Estimating Potential Output, Core Inflation and the NAIRU as Latent Variables

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This paper proposes a new method to obtain estimates of the NAIRU, the core inflation rate and the trend investment rate for the United States using an unobserved components model which is compatible with the usual decomposition of real gross domestic product into trend and cycle. The model includes an Okun's law, a forward-looking Phillips curve and an accelerator-type investment equation, and accounts for some volatility breaks in two components. The unknown parameters in the model are estimated by maximum likelihood using a Kalman filter initialized with a partially diffuse prior, and the unobserved components are estimated using a smoothing algorithm. Our results show that the output gap is positively correlated with the deviations of the investment rate from its trend and the inflation rate from core inflation, and negatively correlated with the deviations of the unemployment rate from the NAIRU.

KEY WORDS: Output gap, forward-looking Phillips curve, Okun's law, investment, Kalman filter, volatility breaks.

1 Introduction

A useful decomposition of output into its trend and cyclical components should account for three central stylized facts in modern macroeconomics:

1. The negative correlation between the deviation of output from its trend and the deviation of the unemployment rate from the structural rate of unemployment, or NAIRU (non–accelerating inflation rate of unemployment) as it is sometimes called. This relationship between the cyclical components of output and unemployment is usually referred to as Okun's Law.

2. The trade-off in the short run between inflation and unemployment, which leads Mankiw (2001) to assert that "it is impossible to make sense of the business cycle ... unless we admit the existence of such a trade-off".

3. The comovement of output and investment. This is one of the most important regularities of business cycles, independently of the detrending method (Stadler, 1994, Canova, 1998, Burnside, 1998). Since investment is more volatile than the gross domestic product (GDP), the investment rate increases in expansions and falls in recessions.

Taking these facts together, it seems that the unemployment, inflation and investment rates contain very important information about the cyclical position of the economy and, therefore, of the output gap. In this paper, we take all this evidence into consideration and propose an unobserved components model for the United States which will allow us to obtain time-varying estimates of the NAIRU, core inflation and the structural investment rate which are compatible with the usual decomposition of the GDP into trend and cycle.

The different cyclical components in the model are specified in terms of the output gap. The model is estimated by maximum likelihood through the use of the Kalman filter initialized with a partially diffuse prior. A smoothing algorithm is used to obtain estimates of the unobserved components based on the whole sample together with their mean squared errors. We also account in our model for some volatility breaks in the output gap (and, therefore, in the cyclical components of the unemployment and investment rates) and core inflation components. Our results are in agreement with those reported by Stock and Watson (2002) and Sensier and Van Dijk (2003).

In contrast to our approach, previous research trying to obtain alternative estimates of the output gap for the US or the European countries has omitted at least one of the three facts mentioned above. For example, Kuttner (1994) uses only the information contained in inflation through a simple backward-looking version of the Phillips curve. Apel and Jansson (1999), Camba-Méndez and Palenzuela (2003) and Fabiani and Mestre (2004) do not consider the investment rate and their estimated Phillips curve does not include any time-varying component which proxies core or expected inflation. Alternatively, Gerlach and Smets (1999) consider only a backward-looking Phillips curve and an aggregate demand equation which relates the output gap to its own lags and the real interest rate. Laubach (2001) has proposed a model consisting only of a Phillips curve linking the first difference of inflation to cyclical unemployment and the equations necessary to model the two unobservable components (the NAIRU and the gap) of the unemployment rate. Their model is similar to the one proposed by Gordon (1997), but allowing the NAIRU to be a non-stationary process in some countries. Using a similar framework, Staigner, Stock and Watson (2001) take advantage of the information contained in the inflation rate and the growth of real wages to compute a time-varying estimate of the NAIRU. Roberts (2001) decomposes output into labor productivity and hours, obtaining the trend and cyclical components using the additional information of the inflation rate through the estimation of a backward-looking version of the Phillips curve. More recently, Rünstler (2002) estimates the real-time output gap in a supply curve, but he further investigates alternative extensions including the unemployment rate, capital stock, productivity and capacity utilization.

In short, to the best of our knowledge, previous research has made no use of the rich information about the business cycle simultaneously contained in the GDP and the unemployment, inflation and investment rates to obtain a better decomposition of these four variables into trend and cyclical movements. However, our results show that the output gap estimated as a latent variable is very significant in the three equations that we use to specify the relationships among the previous four variables, namely, the Okun's law, a forward-looking Phillips curve and an accelerator-type investment equation.

Besides the contribution in terms of the model specification to include additional information from relevant macroeconomic variables, we use a very flexible methodology that has a solid foundation and is specially designed for nonstationary state-space models, where the initial conditions for the Kalman filter are not well defined. Specifically, the initial state vector is modelled as partially diffuse and a "diffuse Kalman filter" (De Jong, 1991) is used for prediction and likelihood evaluation. At a later stage, we use a smoothing algorithm to obtain estimates of the unobserved components together with their confidence intervals. This contrasts with the situation usually found in the literature as regards the initialization problem in the Kalman filter in the presence of nonstationary series. More often than not the model assumptions are not elucidated, the initial state is not explicitly defined, and the initial conditions for the Kalman filter are obtained by using some approximation such as a backcasting device. This may cause problems in the optimization routine and many parameters may have to be fixed to some pre–specified, and somewhat arbitrary, values.

Other contributions of the article are: i) a thorough theoretical discussion of the new hybrid Phillips curve as defined by Galí and Gertler (1999, p. 203), and the result that it can be given a model–based interpretation according to which the model behind it is a simplified version of our proposed Phillips curve equation, ii) the possibility of incorporating volatility breaks in our model, and iii) a theoretical result showing that the recent filter proposed by Cogley (2002) to estimate the core inflation also admits a model–based interpretation.

The paper is structured as follows. In section two we present the unobserved components model used to decompose each variable into a trend and a cyclical component, and we discuss some estimation issues. The third section presents the results of the estimation of our model and some basic features of the estimated unobserved components. In order to evaluate the validity of our decomposition, the fourth section analyzes some properties of our estimates in terms of revisions and inflation forecasts, compared with alternative procedures. Finally, section four summarizes the conclusions.

2 The potential output model

2.1 Output decomposition

To model the log of real GDP, y_t, we start with Watson's (1986) decomposition

$$y_t \equiv \overline{y}_t + y_t^c, \tag{1}$$

where $\bar{y}_t$ is the trend and y_t^c is the cyclical output, that is also used by Kuttner (1994) and others. The cycle is assumed to follow a stationary AR(2) model with complex roots

$$y_t^c = 2\theta_1 \cos(\theta_2) y_{t-1}^c - \theta_1^2 y_{t-2}^c + \omega_{yt}, \tag{2}$$

where $\{\omega_{yt}\}$ is assumed to be an i.i.d. $N(0, \sigma_{\omega y}^2)$ sequence, $\theta_2 \in [\pi/20, \pi/3]$, and $0 < \theta_1 < 1$. To model the trend, we first perform some unit root tests to see whether y_t is $I(0)$ or $I(1)$, where $I(0)$ and $I(1)$ refer to integrated processes of order zero and one. In Table 1, we report the results obtained when we apply some augmented Dickey–Fuller (ADF) tests for nonstationarity. The tests clearly reject the hypothesis of a stationary process around a deterministic trend, in agreement with the results of Kuttner (1994). We can also see in Table 1 the first twelve autocorrelations of the first difference of GDP.

A sufficient condition for model (1) to be identified (Harvey, 1987, p. 206) is that the order of the moving average component of $\bar{y}_t$ is less than that of its autoregressive part, including the unit roots. The previous considerations then lead to the specification

$$\nabla \bar{y}_t = \bar{\gamma}_y + \omega_{\gamma t} \tag{3}$$

for the trend $\bar{y}_t$, where $\nabla = 1 - L$, L is the lag operator, $L\overline{y}_t = \overline{y}_{t-1}$, $\bar{\gamma}_y$ is a drift term, and $\{\omega_{\gamma t}\}$ is an i.i.d. $N(0, \sigma_{\omega\gamma}^2)$ sequence uncorrelated with $\{\omega_{yt}\}$.

Table 1. Unit-root Tests for Real GDP

	ADF t statistics		Autocorrelations				
Lags	Constant	Constant, trend	Lags	Correlation coefficient			
2	-1.25	-2.95	1–2	0.34	0.19		
4	-1.24	-2.32	3–4	0.00	-0.12		
8	-1.80	-2.54	5–8	-0.17	-0.10	-0.08	-0.05
12	-1.64	-2.20	9–12	0.05	0.07	0.03	-0.13

NOTE: Results are based on 225 quarterly observations from 1947:I through 2003:I

It remains to see whether we should include an additional stationary component y_t^s that follows an ARMA model so that $y_t = \bar{y}_t + y_t^c + y_t^s$ and the model is identified. To this end, we have first estimated a univariate model with the two first components and then we have examined the residuals. The results are reported in Table 2. Some Q statistics obtained from the residuals are $Q(17) = 16.58$ and $Q(20) = 18.46$. The residual sample autocorrelations show no significant structure and thus the extra component y_t^s is not needed.

Although the residuals seem to have no autocorrelation, they do show some het-

Table 2. Estimated Univariate Model for Real GDP

$\bar{\gamma}_y$	θ_1	θ_2	$\sigma_{\omega\gamma}$	$\sigma_{\omega y}$
0.0084	0.7981	0.2845	0.0067	0.0058
(18.33)	(15.11)	(2.73)		(3.46)

NOTE: Results are based on 225 quarterly observations from 1947:I through 2003:I. t–values are in parenthesis. The parameters $\sigma_{\omega\gamma}$ and $\bar{\gamma}_y$ are concentrated out of the likelihood.

eroscedasticity. This is in agreement with Stock and Watson (2002), who found no change in the autoregressive parameters of the output gap, but found a break in the output gap volatility in 1983:II. One nice feature of our four variables model is that it can incorporate these volatility breaks. We accomplish this by allowing the parameter $\sigma_{\omega y}$ to vary with time. That is, instead of $\sigma_{\omega y}$, we use $\sigma_{\omega yt} = \sigma_{\omega y1}$ if $t <$ 1983:II and $\sigma_{\omega yt} = \sigma_{\omega y2}$ if $t \geq$ 1983:II.

A useful insight into our proposal can be obtained by comparing our specification with that of the Hodrick and Prescott (1997) filter (henceforth HP filter). In the model-based interpretation of this filter (Gómez, 1999), output is also expressed as a trend–plus–cycle model (1), but with the trend following the model $\nabla^2 \overline{y}_{hp,t} = \omega_{\gamma t}$ instead of (3) and the cycle being white noise, $y_t^c = \{\omega_{yt}\}$, instead of following the model (2). In addition, the noise to signal ratio, $\sigma^2_{\omega y}/\sigma^2_{\omega\gamma}$, is assumed to be fixed and equal to 1600. However, it has been suggested that this value is too small and a larger value should be used. The larger the value, the smoother the trend.

Instead of using fixed filters, like the HP filter, to estimate the unobserved components, we propose to use a model–based approach. Some advantages of a model–based approach are that the filters implied by the model are consistent with each other and with the data. In addition, they automatically adapt to the ends of the sample and, if desired, root mean squared errors can be calculated.

2.2 The Phillips curve

Assuming proportionality between marginal cost and the output gap, a simple specification of the new Phillips curve is (Galí and Gertler, 1999, p. 201)

$$\pi_t = \alpha y_t^c + \beta E_t(\pi_{t+1}),$$

where π_t is the inflation rate, y_t^c is the output gap, $E_t(.)$ is the expectation operator based on information up to and including time t, and α and β are constants. Thus, the $E_t(\pi_{t+1})$ term summarizes the rational expectations of future inflation at time t. A key difference with the old Phillips curve is that it is $E_t(\pi_{t+1})$ as opposed to $E_{t-1}(\pi_t)$ (generally assumed to equal π_{t-1}) that matters.

As stated by Galí and Gertler, (1999), p. 203, some empirical limitations of the new Phillips curve have led a number of researchers to consider a hybrid version of the new and old

$$\pi_t = \mu E_t(\pi_{t+1}) + (1-\mu)\pi_{t-1} + \delta y_t^c,$$

where $0 \leq \mu \leq 1$. The idea is to let inflation depend on a convex combination of expected future inflation and lagged inflation.

The previous equation can be generalized by including a white noise term v_t uncorrelated with y_t^c to get

$$\pi_t = \mu E_t(\pi_{t+1}) + (1-\mu)\pi_{t-1} + \delta y_t^c + v_t. \tag{4}$$

This generalization does no harm and, in any case, one can always assume in the discussion that follows that v_t is zero without the main results being affected. It is shown in Appendix C that if we make some reasonable assumptions, a model–based interpretation of (4) is obtained. More specifically, the following theorem holds.

Theorem 1. *Under the assumptions 1)–5) in Appendix C, the solution for inflation* π_t *in equation (4) coincides with the solution given by the model*

$$\pi_t = \tilde{\mu}\overline{\pi}_t + (1-\tilde{\mu})\pi_{t-1} + \eta_y y_t^c + v_{\pi t} \tag{5}$$

$$\nabla\overline{\pi}_t = \omega_{\pi t} \tag{6}$$

$$\phi(L)y_t^c = \omega_{yt}, \tag{7}$$

where model (7) coincides with (2), $\tilde{\mu} = 1-(1-\mu)/\mu$, $\eta_y = \delta\sqrt{1+\theta_1^2}/(\mu\sqrt{\alpha})$, $Var(\omega_{\pi t}) = [(\delta\sigma_{y\omega})^2 + (\alpha-1)^2(\sigma_v)^2]/(\tilde{\mu}\varphi_0)^2$, $Var(v_{\pi t}) = \alpha\sigma_v^2/\varphi_0^2$, $\varphi_0 = 1-\mu\beta_0$, *and* $\{v_{\pi t}\}$ *and* $\{\omega_{\pi t}\}$ *are uncorrelated white noise sequences with zero mean that are both uncorrelated with* $\{y_t^c\}$*. Also,* $0 \leq \tilde{\mu} \leq 1$ *in (5).*

The previous considerations lead us to propose the following model for inflation

$$\pi_t = (1-\sum_{i\geq 1}\mu_{\pi i})\overline{\pi}_t + \mu_\pi(L)\pi_{t-1} + \eta_y y_t^c + v_{\pi t}, \tag{8}$$

where y_t^c is the output gap, that follows model (2), η_y is a constant, $\{v_{\pi t}\}$ is an i.i.d. $N(0,\sigma_{v\pi}^2)$ sequence, $\mu_\pi(L) = \sum_{i\geq 1}\mu_{\pi i}L^i$ is a polynomial in the lag operator, and $\overline{\pi}_t$ is the long-run inflation rate, that follows model (6) with $\{\omega_{\pi t}\}$ assumed to be an i.i.d. $N(0,\sigma_{\omega\pi}^2)$ sequence. In addition, $\{v_{\pi t}\}$, $\{\omega_{\pi t}\}$ and $\{y_t^c\}$ are assumed to be mutually uncorrelated.

In equation (8), we use several lags for inflation to account for backward looking behavior. In addition, the importance of forward looking behavior is measured by the coefficient $1-\sum_{i\geq 1}\mu_{\pi i}$ of core inflation $\overline{\pi}_t$. To see this, consider first that $\overline{\pi}_t$ is unobservable and is thus different from $E_t(\pi_{t+1})$ in (4), and also different from its concurrent estimator, $E_t(\overline{\pi}_t)$. In fact, the precise relationship between the estimator $E_t(\overline{\pi}_t)$ and expected future inflation is given by the following theorem. The proof is in Appendix C.

Theorem 2. *In model (8), the following relation holds*

$$E_t(\pi_{t+k}) = E_t(\overline{\pi}_{t+k}) = E_t(\overline{\pi}_t),$$

where $E_t(\pi)$ denotes conditional expectation of the random variable π with respect to the information set $I_t = \{z_s : s \leq t\}$, z_s are the four dimensional observations of our proposed model, and k is a sufficiently large integer.

Then, by theorem 2, if we take conditional expectations in (8), we get

$$\pi_t \quad = \quad (1 - \sum_i \mu_{\pi i}) E_t(\pi_{t+k}) + \mu_\pi(L)\pi_{t-1} + \eta_y E_t(y_t^c) + E_t(v_{\pi t}). \tag{9}$$

It is clear by inspection of equations (9) and (4) that our specification also includes rational expectations of inflation, but with a suitable forecast horizon k instead of just one period ahead. Another consequence of (9) is that it proves our earlier claim about the importance of forward looking behavior being measured by the coefficient $1 - \sum_{i \geq 1} \mu_{\pi i}$ of $\overline{\pi}_t$ in (8). We note that it is equation (9) and not equation (8) that should be compared to (4).

We note that, by theorem 2, our concurrent estimator $E_t(\overline{\pi}_t)$ satisfies the usual definition of core inflation $\bar{\pi}_{t|t}$ of Bryan and Cecchetti (1994), $\bar{\pi}_{t|t} = E_t(\pi_{t+k})$, where π_t is actual inflation and k is a suitably long forecast horizon.

We will include later in the simulation experiment the filter recently proposed by Cogley (2002), that is a very simple one–sided filter depending on one parameter. It is shown in Cogley (2002) that this filter performs better than the best known and most widely used measures of core inflation, namely the "ex food and energy" series produced by the U.S. Bureau of Labor Statistics and Bryan and Cecchetti's median core and trimmed mean series. It is interesting to note that the Cogley filter can be given a model–based interpretation as the following theorem shows. The proof is in Appendix C.

Table 3. Unit-root Tests for Inflation

	ADF t statistics		Autocorrelations				
Lags	Constant	Constant, trend	Lags	Correlation coefficient			
2	-2.71	-2.75	1–2	-0.27	-0.31		
4	-2.92	-2.98	3–4	0.36	-0.09		
8	-2.60	-2.68	5–8	-0.14	0.17	0.05	-0.26
12	-2.04	-2.10	9–12	0.03	0.10	-0.10	-0.06

NOTE: Results are based on 209 quarterly observations from 1951:I through 2003:I

Theorem 3. *Given the semi–infinite sample* $\{\pi_s : s \leq t\}$*, the series filtered with the Cogley filter,*

$$\hat{\pi}_t = \frac{1+\theta}{1+\theta L}\pi_t,$$

coincides with the concurrent estimator of s_t *in the signal plus noise model* $\pi_t = s_t + n_t$*, where* s_t *follows the model* $\nabla s_t = b_t$ *and* $\{b_t\}$ *and* $\{n_t\}$ *are two white noise series mutually uncorrelated.*

To test our specification for inflation π_t for consistency with the data, we first perform some unit root tests. By visual inspection, we detect some outliers in the first part of the sample. For this reason, we drop some observations at the beginning of the sample before performing some ADF tests for nonstationarity. The results are reported in Table 3. The tests clearly reject the hypothesis of a stationary process around a deterministic trend, also in agreement with the results of Kuttner (1994).

The ADF tests suggest that we specify for the trend $\bar{\pi}_t$ the model $\nabla\bar{\pi}_t = \omega_{\pi t}$. Table 3 displays the first twelve autocorrelations of the first difference of inflation. To identify both the outliers at the beginning of the sample and an ARIMA model for inflation, we use the automatic identification facility of the TRAMO program of Gómez and Maravall (1997) (downloadable at **http://www.bde.es**).

Including as a regressor the filtered estimate of the output gap, $E_t(y_t^c)$, obtained with the previous univariate model for GDP, the program identifies an ARIMA$(3,1,0)(0,0,1)_4$ model for inflation with some outliers. However, the moving average parameter is not significant and so we accept the ARIMA$(3,1,0)(0,0,0)_4$ model. With this last model, the following outliers were identified: i) AO, 1951:I, ii) TC, 1948:IV, and iii) AO, 1951:III, where AO and TC stand for additive outlier and temporary change. An additive outlier is modelled by a dummy variable taking the value one at the appropriate date and zero otherwise. A temporary change is modelled by a dummy variable taking the value one at the start T of the effect and $.7^j$ at later times $T+j$, $j=1,2,\ldots$. Some Q statistics obtained from the residuals are $Q(17)=16.43$ and $Q(20)=18.27$.

We consider the previous ARIMA model as a tentative reduced form specification. Based on it, we use the specification (8) for inflation π_t, but we include the three outliers detected with the TRAMO program and we replace the output gap with the filtered estimate, $E_t(y_t^c)$, obtained with the univariate model for GDP. Given the previously mentioned identification issues, it remains to see whether $\{v_{\pi t}\}$ should be specified as an i.i.d. sequence or as a stationary component following an ARMA model. As in the GDP case, we have first estimated a univariate model and then we have examined the residuals. The results are reported in Table 4.

Some Q statistics obtained from the residuals are $Q(17)=19.09$ and $Q(20)=22.98$. The residuals show no autocorrelation and thus we specify $\{v_{\pi t}\}$ as an i.i.d. sequence.

As with GDP, although the residuals seem to have no autocorrelation, they do show

Table 4. Estimated Univariate Model for Inflation

o_1	o_2	o_3	$\sigma_{\omega\pi}$	μ_1	μ_2	μ_3	μ_4	η_y	$\sigma_{v\pi}$
0.1041	-0.988	-0.0318	0.0123	0.0000	-0.1476	0.2544	0.0000	0.6714	0.0080
(8.54)	(-6.95)	(-2.61)	(4.16)		(-2.59)	(5.21)		(3.85)	

NOTE: Results are based on 224 quarterly observations from 1947:II through 2003:I. t–values are in parenthesis. The parameters $\sigma_{v\pi}$ and o_i, $i = 1, 2, 3$, are concentrated out of the likelihood.

some heteroscedasticity. This is in agreement with Sensier and Van Dijk (2003), who report to have found several volatility breaks for inflation. We have in fact found two breaks in inflation volatility, in 1972:I and 1983:II, that we have incorporated into our model. We have accomplished this by allowing the parameter $\sigma_{\omega\pi}$ to vary with time. That is, instead of $\sigma_{\omega\pi}$, we use $\sigma_{\omega\pi t} = \sigma_{\omega y1}$ if $t <$ 1972:I, $\sigma_{\omega\pi t} = \sigma_{\omega\pi 2}$ if 1972:I $\leq t <$ 1983:II, and $\sigma_{\omega\pi t} = \sigma_{\omega\pi 3}$ if $t \geq$ 1983:II.

2.3 Okun's Law

Some empirical evidence suggests that there is a relationship between movements in output and unemployment. This relationship, known as Okun's law, has been used by several authors to asses the cyclical position of the economy (Clark, 1989, Blanchard and Quah, 1989), and it is also used by the Congressional Budget Office to estimate the output gap.

The previous empirical evidence can be explained by some models, like the one proposed by Blanchard and Quah (1989). According to a more general version of this last model, in the steady-state balanced growth path, output is on its trend, the unemployment rate is equal to its structural level, inflation is steady and the investment rate is on its long-run level. In this model, Okun's law is simply the relationship between output relative to trend and the deviations of the unemployment rate from its trend component.

Although we do not propose a fully fleshed out macro model in this article, we account for the negative correlation between the output gap and cyclical unemployment by means of the following equation

$$U_t = \phi_u U_{t-1} + (1 - \phi_u)\overline{U}_t + \phi_y(L)y_t^c + v_{ut},$$

where $\overline{U}_t$ is the trend component, $\{v_{ut}\}$ is an i.i.d. $N(0, \sigma^2_{uv})$ sequence and $\phi_y(L)$ is a polynomial in the lag operator such that $\phi_y(1) < 0$. Since the output gap follows an $AR(2)$ process, our cyclical unemployment specification is rather flexible. In contrast to the assumptions of Apel and Jansson (1999) and Camba-Méndez and Palenzuela (2003), we allow in principle the output gap to affect the unemployment rate with some lags, as suggested by some empirical evidence showing that firms usually adjust employment slowly.

The non–accelerating inflation rate of unemployment or NAIRU, $\overline{U}_t$, is allowed to follow either an $I(2)$ or an $I(1)$ process, where $I(2)$ means that a process is integrated of order two. That is,

$$\overline{U}_t = \gamma_{ut} + \overline{U}_{t-1},$$

where

$$\gamma_{ut} = \rho_u \gamma_{ut-1} + \omega_{ut},$$

$0 \leq \rho_u \leq 1$, and $\{\omega_{ut}\}$ is an i.i.d. $N(0, \sigma^2_{\omega u})$ sequence. Thus, if $\rho_u = 1$, then $\nabla\overline{U}_t$ is $I(1)$. But if $\rho_u = 0$, then $\overline{U}_t$ is just a random walk. As pointed out by Laubach (2001), the assumption that the NAIRU follows a random walk could be convenient for the US but not for other countries such as, for example, the European ones that are believed to have

NAIRUs following $I(2)$ processes.

Proceeding as in the cases of the GDP and the inflation rate, we have tested whether the data support this specification. The results (not shown) indicate that $\overline{U}_t$ follows a random walk without drift and that there are no lags in the polynomial $\phi_y(L)$.

2.4 Investment

One of the most important regularities that the empirical research on business cycles has found is that investment strongly comoves with output but with more volatility (Canova, 1998, Burnside, 1998, Harvey and Trimbur, 2003). This stylized fact implies that the deviation of the investment rate, $x_t \equiv \text{investment}_t/\text{output}_t$ (see the data definitions in Appendix A), from its long-run trend, $\overline{x}_t$, is markedly procyclical. For this reason, we model the comovement of the investment rate with the output gap by means of the following equation

$$x_t = \beta_x x_{t-1} + (1 - \beta_x)\overline{x}_t + \beta_y(L)y_t^c + v_{xt}, \tag{10}$$

where $\{v_{xt}\}$ is an i.i.d. $N(0, \sigma_{xv}^2)$ sequence and, given that the investment rate is procyclical, $\beta_y(L)$ is a polynomial in the lag operator such that $\beta_y(1) > 0$.

Note that, although (10) resembles an accelerator investment model, it is not a proper investment equation, but a reduced form that may help to estimate more accurately the output gap because it accounts for the correlation between the cyclical components of the investment rate and output.

We have considered the possibility of including the real interest rate as a regressor in (10), as suggested by a referee. However, after estimating our full model again with the additional regressor as described in the next section we found that it was not significant.

Therefore, we dropped it from the investment rate equation.

As in the case of the NAIRU, the trend component of the investment rate is allowed to follow either an $I(1)$ or an $I(2)$ process. That is,

$$\overline{x}_t = \gamma_{xt} + \overline{x}_{t-1},$$

where

$$\gamma_{xt} = \rho_x \gamma_{x,t-1} + \omega_{xt},$$

$0 \leq \rho_x \leq 1$, and $\{\omega_{xt}\}$ is an i.i.d. $N(0, \sigma^2_{\omega x})$ sequence. Note that if $\rho_x = \sigma^2_{\omega x} = 0$, then $\overline{x}_t$ is equal to a constant. Thus, we let the data speak to see how flexible the investment trend is.

Proceeding as with the previous variables, we have tested whether the data support this specification. The results (not shown) indicate that $\overline{x}_t$ follows a random walk without drift and that there is one lag in the polynomial $\beta_y(L)$.

2.5 Model Estimation

Our approach to estimate the unknown parameters in the model is to cast it into state-space form and use the Kalman filter for likelihood evaluation. Then, at a later stage, we use a smoothing algorithm to obtain estimates of the unobserved components together with their mean squared errors. We use quarterly data for the US economy from 1946:I to 2003:I. The data are described in Appendix A. There are several missing values, but that poses no problem for the Kalman filter.

Based on the analysis of the previous sections, we specify all four variables as $I(1)$, with only the output trend having a drift term, and we let the parameters $\sigma_{\omega y}$ and $\sigma_{\omega\pi}$ vary with time in the way described in sections 2.1 and 2.2. In addition, we specify degree zero for the polynomial $\phi_y(L)$, degree one for the polynomial $\beta_y(L)$, degree four for the polynomial $\mu_\pi(L)$, and we include in the model the three outliers identified for inflation.

Our model, with the previous specification, can be put into state-space form as follows. Define the following matrices

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -\theta_1^2 & 2\theta_1 \cos\theta_2 \end{bmatrix},$$

$$\alpha_t = \begin{bmatrix} \overline{y}_t \\ \overline{U}_t \\ \overline{x}_t \\ \overline{\pi}_t \\ y^c_{t-2} \\ y^c_{t-1} \\ y^c_t \end{bmatrix}, \quad H_t = \begin{bmatrix} \sigma^*_{\omega\gamma} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^*_{\omega u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^*_{\omega x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^*_{\omega\pi t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^*_{\omega yt} \end{bmatrix},$$

$$Z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1-\phi_u & 0 & 0 & 0 & 0 & \phi_0 \\ 0 & 0 & 1-\beta_x & 0 & 0 & \beta_{y_1} & \beta_{y_0} \\ 0 & 0 & 0 & 1-\sum_{i=1}^4 \mu_{\pi_i} & 0 & 0 & \eta_y \end{bmatrix},$$

$$X_t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & o_{1t} & o_{2t} & o_{3t} \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^*_{uv} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^*_{xv} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

where $\sigma^*_{\omega\gamma} = \sigma_{\omega\gamma}/\sigma_{\pi v}$, $\sigma^*_{\omega u} = \sigma_{\omega u}/\sigma_{\pi v}$, $\sigma^*_{\omega x} = \sigma_{\omega x}/\sigma_{\pi v}$, $\sigma^*_{\omega\pi t} = \sigma_{\omega\pi t}/\sigma_{\pi v}$, $\sigma^*_{\omega yt} = \sigma_{\omega yt}/\sigma_{\pi v}$, $\sigma^*_{uv} = \sigma_{uv}/\sigma_{\pi v}$, $\sigma^*_{xv} = \sigma_{xv}/\sigma_{\pi v}$, and the o_{it} variables, $i = 1, 2, 3$, model the three outliers that affect the inflation rate. Then, α_t is the state vector, the parameter $\sigma^2_{\pi v}$ is concentrated out of the likelihood, and the state-space equations are

$$\begin{aligned} \alpha_{t+1} &= W\gamma + T\alpha_t + H_t\epsilon_t \\ z_t &= X_t\gamma + Z\alpha_t + G\epsilon_t, \end{aligned}$$

where $z_t = [y_t, U_t - \phi_u U_{t-1}, x_t - \beta_x x_{t-1}, \pi_t - \sum_{i=1}^4 \mu_{\pi_i}\pi_{t-i}]'$, $\gamma = (\overline{\gamma}_y, o_1, o_2, o_3)'$ is the vector of regression coefficients and $\text{Var}(\epsilon_t) = \sigma^2_{\pi v} I$. The parameters in γ are also concentrated out of the likelihood. The filter starts filtering at $t = 5$, so that we condition on the first four non missing observations of each series.

The previous state-space model is non-stationary and the initial conditions for the Kalman filter are not well defined. To overcome this difficulty, we use the approach of De Jong (1991). According to this approach, the initial state vector α_1 is modelled as partially diffuse and an augmented Kalman filter algorithm called the "diffuse Kalman filter" (DKF) is used to handle the diffuse part. As shown by De Jong and Chu-Chun-Lin (1994), the DKF can be collapsed to the ordinary Kalman filter after a few iterations. The DKF can be used to evaluate the likelihood and thus the model parameters can be estimated by maximum likelihood.

After having estimated the model parameters, we can use a smoothing algorithm to obtain two-sided estimates of the unobserved components and their mean squared errors. We use the algorithm proposed by De Jong and Chu-Chun-Lin (2003). The diffuse part is $\delta = [\overline{y}_0, \overline{U}_0, \overline{x}_0, \overline{\pi}_0]'$, so that the initial state is $\alpha_1 = A\delta + W\gamma + [0, x_1']'$, where $A = [I, 0]'$ and $x_1 = [y_{-1}^c, y_0^c, y_1^c]'$ has a known (stationary) distribution. Some technical details on the methodology and the optimization method are given in Appendix B.

3 Estimation Results

In Table 5 we present the estimates of the different model parameters, together with their $t-$statistics in parenthesis. It is seen that our estimation of the output gap is very significant in the Okun's law (ϕ_{y0}), the Phillips curve (η_y) and the investment equation ($\beta_{yi},\ i = 0, 1$). This suggests that the unemployment, inflation and investment rates contain very useful information about the cyclical position of the economy. The Q statistics obtained from the residuals (not shown) indicate no evidence of residual autocorrelation.

The results in the first two columns of Table 5 show that there is indeed a break at 1983:II in the output gap volatility, measured by the standard deviation $\sigma_{\omega yt}$, in agreement with the results of Stock and Watson (2002). This standard deviation has sharply declined from 0.0092 before 1983:II to 0.0029 afterwards.

The results for the Okun's law indicate that there is a significant and sizable direct contemporaneous effect of business cycles on the unemployment rate. Another noteworthy result is that the magnitude of σ_{vu} is so small that Okun's law almost fits completely the unemployment rate. In the case of the investment rate, we obtain a similar picture. The contemporaneous correlation with the output gap is very significant, and there is also a substantial inertia in the investment rate since β_x is relatively high. Because the standard

Table 5. Maximum Likelihood Parameter Estimates

Equation											
Output		Okun's Law		Investment		Phillips Curve					
θ_1	0.7673	ϕ_{y0}	-0.2981	β_{y0}	0.6403	η_y	0.2710	o_1	0.1073
	(20.60)		(-14.48)		(12.75)		(3.05)		(8.59)
θ_2	0.2412	ϕ_u	0.4380	β_{y1}	-.6086	μ_1	0.1633	o_2	-0.0907
	(3.72)		(10.42)		(-11.97)		(2.18)		(-7.01)
$\overline{\gamma}_y$	0.0083	$\sigma_{\upsilon u}$	0.0012	β_x	0.8289	μ_2	-0.1435	o_3	-0.0266
	(26.03)		(6.09)		(13.67)		(-2.35)		(-2.13)
$\sigma_{\omega y1}$	0.0092	$\sigma_{\omega u}$	0.0046	$\sigma_{\upsilon x}$	0.0048	μ_3	0.2574	$\sigma_{\omega\pi 1}$	0.0068
	(7.85)		(3.80)		(8.47)		(4.85)		(3.28)
$\sigma_{\omega y2}$	0.0029			$\sigma_{\omega x}$	0.0027	μ_4	-0.1440	$\sigma_{\omega\pi 2}$	0.0161
	(5.25)				(2.39)		(-2.83)		(4.38)
$\sigma_{\omega\gamma}$	0.0046					$\sigma_{\upsilon\pi}$	0.0106	$\sigma_{\omega\pi 3}$	0.0032
	(7.34)								(1.96)

NOTE: Results are based on 229 quarterly observations from 1946:I through 2003:I. t–values are in parenthesis. The parameters $\sigma_{\upsilon\pi}$, $\overline{\gamma}_y$ and o_i, $i = 1, 2, 3$, are concentrated out of the likelihood.

deviation of υ_x is very small (close to 0.5 per cent), the decomposition between trend and cycle accounts almost entirely for the variation of the investment rate.

The last four columns of Table 5 present the estimation results for the Phillips curve. Again, the model performs extremely well in explaining the dynamics of inflation in the United States. The output gap is statistically significant suggesting that most of the business cycles fluctuations have been associated with a procyclical behaviour of inflation. Although the models are not directly comparable, the estimated value of η_y is higher than that estimated by Rudebusch and Svensson (1999).

As mentioned in Section 2.2, in the Phillips curve we use several lags for inflation to account for backward looking behavior whereas forward looking behavior is measured by the coefficient $1 - \sum_{i=1}^{4} \mu_i$ of $\overline{\pi}_t$ in (8) or, equivalently, the coefficient of $E_t(\pi_{t+k})$ in (9). From the results in Table 5 we see that forward looking behavior is more important than backward looking behavior, since the estimated forward looking coefficient $1 - \sum_{i=1}^{4} \mu_i$ $= .8668$ is well above the estimated backward looking coefficients μ_i. Note that some of the coefficients of lagged inflation are negative. This should not be worrisome however for two reasons. First, economic theory does not predict that the coefficients should all be positive and second, the negative sign in some lagged values is probably the result of accounting for some autocorrelation in the error term.

As with GDP, and in agreement with the results of Sensier and Van Dijk (2003), we have found two breaks in inflation volatility, measured by the standard deviation $\sigma_{\omega\pi}$. The breaks occur in 1972:I and 1983:II. From the results in Table 5, a hump–shaped pattern is observed, with an increase in volatility from 0.0068 to 0.0161 at 1972:I followed by a sharp decrease to 0.0032 at the second break in 1983:II.

Figure 1 displays the output gap, the NAIRU, the core inflation rate and the trend investment rate for the US economy, as well as estimated ninety per cent confidence intervals. In Figure 2, our estimation of the output gap is also compared with the cyclical component estimated with the HP filter. The correlation between both estimates of the output gap is relatively high (0.835), but we observe some important discrepancies. Thus, when we use the HP filter, the 1990–91 recession appears very mild compared to other post World War II episodes, whereas the growth in GDP during the second part of the

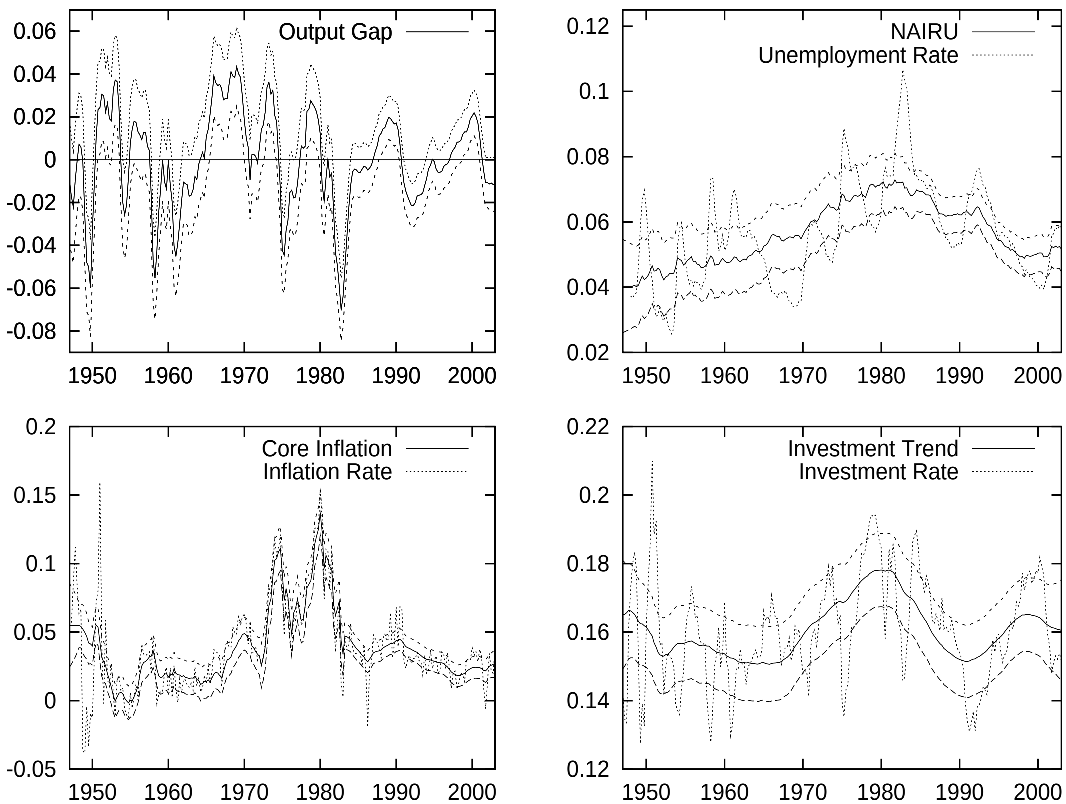

Figure 1: Output gap, NAIRU, Core inflation rate, and Investment rate trend. United States, 1947:I–2003:I.

nineties is compatible with a small output gap. However, our estimation of the output gap shows a more severe recession in 1990–91 and also an important cyclical expansion after these years, reaching a maximum at the beginning of 2000 that is similar to the one observed during the latest eighties. In addition, our output gap measure differs from the cyclical component of GDP estimated with the HP filter at the end of the sixties and beginning of the seventies, and during the recession of the first half of the eighties, that was more intense according to our model. In both cases, the differences can be explained by the behavior of the unemployment rate during these episodes. Thus, at the beginning of the seventies the unemployment rate approached the NAIRU from unusually low levels, whereas after the second oil crisis the unemployment rate reached the highest level in the second half of the last century.

Figure 1 is also very illustrative about the performance of the NAIRU, which has remained quite stable from mid nineties onwards, around five per cent. This level is similar to the one observed in the fifties and sixties. Additionally, the confidence intervals indicate that expansions and recessions are precisely identified and, therefore, the difference between the current unemployment rate and the NAIRU is very useful for the conduct of economic policy. Our reading of these results is that they cast some doubts on recent criticisms (Staigner, Stock and Watson, 2001) about the statistical uncertainty in the estimation of the NAIRU and its usefulness for policy makers. As regards the performance of the core inflation rate, Figure 1 shows that disinflationary policies were very aggressive in the first half of the eighties, as Ball (1997) has pointed out, with a significant reduction of core inflation. In more recent years, core inflation has remained relatively stable. In Figure 1, one can also see the changes in volatility in the GDP and in inflation, with the confidence bands getting broader or narrower in periods of greater or lower volatility.

4 Revision and Simulation Exercises

In this section, we perform two standard additional exercises aimed at analyzing some properties of our decomposition in comparison to alternative methods. In particular, we are interested in how important the revisions of our estimates are after new information becomes available and how good our model is in recovering the unobserved output gap from a simulated series. We are also interested in investigating whether our model can

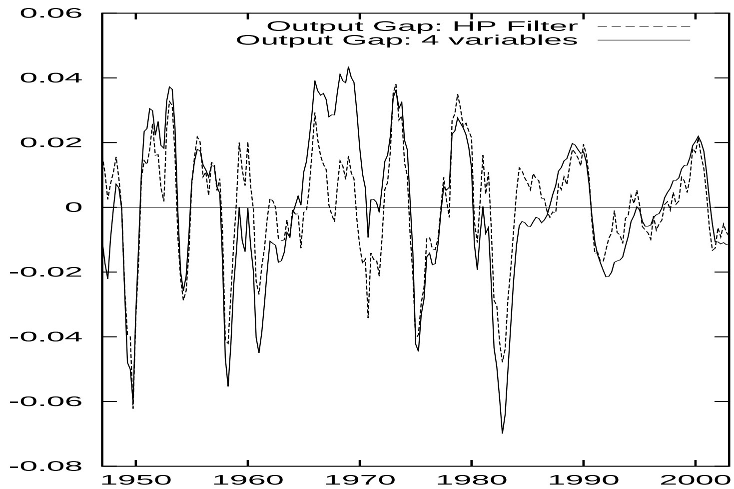

Figure 2: Output gap estimated with the HP filter and the four variables model. United States, 1947:I–2003:I.

generate spurious cycles.

Several authors (Rünstler, 2002, Orphanides and van Norden, 2003, Camba-Mendez and Rodriguez-Palenzuela, 2003) have proposed comparing alternative models using their revision properties. That is, analyzing to what extent the availability of new information introduces changes into the previously estimated unobserved components. Let us define $z_{t|t+j}$ as the estimator of the unobserved component z_t based on all available observations up to time $t+j$. Thus, if $j = 0$ then $z_{t|t}$ is the real–time or concurrent estimate of z_t. As new data become available ($j = 1, 2, ...$), the model yields newer estimates of $z_{t|t+j}$ and, therefore, the difference between $z_{t|t}$ and $z_{t|t+j}$ is a measure of the revision made. As Orphanides and van Norden (2003) have shown, most of the revisions are due to the unreliability of end-of-sample estimates of the output gap. For this reason, we are interested in revisions due to the use of the information contained in the full sample, that is, in $z_{t|t} - z_{t|T}$, where T is the last observation in the sample. This revision is the quasi-final time estimate used by Orphanides and van Norden (2003), which is simply the

Table 6. Revisions 1960:I–1994:IV

	(1)	(2)	(3)	(4)	(5)	(6)
	HP	BK	Our model	Our model	Kuttner	Our model
	($\lambda = 1600$)	(4,32)	4 vab.	3 vab.	2 vab.	2 vab.
Relative std. dev. of revisions in cyclical components (SR)						
Output	0.992	0.696	0.321	0.394	0.672	0.554
Inflation	0.732	0.591	0.401	0.438	0.603	0.463
Unemployment	0.993	0.670	0.346	0.418	–	–
Investment	0.771	0.611	0.491	–	–	–
Correlation between concurrent and full sample estimates						
Output gap	0.526	0.718	0.954	0.932	0.805	0.843
Δ Output gap	0.900	0.785	0.979	0.979	0.914	0.964

rolling estimate based on the final data series but holding constant the set of estimated parameters for the whole sample.

In the revision exercise, we compute the revisions in relative terms according to the formula $SR = \sigma(z_{t|t} - z_{t|T})/\sigma(z_{t|T})$, where $\sigma(x)$ stands for the empirical standard deviation of a sample of the variable x, and we compare the results of our model with several alternatives.

We use the HP filter and the filter proposed by Baxter and King (1995) (henceforth BK filter) as they have become standard examples of univariate methods. Additionally, we use two alternative multivariate models. The first one is our model with only two variables, inflation and output. This model is similar to the model proposed by Kuttner (1994) and will be referred to as Kuttner's model in the sequel. The second one is based on our model but excluding the investment equation.

In Table 6 we present our measure SR of revisions of the trend components over the period 1960:I to 1994:IV, therefore excluding more than eight years at the beginning and at the end of the sample. This sample size allows us to use the HP and BK filters without ARIMA extrapolations at both ends of the sample. The results are very illustrative. The revisions for the HP filter are larger than for any other alternative method in each of the four variables. On the contrary, our preferred unobserved components model with four variables produces generally the smallest revisions. For the BK filter, the revisions of the output gap are half way between those obtained with the HP filter and our model, and close to the ones computed with Kuttner's model. Therefore, these results show that the estimates of the trend components are more stable when new observations become available in our preferred unobserved components model than in the alternative ones.

Sometimes, looking at the magnitude of the output gap revisions is a somewhat misleading way to asses the value–added of incorporating additional variables because the size of revisions will depend on the degree of "flexibility" in the underlying trend, which is determined by the ratio of the variances in the innovations to the trend and to the cycle. To investigate whether it was the degree of "flexibility" that accounted for the reduction in the size of the estimates' revisions, as opposed to the inclusion of additional variables, we have re–done the estimates of the unobservables using the parameter estimates from the full model but deleting the portions of the measurement equation involving the extra indicators (the unemployment and the investment rates). The results are presented in the last column of Table 6. They clearly show that these two variables do account for a reduction in the revisions.

Table 6 also shows the correlations between concurrent and full sample estimates of the

Table 7. Revisions 1985:1–1994:4

	HP	BK	Our model
	($\lambda = 1600$)	(4,32)	4 vab.
Relative std. dev. of revisions (SR)			
Output	1.2383	0.6711	0.2520
Inflation	0.7183	0.4226	0.2628
Unemployment	1.3202	0.7441	0.2498
Investment	1.2023	0.8141	0.5155
Correlation between concurrent and full sample estimates			
Output gap	0.4288	0.74241	0.9793
Δ Output gap	0.8507	0.77303	0.9726

output gap for the five alternative models considered. The best results correspond to our unobserved components model with four variables, which exhibits a very high correlation. It is to be noted that the three unobserved components model yields a higher correlation than the HP and the BK filters. Finally, the last row in Table 6 shows the correlation between the change in the concurrent and full sample estimates of the output gap, since Walsh (2003) has pointed out that monetary policies which focus on the change of the output gap (speed limit policies) stabilize inflation and economic activity better than policies that focus directly on the output gap level. Again, our unobserved components model with four variables yields the highest correlation, making the results of our model very attractive as an input of stabilization policies.

In the previous exercise, the revisions of the unobserved components model have been computed using the parameters estimated with the whole sample. Another possibility is to estimate the parameters using a smaller sample, and then compute the revisions for the rest of the sample. We have estimated our model with observations from 1948:I to 1984:IV, and then we have repeated the preceding exercise for the period 1985:I to 1994:IV, which contains a complete cycle according to the three alternative decompositions. Thus, in this exercise the revisions are computed for quarters that have not been previously used in the estimation of the parameters of our model. As we can see in Table 7, the results show again that the smallest revisions of the output gap are obtained with our model.

If we have to evaluate several methods for estimating unobserved components, we believe that the best way to proceed is to use simulated series. This motivates our simulation exercise. We have first simulated four series of 250 observations that follow a simplified version of our model. More specifically, the output series is the sum of a cycle y_t^c with $\theta_1 = .9$, $\theta_2 = \pi/10$ (five years) and $\omega_{yt} \sim N(0,1)$, plus a trend $\overline{y}_t$ with $\bar{\gamma}_y = .5$ and $\omega_{\gamma t} \sim N(0,.5^2)$. The inflation series is the sum of a core inflation $\overline{\pi}_t$ with $\omega_{\pi t} \sim N(0,.6^2)$, $\mu_\pi(L) = 0$, plus $\eta_y y_t^c$ with $\eta_y = .8$ plus $v_{\pi t} \sim N(0,.4^2)$. The unemployment series is the sum of a trend $\overline{U}_t$ with $\{\omega_{ut}\} \sim N(0,.7^2)$ plus $\phi_{y0} y_t^c$ with $\phi_{y0} = -.5$ plus $v_{ut} \sim N(0,.5^2)$. Finally, the investment series is the sum of a trend $\overline{x}_t$ with $\omega_{xt} \sim N(0,.8^2)$ plus $\beta_{y0} y_t^c$ with $\beta_{y0} = .5$ plus $v_{xt} \sim N(0,.3^2)$.

Then, we have estimated our four variables model. The parameters that are zero are estimated as not significant. After setting nonsignificant parameters to zero, we have estimated the model again. The fit is rather good. We do not present the results for lack of space but they are available upon request. All estimated parameters are close to the true ones and the residuals show no autocorrelation and can be accepted as normally distributed.

We compare our results with the results obtained when we apply the HP filter, the

Table 8. Correlations between original and estimated cycles

	HP	BK	Cogley	Our model
	($\lambda = 1600$)	(4,32)		4 vab.
Output gap	0.8837	0.8611	0.7845	0.9816
Core inflation	0.7543	0.7259	0.7518	0.9622

Baxter and King filter and the Cogley filter. The goal of the experiment is to see how well the different methods recover core inflation $\overline{\pi}_t$ and the output gap y_t^c. In Table 8 we can see the correlations between the original variables and the estimates obtained with the different methods. It is seen that our model gives the highest correlations.

Our second simulation experiment investigates which of the alternative methods that we are comparing can produce spurious cycles. To this end, we have simulated four series of 250 observations that follow the same model of our first simulation experiment but with the cycle y_t^c replaced with a white noise series distributed $N(0, 10^2)$. Proceeding as before, we have estimated our four variables model. The parameters that are zero are estimated as not significant. After setting nonsignificant parameters to zero, except those of the cycle, we have estimated the model again. The fit is rather good. Again, we do not present all of the results for lack of space but they are available upon request. All estimated parameters are again close to the true ones and the residuals show no autocorrelation and can be accepted as normally distributed. The parameters estimated for the cycle are $\theta_1 = 0.0240$, and $\theta_2 = 0.1963$, with t–values of 0.78 and 0.13. It is clear that we would not estimate a cycle with our model in this case because it is not significant. However, all the alternative procedures would estimate spurious cycles.

5 Conclusions

In this paper we have proposed an unobserved components model that provides estimates of the NAIRU, core inflation and the output gap for the United States. The model exploits the rich information about the business cycle simultaneously contained in the GDP and the unemployment, inflation and the investment rate, to decompose these four variables into trend and cyclical movements. The unknown parameters in the model have been estimated by maximum likelihood using a Kalman filter initialized with a partially diffuse prior, and the unobserved components have been estimated using a smoothing algorithm. Although the correlation between the output gap estimated with this method and that obtained with the HP filter is relatively high, there are some important discrepancies, particularly in the second half of the nineties. Contrary to the HP filter, our method also works well at the end of the sample and, thus, it is very appropriate to infer how current economic conditions affect output, inflation and the unemployment rate.

Our results also show that the output gap estimated with our model is a very significant variable in Okun's law, the Phillips curve and the investment rate equation. These results confirm that the dependent variables in these equations improve the precision of the GDP decomposition into its trend and output gap components. Finally, we have verified that the revisions of output gap estimates when new information becomes available are lower with our preferred model than with the HP and the BK filters. The results obtained in this paper illustrate the usefulness of our decomposition for the conduct of stabilization economic policies in real time.

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APPENDIX A: DATA SOURCES

The data set is available at **http://iei.uv.es/˜rdomenec/output/output.htm**. There are 229 quarterly observations from 1946:I to 2003:I. The variables contained in this file are the following:

- $\ln GDP$: log of Real Gross Domestic Product, Billions of Chained 1996 Dollars, SAAR. Source: BEA, Table 1.10, Line 1.
- π: quarterly inflation rate, defined as $4(\ln P_t - \ln P_{t-1})$, where P_t is the geometric average of the monthly price levels. Source: BLS, CPI Urban Consumer, all items, 1982-84=100, SA.
- U : unemployment rate, defined as the average of the monthly unemployment rates. Source: BLS, household survey, SA.
- x: nominal investment rate, defined as I/Y, where I is the nominal gross private domestic investment (Bil. \$, SAAR), source: BEA, Table 5.4, Line 1, and Y is the nominal gross domestic product, (Bil. \$, SAAR), source: BEA, Table 1.9 Line 1.

APPENDIX B: TECHNICAL DETAILS

To implement the methodology used in this paper, we have used a set of MATLAB programs written by V. Gómez. The code and the data sets are available at the Internet address: **http://iei.uv.es/~rdomenec/output/output.htm**.

The estimation of the model parameters is performed by maximizing the so called "diffuse likelihood" (De Jong, 1991). It can be shown that maximizing the concentrated log–likelihood is equivalent to minimizing a nonlinear sum of squares function. To this end, a routine has been written in MATLAB that implements the Levenverg–Marquardt method, although the standard MATLAB routine **lsqnonlin** of the OPTIMIZATION TOOLBOX can also be used.

Letting θ be the vector of parameters to be estimated, the nonlinear sum of squares function that is minimized can be written as $F(\theta) = e'(\theta)e(\theta)$, where $e(\theta)$ is a vector of residuals that can be computed by the diffuse Kalman filter. In terms of $F(\theta)$, the log–likelihood $L(\theta)$ can be expressed as $L(\theta) = \text{const.} - n \ln F(\theta)/2$, where n is the total number of observations. Under the usual assumptions, the estimator $\hat{\theta}$ of θ is asymptotically distributed as $N(\theta, I^{-1}/n)$, where I is the information matrix. This last matrix can be estimated by $\hat{I} = -\frac{1}{n}\frac{\partial^2 L(\theta)}{\partial\theta\partial\theta'}_{|\theta=\hat{\theta}} = \frac{1}{2}\frac{\partial^2 \ln F(\theta)}{\partial\theta\partial\theta'}_{|\theta=\hat{\theta}}$, where the derivatives can be computed numerically. This is the method that has been implemented in MATLAB and used to compute standard errors for the model parameters.

The smoothing algorithm provides the standard errors of the unobserved components, with which the confidence intervals can be calculated. Other sources of uncertainty, like that due to parameter estimation could be taken into account through simulation, but this has not been implemented for this paper.

APPENDIX C: ASSUMPTIONS AND PROOFS

To obtain a model–based interpretation of (4), we will make the following reasonable and unrestrictive assumptions

1) $.5 \leq \mu \leq 1$

2) y_t^c and v_t are unobserved components. $\{y_t^c\}$ follows the model $\phi(L)y_t^c = \omega_{yt}$, where L is the lag operator, $Ly_t^c = y_{t-1}^c$, $\{\omega_{yt}\}$ is a white noise sequence with zero mean and $\mathrm{Var}(\omega_{yt}) = \sigma_y^2$, $\phi(z) = 1 - 2\theta_1 \cos(\theta_2)z + \theta_1^2 z^2$, $0 < \theta_1 < 1$ and $\theta_2 \in [\pi/20, \pi/3]$. $\{v_t\}$ is a white noise sequence with zero mean and $\mathrm{Var}(v_t) = \sigma_v^2$ uncorrelated with $\{y_t^c\}$.

3) the rational expectations $E_t(\pi_{t+1})$ are linear in $\{\pi_s : s \leq t\}$. That is,$E_t(\pi_{t+1}) = \sum_{j=0}^{\infty} \beta_j \pi_{t-j}$ with $\sum_{j=0}^{\infty} |\beta_j| < \infty$

4) $0 < \beta_0 < 1/\mu$

5) $\alpha = 1/\mu - \beta_0 = 2\phi(1)$

Some remarks on the previous assumptions are in order. First, assumption 1) seems reasonable because it gives at least the same weight to expected future inflation than to past inflation. Note that this assumption implies $1 \leq 1/\mu \leq 2$ and $0 \leq (1-\mu)/\mu \leq 1$. Second, the interval $[\pi/20, \pi/3]$ corresponds to the quarterly frequencies usually accepted as appropriate for the output gap, namely those with periods between a year and a half and eight years. Thus, the output gap y_t^c is assumed to follow a stationary autoregressive process of order two that has two complex roots to ensure a cyclical behavior. Third, assumption 3) is obvious if one wants to keep the model linear and assumption 4) is necessary to avoid having π_t on both sides of (4). Fourth, by lemma C1 later in this appendix, $0 \leq \phi(1) \leq 1$ for all $0 \leq \theta_1 \leq 1$ and $\theta_2 \in [\pi/20, \pi/3]$. Thus, $0 \leq \alpha \leq 2$ in assumption 5), what is in agreement with assumptions 1) and 4). Some motivation for assumption 5) will be given later in this Appendix.

The following three lemmas will be useful to prove theorem 1.

Lemma C1. *The function* $\phi(1) = 1 - 2\theta_1 \cos(\theta_2) + \theta_1^2$, *where* $0 \leq \theta_1 \leq 1$ *and* $\theta_2 \in [\pi/20, \pi/3]$, *satisfies* $0 \leq \phi(1) \leq 1$ *for all* θ_1 *and* θ_2.

Proof. First note that $-\cos(\pi/20) \leq -\cos(\theta_2) \leq -\cos(\pi/3) = -\sqrt{3}/2$. Then, $\phi(1) \leq 1 - \sqrt{3}\theta_1 + \theta_1^2$. Define $f(\theta_1) = 1 - \sqrt{3}\theta_1 + \theta_1^2$. Then, $f'(\theta_1) = 2\theta_1 - \sqrt{3}$ implies that f is decreasing in $[0, \sqrt{3}/2]$ and increasing in $[\sqrt{3}/2, 1]$. Since $f(0) = 1$, $f(\sqrt{3}/2) < 1$, $f(1) = 2 - \sqrt{3} < 1$ and $\phi(1) = (1-\theta_1)^2 + 2\theta_1[1 - \cos(\theta_2)] \geq 0$, the lemma is proved. $\square$

Lemma C2. *Under assumptions 1)–4), let* $\alpha = 1/\mu - \beta_0$ *and let* $\tilde{\phi}(z) = 1 + (\phi_1 - \alpha)z + \phi_2 z^2$, *where* $\phi(z) = 1 + \phi_1 z + \phi_2 z^2$, $\phi_1 = -2\theta_1 \cos(\theta_2)$, *and* $\phi_2 = \theta_1^2$. *Then, the following equality*

$$\left|\frac{\tilde{\phi}(e^{-ix})}{\phi(e^{-ix})}\right|^2 = \left|\frac{\tilde{\phi}(1)}{\phi(1)}\right|^2 + k\left|\frac{1 - e^{-ix}}{\phi(e^{-ix})}\right|^2$$

holds if, and only if, $\alpha = 2\phi(1)$. *In this case,* $k = 2\phi(1)(1 + \phi_2) > 0$, $|\tilde{\phi}(1)/\phi(1)|^2 = 1$ *and the function* $|\tilde{\phi}(e^{-ix})/\phi(e^{-ix})|^2$ *has a global minimum at* $x = 0$ *that is equal to* 1.

Proof. Let $\tilde{\phi}_1 = \phi_1 - \alpha$. Then, $|\tilde{\phi}(e^{-ix})|^2 = (1 + \tilde{\phi}_1^2 + \phi_2^2) + 2\tilde{\phi}_1(1 + \phi_2)\cos(x) + \phi_2 \cos(2x)$, $|\phi(e^{-ix})|^2 = (1 + \phi_1^2 + \phi_2^2) + 2\phi_1(1 + \phi_2)\cos(x) + \phi_2 \cos(2x)$ and the equality of the lemma is satisfied if, and only if, $\phi_2 - k_0\phi_2 = 0$, $\tilde{\phi}_1 + \tilde{\phi}_1\phi_2 - k_0(\phi_1 + \phi_1\phi_2) = -k$, and $1 + \tilde{\phi}_1^2 + \phi_2^2 - k_0(1 + \phi_1^2 + \phi_2^2) = 2k$, where $k_0 = |\tilde{\phi}(1)/\phi(1)|^2$. The first equation implies $k_0 = 1$ and it is not difficult to verify that the unique solution is $\alpha = 2\phi(1)$. In addition, $k = 2\phi(1)(1 + \phi_2) > 0$ is the only number that satisfies the last two relations when $k_0 = 1$. $\square$

Lemma C3. *Let $\alpha > 0$ and let $u_t = v_t - \alpha v_{t-1}$, where $\{v_t\}$ is a white noise process with zero mean and Var$(v_t) = 1$. Then, the function $f(x) = |1-\alpha e^{-ix}|^2$ satisfies $|1-\alpha e^{-ix}|^2 = (\alpha-1)^2+\alpha|1-e^{-ix}|^2$ and thus $f(x)$ has a global minimum at $x = 0$ that is equal to $(\alpha-1)^2$.*

Proof. Since $f(x)= 1 + \alpha^2 - 2\alpha\cos(x)$, $f(0) = (\alpha-1)^2$, $f'(x) = 2\alpha\sin(x) \geq 0$, and f is increasing in $[0, \pi]$. It is not difficult to verify that k satisfies $1 + \alpha^2 - 2\alpha\cos(x) = (\alpha-1)^2 + 2k(1-\cos(x))$ if, and only if, $k = \alpha$. $\square$

Proof of Theorem 1. By assumption 3), $E_t(\pi_{t+1}) = \sum_{j=0}^{\infty}\beta_j\pi_{t-j}$. Therefore, we can write $\pi_t - \mu E_t(\pi_{t+1}) - (1-\mu)\pi_{t-1} = \sum_{j=0}^{\infty}\varphi_j\pi_{t-j}$, $\sum_{j=0}^{\infty}|\varphi_j| < \infty$.

By stationarity, $\delta y_t^c + v_t = \delta\sum_{j=0}^{\infty}\psi_j\omega_{y,t-j} + v_t$ and, from equation (4), it is obtained that $\sum_{j=0}^{\infty}\varphi_j\pi_{t-j} = \delta\sum_{j=0}^{\infty}\psi_j\omega_{y,t-j} + v_t$. Inverting the previous equation yields $\pi_t = \delta\sum_{j=0}^{\infty}\psi_{yj}\omega_{y,t-j} + \sum_{j=0}^{\infty}\psi_{vj}v_{t-j}$, where $\psi_{y0} = \psi_{v0} = 1/\varphi_0$ and, by assumption 4), $\varphi_0 = 1-\mu\beta_0 > 0$. Letting $a_{t+1} = \pi_{t+1} - E_t(\pi_{t+1})$, we can write

$$a_t = (\delta/\varphi_0)\omega_{yt} + (1/\varphi_0)v_t. \tag{C.1}$$

Replacing $E_t(\pi_{t+1})$ by $\pi_{t+1} - a_{t+1}$ in (4), we get $\pi_t = \mu\pi_{t+1} - \mu a_{t+1} + (1-\mu)\pi_{t-1} + \delta y_t^c$, and we can write

$$\pi_t = \frac{1}{\mu}\pi_{t-1} - \frac{1-\mu}{\mu}\pi_{t-2} - \frac{\delta}{\mu}y_{t-1}^c - \frac{1}{\mu}v_{t-1} + a_t. \tag{C.2}$$

Since $1 - (1/\mu)z + [(1-\mu)/\mu]z^2 = \{1 - [(1-\mu)/\mu]z\}(1-z)$, from (C.1) and (C.2), it is obtained that

$$\begin{aligned}(1 - \frac{1-\mu}{\mu}L)\nabla\pi_t &= -\frac{\delta}{\mu}y_{t-1}^c - \frac{1}{\mu}v_{t-1} + \frac{\delta}{\varphi_0}\omega_{yt} + \frac{1}{\varphi_0}v_t \\ &= \frac{\delta}{\varphi_0}\left[\omega_{yt} - \frac{\varphi_0}{\mu}y_{t-1}^c\right] + \frac{1}{\varphi_0}\left[v_t - \frac{\varphi_0}{\mu}v_{t-1}\right] \\ &= \frac{\delta}{\varphi_0}\left[\frac{\phi(L)-\alpha L}{\phi(L)}\right]\omega_{yt} + \frac{1}{\varphi_0}\left[v_t - \alpha v_{t-1}\right], \end{aligned} \tag{C.3}$$

where $\alpha = \varphi_0/\mu = 1/\mu - \beta_0$. Let $\tilde{\phi}(L) = \phi(L) - \alpha L$. Then, under assumptions 1)–5), by lemmas C2 and C3, we get the canonical decompositions (Hillmer and Tiao, 1982) $\left[\frac{\phi(L)-\alpha L}{\phi(L)}\right]\omega_{yt} = \sqrt{k}\nabla y_t^c + \omega_{ct}$ and $v_t - \alpha v_{t-1} = \sqrt{\alpha}\nabla v_t + \omega_{vt}$, where $\{\omega_{ct}\}$ is a white noise sequence with zero mean and $\mathrm{Var}(\omega_{ct}) = \sigma_y^2$, uncorrelated with y_t^c, $\{\omega_{vt}\}$ is a white noise sequence with zero mean and $\mathrm{Var}(\omega_{vt}) = (\alpha-1)^2\sigma_v^2$, uncorrelated with v_t, and $k = 2\phi(1)(1+\phi_2)$. Substituting the previous expressions into (C.3) yields $(1 - \frac{1-\mu}{\mu}L)\nabla\pi_t = \frac{\delta}{\varphi_0}\left[\sqrt{k}\nabla y_t^c + \omega_{ct}\right] + \frac{1}{\varphi_0}[\sqrt{\alpha}\nabla v_t + \omega_{vt}]$. Define $\tilde{\omega}_{\pi t} = (\delta/\varphi_0)\omega_{ct} + (1/\varphi_0)\omega_{vt}$ and let $\tilde{\mu} = 1 - (1-\mu)/\mu$. Then, $\{\tilde{\omega}_{\pi t}\}$ is a white noise sequence with zero mean and $\mathrm{Var}(\omega_{\pi t}) = [(\delta\sigma_y/\varphi_0)^2 + (\alpha-1)^2(\sigma_v/\varphi_0)^2]$ that is uncorrelated with both $\{y_t^c\}$ and $\{v_t\}$ and we can write $[1-(1-\tilde{\mu})L]\nabla\pi_t = \frac{\sqrt{k}\delta}{\varphi_0}\nabla y_t^c + \frac{\sqrt{\alpha}}{\varphi_0}\nabla v_t + \tilde{\omega}_{\pi t}$. Defining $\omega_{\pi t} = \tilde{\omega}_{\pi t}/\tilde{\mu}$, $v_{\pi t} = (\sqrt{\alpha}/\varphi_0)v_t$ and $\eta_y = \sqrt{k}\delta/\varphi_0$, and dividing by ∇ in the previous equation, the theorem is proved. □

Assumption 5) ensures that the first term on the right hand side of equation (C.3), that is a cyclical component, has a global minimum in the spectrum at the zero frequency. This seems a reasonable assumption, since inflation appears on the left hand side in first differences and, therefore, all low frequency components should have been removed.

Proof of Theorem 2. Taking first conditional expectations in (6), we get $E_t(\overline{\pi}_t) = E_t(\overline{\pi}_{t+k})$. Then, taking conditional expectations in (8) yields $E_t(\pi_{t+k}) = (1 - \sum_{j=1}^4 \mu_j) \times E_t(\overline{\pi}_{t+k}) + \sum_{j=1}^4 \mu_j E_t(\pi_{t+k-j})$, since, due to the damping factor $\theta_1 < 1$ in the cycle equation, $E_t(y_{t+k}^c)$ is zero for k sufficiently large. From this it is concluded that $E_t(\pi_{t+k}) = E_t(\overline{\pi}_{t+k})$. □

Proof of Theorem 3. The model $y_t = s_t + n_t$ can be cast into state space form by defining the state vector $\alpha_t = s_t$. The state space equations are $y_t = \alpha_t + n_t$ and

$\alpha_{t+1} = \alpha_t + b_{t+1}$. The reduced form model of y_t is $\nabla y_t = (1+\theta L)a_t$, where θ and Var(a_t) $= \sigma_a^2$ can be obtained by equating the covariances in $b_t + \nabla n_t = (1+\theta L)a_t$. Proceeding as in Burridge and Wallis (1988), pp. 71–73, setting $\phi = 1$ in that article, we can show that $\hat{s}_{t|t} = (1+\theta)/(1+\theta L)\pi_t$. $\square$

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