# The Generalized Arc Routing problem

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## 1 Introduction

The Generalized Traveling Salesman Problem (GTSP) is a well-known extension of the TSP in which the set of nodes is partitioned into clusters and a route is sought that visits at least one node of each cluster. The Generalized Arc Routing Problem (GARP), which extends the Rural Postman Problem, is the arc routing version of the GTSP. Like its node routing counterpart, the GTSP is an NP-hard problem with potential applications in routing and scheduling. To the best of our knowledge the GARP has not been studied in the literature. In this work we present the problem and describe possible variations and particular cases. An integer programming formulation for the general case is presented and several families of valid inequalities are discussed. Preliminary computational results are presented and analyzed.

The GARP is defined on an undirected connected graph $G = (V, E)$ with a distinguished vertex $v_d \in V$, the depot, and a subset of *demand edges*, $D \subset E$. A set of connected subgraphs of $G$, $C_k = (V_k, D_k)$ $(k \in K)$, is given, where $V = \bigcup_{k \in K} V_k$ and $D = \bigcup_{k \in K} D_k$. We assume $v_d \in V_1$. The subgraphs $C_k = (V_k, D_k)$ $(k \in K)$ are referred to as *clusters*. Associated with each edge $(u, v) \in E$ there is a non-negative cost, $c_{uv}$. The GARP is to find a minimum cost tour going through $v_d$ that traverses at least one edge of each $D_k$ $(k \in K)$. That is the GARP is to find a tour $T^*$, passing through $v_d$, with $T^* \cap D_k \neq \emptyset$ $(k \in K)$, of minimum cost

$$\sum_{e \in T^*} t_e c_e,$$

where $t_e$ denotes the number to times edge $e$ is traversed in $T^*$.

**Remarks:**

- In contrast to other arc routing problems, in the GARP it is not assumed that the subsets of demand edges $D_k$ $(k \in K)$ are pairwise disjoint.
- The GARP is an extension of the Rural Postman Problem (RPP). Consider a RPP on $G$ with set of required edges $R \subset E$. For each $e = (u, v) \in R$, define $C_e = (V_e, D_e)$ with $V_e = \{u, v\}$ and $D_e = \{e\}$. The RPP on $G$ with set of required edges $R \subset E$ is equivalent to the GARP on $G$ with set of clusters $C_e$, $(e \in R)$.
- The GARP is NP-hard, as it reduces RPP when $|D_k| = 1$ for all $k \in K$.
- The GARP can be easily extended to consider situations in which the number of traversed edges of each cluster ranges within some prespecified values $l_k$ and $u_k$ $(k \in K)$. In this case, any feasible tour $T$ must satisfy $l_k \leq |T \cap D_k| \leq u_k$ $(k \in K)$.

Like in other single vehicle arc routing problems on undirected graphs, it is easy to prove that, for a given GARP instance, an optimal solution exists in which no edge is traversed more than twice. This allows to formulate the GARP using a mixed integer programm with binary variables to represent the first and second traversal of edges. In particular, for each $e \in E$, let $x_e$ and $y_e$ represent the first and second traversals of edge $e$, respectively. Then, a formulation for the GARP is as follows:

$$\min \sum_{e \in E} c_e(x_e + y_e) \tag{1}$$

$$x(\delta(v_d)) + x(\delta(v_d)) \geq 2 \tag{2}$$

$$x(D_k) \geq 1 \qquad k \in K \tag{3}$$

$$x(\delta(S)) + y(\delta(S)) \geq 2x_e \qquad S \subseteq V \setminus \{v_d\},\ e \in E(S) \tag{4}$$

$$x(\delta(S) \setminus F) + y(F \setminus L) \geq x(F) + y(L) - (|F| + |L|) + 1, \qquad S \subset V,\ F \subseteq \delta(S),\ L \subseteq F,\ (|F| + |L|) \text{ odd} \tag{5}$$

$$y_e \leq x_e, \qquad e \in E \tag{6}$$

$$x_e, y_e \in \{0, 1\},\ (e \in E) \tag{7}$$

Inequality (2) ensures that solutions go through the depot, whereas inequalities (3) guarantee that at least one edge of each cluster is traversed. When the number of traversed edges of each cluster must range within $l_k$ and $u_k$ these constraints become

$$l_k \leq x(D_k) \leq u_k, \qquad k \in K$$

Connectivity with the depot of solutions is implied by constraints (4). Since at least one edge of each cluster must be traversed, the following constraints are also valid:

$$x(\delta(S)) + y(\delta(S)) \geq 2 \qquad S = \bigcup_{k\widehat{K}} V_k, \quad \widehat{K} \subset K \setminus \{1\}.$$

Constraints (5) are an adaptation to the GARP of co-circuit inequalities [3], which ensure even degree with respect to the solution of the visited vertices. Broadly speaking, they impose that if a solution uses an odd number of edges incident to a set of vertices $S$, the solution uses at least one additional edge of the cutset of $S$. Like in [1, 2] we further exploit the precedence relationship of $x$ variables with respect to $y$ variables, which is captured by inequalities (6).

Formulation (1)–(7) involves $2|E|$ variables and a number of constraints of types (4) and (5) which is exponential on $|V|$. Both families of inequalities can be separated in polynomial time.

## References


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[3] F. Barahona and M. Grötschel. 1986. On the cycle polytope of a binary matroid. *J. Comb. Theory* 40 40–62.