangle = 0. \tag{23}$$

Taking this into account, it is possible to obtain an equivalent expression for the response function of the uniformly accelerating detector in the Minkowski vacuum, $|0_M\rangle$, by subtracting the previous quantity from the right-hand-side of Eq. (13)

$$F(\Delta E > 0) = \int_{-\infty}^{+\infty} d\tau_1 d\tau_2 e^{-i\Delta E\Delta\tau} \times [\langle 0_M|\Phi(x(\tau_1))\Phi(x(\tau_2))|0_M\rangle - \langle 0_A|\Phi(x(\tau_1))\Phi(x(\tau_2))|0_A\rangle]. \tag{24}$$

This expression presents several advantages over (13). It explicitly shows that the difference between two-point correlation functions of the field in the vacuum states $|0_M\rangle$ and $|0_A\rangle$ is at the root of a nonvanishing response function. (Notice that although the integral of $\langle 0_A|\Phi(x(\tau_1))\Phi(x(\tau_2))|0_A\rangle$ in the response function is zero, the correlation function itself is not zero.) Moreover, the integrand is now a smooth and symmetric function, thanks to the universal short-distance behavior of the two-point functions. Thus, the usual "$i\epsilon$-prescription" in the two-point functions is now redundant and can be omitted. Additionally, expression (24) shows a remarkable fact when an invariant cutoff is considered. It manifestly produces a vanishing result in the limit $a \rightarrow 0$, respecting in that way the principle of relativity that we want to preserve.

Now, one can consistently implement the invariant and universal cutoff condition

$$|\langle 0_M|\Phi(x(\tau_1))\Phi(x(\tau_2))|0_M\rangle| < G_N^{-1}, \tag{25}$$

and

$$|\langle 0_A|\Phi(x(\tau_1))\Phi(x(\tau_2))|0_A\rangle| < G_N^{-1} \tag{26}$$

in (24). The first inequality is equivalent to (22) and the second one to $\Delta\tau^2 > \ell_P^2/4\pi^2$. Moreover, both inequalities are essentially equivalent since all quantum states (in particular $|0_M\rangle$ and $|0_A\rangle$) have the same short distance behavior, as is seen explicitly from the short distance asymptotic form of (22).

In the black hole case, the same argument can be applied for the computation of the mean particle number [9,12], and $G(x_1, x_2)$ in Eq. (6) can be substituted by

$$G(x_1, x_2) - \langle \text{out}|\Phi(x_1)\Phi(x_2)|\text{out}\rangle, \tag{27}$$

where $|\text{out}\rangle$ is, as usual, the vacuum state defined by the modes $u_j^{\text{out}}(x)$. This leads to an expression for the mean particle number per unit time

$$\langle \dot{N}_w \rangle = -\frac{1}{4\pi^2 w}\frac{d}{du}\bigg[\int_{-\infty}^{0} dU_1 dU_2 \frac{e^{-iw(u(U_1)-u(U_2))}}{(U_1-U_2)^2} - \int_{-\infty}^{+\infty} du_1 du_2 \frac{e^{-iw(u_1-u_2)}}{(u_1-u_2)^2}\bigg], \tag{28}$$

where now we want to restrict the ranges of integration, so $(U_1 - U_2)^2 > \ell_P^2 \kappa^2 U_1 U_2 / 4\pi^2$ and $(u_1 - u_2)^2 > \ell_P^2/4\pi^2$. The explicit evaluation of these integrals, with the corresponding bounds for $(U_1 - U_2)^2$ and $(u_1 - u_2)^2$, leads to

$$\langle \dot{N}_w \rangle \approx \frac{1}{2\pi}\frac{1}{e^{2\pi\kappa^{-1}w} - 1} - \frac{\kappa\ell_P}{96\pi^4(w/\kappa)} + O(\kappa\ell_P)^3. \tag{29}$$

For black hole radii much bigger than the Planck length ($\kappa \ll \ell_P^{-1}$) and for reasonable values of the frequency, the correction terms are negligible, which shows the irrelevance of ultrahigh energy physics in the derivation of the Hawking effect.

In summary, we have shown that a universal invariant cutoff condition for two-point functions is able to preserve the bulk of the thermal Hawking radiation.

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