Solving a Vehicle Routing Problem with a non-linear load dependant cost function

Simon Spoorendonk†  Allan Larsen§  David Pisinger†  Stefan Røpke§

† DTU Management Engineering, {spoo,pisinger}@man.dtu.dk
§ DTU Transport {ala,sr}@transport.dtu.dk

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Another VRP!

The problem:

- A real-life application in the paper waste collection industry.
- Collecting paper of various quality from customers in the area of Copenhagen, Denmark.
- A bonus is gained for each load depending on the quantity and quality, e.g., white paper is more valuable than old newspapers.
- The objective is to minimize the travel cost subtracted by the paper bonuses.
- Vehicle routes originates from a central depot and must obey time windows and vehicle capacity.
The twist and the focus of talk.

The difference from VRPTW:
• The objective function is non-linear. More precisely it a non-continuous stepwise linear function.

What I am going to talk about:
• An exact solution method based on column generation.
• The workings of the underlying dynamic programming algorithm (dominance, fathoming).
• Look at performance when considering different weight between travel cost and paper bonuses.
Overview

Introduction

A set partitioning model

The pricing problem
  Dominance criteria
  Fathoming Rules
  Paper cost bounds

Experimental results

Conclusion
A set partitioning model

\[
\min \sum_{p \in P} c_p \lambda_p \\
\text{s.t. } \sum_{p \in P} a_{ip} \lambda_p = 1 \quad \forall i \in C \\
\lambda_p \in \{0, 1\} \quad \forall p \in P.
\]

The path cost is calculated as

\[
c_p = \sum_{(i,j) \in p} c_{ij} - \sum_{i \in p} d_i \cdot p^{\text{paper}}(q_p)
\]

where the paper quality (low value is good) is determined as

\[
q_p = \max_{i \in p} \{q_i\}.
\]

A feasible path must obey the time windows and the vehicle capacity.
The paper cost function

Definition (1)
The function $p^{\text{paper}} : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ is strictly decreasing such that

$$p^{\text{paper}}(k - 1) > p^{\text{paper}}(k), \quad k > 0$$
The pricing problem

Another elementary shortest path problem with resource constraints.

\[
\min_{p \in P} \left( \sum_{(i,j) \in p} c_{ij} - \sum_{i \in p} \pi_i - \sum_{i \in p} d_i \cdot p^{paper}(q_p) \right)
\]

We will talk about the effect of the non-linear load dependant term in the objective.
Dynamic programming

Resource vector

\[ \{T_{\text{travel}}^i, T_{\text{time}}^i, T_{\text{load}}^i, T_{\text{unreach}}^i \}_\ell \subseteq \mathbb{C}, T_{\text{paper}}^i, T_{\text{quality}}^i \] for label \( L_i \).

Resource extension functions (REF) related to the paper

\[
\begin{align*}
  f_{ij}(T_i^{\text{paper}}) &= p_{\text{paper}}(T_j^{\text{quality}}) \cdot T_j^{\text{load}} \\
  f_{ij}(T_i^{\text{quality}}) &= \max\{q_j, T_i^{\text{quality}}\}.
\end{align*}
\]
Dominance criteria

A very basic (and not so practical) dominance criterion

**Theorem (1)**

*Label* $L_i$ *dominates label* $\bar{L}_i$ *if*

$$f_{ie}(T_i^{travel}) - f_{ie}(T_i^{paper}) \leq f_{ie}(\bar{T}_i^{travel}) - f_{ie}(\bar{T}_i^{paper})$$

$$\epsilon \in \bar{E}$$

$$\mathcal{E} \supseteq \bar{\mathcal{E}}.$$

Some observations:

- Extension superset implied by resource inequalities

  $$T_i^r \leq \bar{T}_i^r \quad \forall r \in \{time, load, unreach\} \{\ell \in C\} \Rightarrow \mathcal{E} \supseteq \bar{\mathcal{E}}.$$  

- The travel cost of $\epsilon \in \bar{E}$ is identical for both labels, i.e.,

  $$f_{ie}(T_i^{travel}) - T_i^{travel} = f_{ie}(\bar{T}_i^{travel}) - \bar{T}_i^{travel}.$$
Dominance using bounds

Bounds on the paper cost are

\[
LB(T_i^{paper}) \leq \min_{\epsilon \in \bar{E}} f_{i\epsilon}(T_i^{paper}) \leq f_{i\epsilon}(T_i^{paper}) \leq UB(T_i^{paper}) \leq \max_{\epsilon \in \bar{E}} f_{i\epsilon}(T_i^{paper}) \quad \epsilon \in \bar{E}.
\]

We get back to the calculation of the bounds.

**Theorem (2)**

*Label \( L_i \) dominates label \( \bar{L}_i \) if*

\[
T_i^{travel} - LB(T_i^{paper}) \leq \bar{T}_i^{travel} - UB(T_i^{paper})
\]

\[
T_i^{r} \leq \bar{T}_i^{r} \quad \forall r \in \{time, load, unreachable\} \{l \in C\}
\]
Dominance with equal quality

Theorem (3)
Label $L_i$ dominates label $\bar{L}_i$ if

\[
T_{i}^{\text{travel}} - T_{i}^{\text{paper}} \leq \bar{T}_{i}^{\text{travel}} - \bar{T}_{i}^{\text{paper}}
\]

\[
T_{i}^{r} \leq \bar{T}_{i}^{r}
\]

\[
T_{i}^{\text{quality}} = \bar{T}_{i}^{\text{quality}}.
\]

\forall r \in \{\text{time, load, unreach}, \ell \in C\}
Proof of Theorem 3.
Rewrite cost inequality to

$$T_{i}^{\text{travel}} \leq \bar{T}_{i}^{\text{travel}} - p^{\text{paper}}(\bar{T}_{i}^{\text{quality}})(\bar{T}_{i}^{\text{load}} - T_{i}^{\text{load}}).$$

For any $\epsilon \in \bar{E}$ the load difference is $\bar{T}_{i}^{\text{load}} - T_{i}^{\text{load}}$ and remains so. Apply REFs

$$T_{i}^{\text{travel}} \leq \bar{T}_{i}^{\text{travel}} - p^{\text{paper}} \left( f_{i\epsilon}(\bar{T}_{i}^{\text{quality}}) \right) \left( \bar{T}_{i}^{\text{load}} - T_{i}^{\text{load}} \right).$$

From Definition 1 and because the REF for paper quality is non-decreasing we have

$$p^{\text{paper}}(\bar{T}_{i}^{\text{quality}}) \geq p^{\text{paper}} \left( f_{i\epsilon}(\bar{T}_{i}^{\text{quality}}) \right) \geq 0.,$$

meaning that the difference in the paper cost becomes smaller when extending.  $\square$
Dominance with better quality of dominating label

Issues:

• A label being cheaper now with a better quality and less load does not necessarily mean it’s extension is cheaper.

• If extension leads to same quality the cost relation may change.

Apply a penalty (a bound on maximal difference) to the cost comparison. Similar to

• visited-bit relaxation of Chabrier (2005)

• dominance with SR-inequalities in the master problem by Jepsen et al. (2008).
Theorem (4)

Label $L_i$ dominates label $\bar{L}_i$ if

\[
T_i^{\text{travel}} - T_i^{\text{paper}} \leq \bar{T}_i^{\text{travel}} - \bar{T}_i^{\text{paper}} - \alpha_1 T_i^{\text{load}}
\]

\[
T_i^r \leq \bar{T}_i^r \quad \forall r \in \{\text{time, load, unreach}_\ell \mid \ell \in \mathcal{C}\}
\]

\[
T_i^{\text{quality}} \leq \bar{T}_i^{\text{quality}}
\]

where

\[
\alpha_1 = p^{\text{paper}}(T_i^{\text{quality}}) - p^{\text{paper}}(\bar{T}_i^{\text{quality}}).
\]
Proof of Theorem 4 (Sketch).

Rewrite cost inequality to

\[ T_i^{travel} \leq \bar{T}_i^{travel} + \left( p_{paper}(T_i^{quality}) - p_{paper}(\bar{T}_i^{quality}) - \alpha_1 \right) T_i^{load} \]

\[ - p_{paper}(\bar{T}_i^{quality})(\bar{T}_i^{load} - T_i^{load}). \]

Apply REFs

\[ T_i^{travel} \leq \bar{T}_i^{travel} + \left( p_{paper}(f_i\epsilon(T_i^{quality})) - p_{paper}(f_i\epsilon(\bar{T}_i^{quality})) \right) f_i\epsilon(T_i^{load}) \]

\[ - p_{paper}(f_i\epsilon(\bar{T}_i^{quality}))(\bar{T}_i^{load} - T_i^{load}). \]

Relations

\[ p_{paper}(T_i^{quality}) - p_{paper}(\bar{T}_i^{quality}) - \alpha_1 = 0 \leq p_{paper}(f_i\epsilon(T_i^{quality})) - p_{paper}(f_i\epsilon(\bar{T}_i^{quality})) \]

and

\[ T_i^{load} \leq f_i\epsilon(T_i^{quality}) \]
Example

Paper prices: $p(1) = 4$ and $p(2) = 1$

An extension $\epsilon \in \overline{E}$: $d_\epsilon = 2$ and $q_\epsilon = 2$ (i.e. quality worsens to 2)

$L_i = \{T_i^{travel} = 0, T_i^{load} = 1, T_i^{quality} = 1, \ldots \}$

$\overline{L}_i = \{\overline{T}_i^{travel} = 0/2, \overline{T}_i^{load} = 3, \overline{T}_i^{quality} = 2, \ldots \}$

$T_i^r \leq \overline{T}_i^r \quad \forall r \in \{time, unreach_\ell \{\ell \in C\}\}$

$\alpha_1 T_i^{load} = (4 - 1)1 = 3$

Not dominating:
- Node $i$: $0 - 4 \leq 0 - 3 - 3$ (not good with penalty)
- End node: $0 - 3 \leq 0 - 5$ (not good)

Dominating:
- Node $i$: $0 - 4 \leq 2 - 3 - 3$ (also good with penalty)
- End node: $0 - 3 \leq 2 - 5$
Dominance with worse quality of dominating label

Issues:

• A label being cheaper now with a worse quality and less load does not necessarily mean it's extension is cheaper.

• If extension leads to no change in quality the cost relation may change.

Solution is similar to before: add a penalty.
Theorem (5)

Label $L_i$ dominates label $\bar{L}_i$ if

$$T_i^{travel} - T_i^{paper} \leq \bar{T}_i^{travel} - \bar{T}_i^{paper} - \alpha_2 \left( p^{paper}(\bar{T}_i^{quality}) - p^{paper}(T_i^{quality}) \right)$$

$$T_i^r \leq \bar{T}_i^r \quad \forall r \in \{time, load, unreach\} \{\ell \in C\}$$

$$T_i^{quality} \geq \bar{T}_i^{quality}$$

where

$$\alpha_2 = \min\{Q - \bar{T}_i^{load}, \sum_{j \in \bar{R}_i} d_j\}$$

and

$$\bar{R}_i = \{ j \in C \mid \bar{T}_i^{unreach_j} = 0 \}.$$
Proof of Theorem 5 (Sketch).

Rewrite cost inequality to

\[ T_i^{travel} \leq \bar{T}_i^{travel} - \left( p^{paper}(\bar{T}_i^{quality}) - p^{paper}(T_i^{quality}) \right) \left( T_i^{load} + \alpha_2 \right) \]

\[ - p^{paper}(\bar{T}_i^{quality})(\bar{T}_i^{load} - T_i^{load}). \]

Apply REFs

\[ T_i^{travel} \leq \bar{T}_i^{travel} - \left( p^{paper}(f_{i\epsilon}(\bar{T}_i^{quality})) - p^{paper}(f_{i\epsilon}(T_i^{quality})) \right) f_{i\epsilon}(T_i^{load}) \]

\[ - p^{paper}(f_{i\epsilon}(\bar{T}_i^{quality}))(\bar{T}_i^{load} - T_i^{load}). \]

Relations

\[ p^{paper}(\bar{T}_i^{quality}) - p^{paper}(T_i^{quality}) \geq p^{paper}(f_{i\epsilon}(\bar{T}_i^{quality})) - p^{paper}(f_{i\epsilon}(T_i^{quality})) \geq 0 \]

and

\[ T_i^{load} + \alpha_2 \geq f_{i\epsilon}(T_i^{quality}) \geq 0 \]
Example

Paper prices: $p(1) = 4$ and $p(2) = 1$ and $Q = 6$
An extension $\epsilon \in \bar{E}$: $d_\epsilon = 2$ and $q_\epsilon = 1$ (i.e. quality worsens to 2)

$L_i = \{T_i^{\text{travel}} = 0, T_i^{\text{load}} = 1, T_i^{\text{quality}} = 2, \ldots \}$

$\bar{L}_i = \{\bar{T}_i^{\text{travel}} = 11/17, \bar{T}_i^{\text{load}} = 3, \bar{T}_i^{\text{quality}} = 2, \ldots \}$

$T_i^r \leq \bar{T}_i^r \quad \forall r \in \{\text{time}, \text{unreach}_\ell \} \cup \{\ell \in C\}$

$\alpha_2 \left( p^{\text{paper}}(\bar{T}_i^{\text{quality}}) - p^{\text{paper}}(T_i^{\text{quality}}) \right) = 2(4 - 1) = 6$

Not dominating:
Node $i$: $0 - 1 \leq 11 - 12 - 6$ (not good with penalty)
End node: $0 - 3 \leq 11 - 20$ (not good)

Dominating:
Node $i$: $0 - 1 \leq 17 - 12 - 6$ (also good with penalty)
End node: $0 - 3 \leq 17 - 20$
Fathoming Rules

Theorem (6)

If

\[ LB(T_i^{\text{travel}}) - UB(T_i^{\text{paper}}) > \beta \]

where \( \beta \) is an upper bound on the path cost when extending label \( L_i \), then \( L_i \) can be discarded.

The lower bound on the travel cost can be computed using a relaxation such as 2-cycle elimination.
Paper cost lower bounds

- Running time is $O(C)$

$$LB_1(T_i^{\text{paper}}) = p_{\text{paper}} \left( \max_{j \in R_i} q_j \right) \cdot T_i^{\text{load}}.$$ 

- Running time is $O(K|C|)$

$$LB_2(T_i^{\text{paper}}) = \min_{k = T_i^{\text{quality}}, ..., K} \left\{ p_{\text{paper}} (k) \cdot \left( T_i^{\text{load}} + \sum_{j \in R_i} d_j \right) \mid \{ j \in R_i \cup \{ i \} \mid q_j = k \} \right\}$$

- Calculate $LB_2$ as a preprocessing step, for each node, quality and load, i.e., $O(K^2|C|^2Q)$.

$$LB_3(T_i^{\text{paper}})$$

is then a constant time look-up.
Theorem (7)

The lower bounds $LB_j(T_{i}^{\text{paper}})$, $j = 1, 2, 3$ have the following relation

$$LB_1(T_{i}^{\text{paper}}) \leq LB_3(T_{i}^{\text{paper}}) \leq LB_2(T_{i}^{\text{paper}}) \leq LB(T_{i}^{\text{paper}}).$$

For large $Q$ calculating $LB_3$ as a preprocessing step may be very time consuming.

- Calculate the bound for every capacity step size $B$.
- Then $LB_3(T_{i}^{\text{paper}})$ is given as the lower value of the steps $B_i$ and $B_{i+1}$ where $B_i \leq T_{i}^{\text{load}} < B_{i+1}$.

Side-effect: The relation

$$LB_2(T_{i}^{\text{paper}}) \leq LB_3(T_{i}^{\text{paper}})$$

does not hold anymore.
Paper cost upper bounds

- Running time \(O(|C|)\)

\[ UB_1(T^\text{paper}_i) = p^\text{paper}(T^\text{quality}_i) \cdot \min\{Q, T^\text{load}_i + \sum_{j \in C_i} d_j\}. \]

- A generalized multiple-choice knapsack problem (Pisinger (2001)) and is \(NP\)-hard

\[ UB_2(T^\text{paper}_i) = \max_{k=T^\text{quality}_i, \ldots, K} \left\{ p^\text{paper}(k) \cdot \left( T^\text{load}_i + \sum_{j \in R_i} d_j \right) \right\} \bigg| q_j \leq k, \{ j \in C_i \cup \{i\} \} \bigg| q_j = k \]

For small instances it is very easy to solve.

- Calculate \(UB_2\) as a preprocessing step, for each node, quality and load, i.e., \(O(K|C|Q)\) steps.

\[ UB_3(T^\text{paper}_i) \]

is then a constant time look-up.
Theorem (8)

The lower bounds $UB_j(T^\text{paper}_i), j = 1, 2, 3$ have the following relation

$$UB_1(T^\text{paper}_i) \geq UB_3(T^\text{paper}_i) \geq UB_2(T^\text{paper}_i) \geq UB(T^\text{paper}_i).$$

The same story with the preprocessing.

In a similar way the relation

$$UB_2(T^\text{paper}_i) \geq UB_3(T^\text{paper}_i)$$

does not hold.
Experimental results

Instances (derived from Solomon instances for the VRPTW):
a) \( p_k = 10(K - k)^2, k = 0, \ldots, K - 1 \)
b) \( p_k = (K - k)^2, k = 0, \ldots, K - 1 \)

Settings:
1. Equal paper quality, no fathoming
2. LE paper quality, no fathoming
3. GE paper quality, no fathoming
4. LEG paper quality, no fathoming
5. Equal paper quality + fast bound, no fathoming
6. Equal paper quality + slow bound, no fathoming
7. Equal paper quality + preproc bound (bucket size 25), no fathoming
8. LEG paper quality, fast bound fathoming
9. LEG paper quality, slow bound fathoming
10. LEG paper quality, preproc bound fathoming (bucket size 25)

Solve root node. Max running time is 60 seconds.
Instances solved (root node only)

Using bounding functions does not seem to be so efficient, especially the slow function.

It appears that less objective weight on the paper cost makes the instances harder.
Average times (on solved instances only)

The calculation of the bounding functions takes a lot of time. Perhaps for larger instances the preprocessed bounding will pay off.

Effect is not so big on b) instances as on a) instances. Perhaps because only the easy instances were solved?
Dominated

Clearly the bounding dominance functions does not perform as well.

Unclear why Dom 6 (with bound 2) is so bad, mistake? Should be better quality than two other bounding functions.

Dominance is slightly better on larger b)-instances than on a). Reverse holds for small instances.
Not very efficient. Less than 1% fathomed. As expected bound 2 appears to most efficient.
Conclusion

• Modelled another VRP!
• Presented an exact solution approach based on column generation
• Introduced several dominance rules and proved their validity + ways to calculate bounds on the paper cost function
• Experiments showed that
  • constant time calculation in dominance checks is most efficient
  • fathoming not so big a deal in this case
  • the problem (at least the root node) is solveable in a majority of the cases

Future:
• Obtain integer solutions
• Test on real-life instances

Dominance techniques using penalties can be generalized:
• objective function is not non-decreasing (Jepsen et al. (2008))
• or to relax resource relations during dominance (Chabrier (2005))