XV PORTUGUESE CATEGORY SEMINAR
UNIVERSITY OF AVEIRO, SEPTEMBER 2025

Free monoids and Riguet congruences

Raúl Ruiz Mora
Departament de Matemàtiques
Universitat de València



01

Preliminaries

Many-sorted sets and the free monoid on a set

Many-sorted sets

Given a set A, an **A-sorted set** is an A-indexed family $X = (X_a)_{a \in A}$ where, for every $a \in A$, X_a is a set.

If X and Y are A-sorted sets, an **A-sorted mapping** from X to Y is an A-indexed family $f = (f_a)_{a \in A}$ where, for every $a \in A$, f_a is a mapping from X_a to Y_a.

A-sorted sets and A-sorted mappings form a category Set^A.

An A-sorted set X is **finite** if $\coprod X$ is finite.

Finite A-sorted sets and A-sorted mappings between them form a category $\mathrm{Set}_{\mathrm{f}}^A$.

Free monoid on a set

Given a set A, a **word** on A is a mapping $\mathbf{a}\colon n \to A$ for some $n \in \mathbb{N}$. We will denote by $A^\star$ the set of all words on A. $|\mathbf{a}|$ is the domain of the word $\mathbf{a}$. Moreover, for a letter $a \in A$ we will denote by $|\mathbf{a}|_a$ the number of ocurrences of a in $\mathbf{a}$.

Two words $\mathbf{a}$ and $\mathbf{b}$ on A can be concatenated $\mathbf{a} \curlywedge \mathbf{b}$.

The empty word will be denoted by λ.

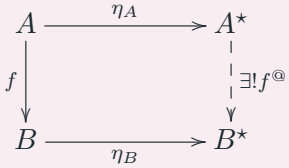
The **free monoid on** A is $(A^\star, \curlywedge, \lambda)$.

EXAMPLE

$\mathbf{a} = baac$ and $\mathbf{b} = caba$ are words on $A = \{a, b, c\}$.

Free monoid on a set

PROPOSITION



Free monoid on a set

EXAMPLE

If we consider the sets $A = \{a, b, c\}$ and $B = \{x, y, z\}$ and the mapping $f\colon A \to B$ defined as follows, $f(a) = x$, $f(b) = y$ and $f(c) = z$, then:

$$\begin{aligned} f^{@}(\mathbf{a}) &= f^{@}(baac) \\ &= f(b)f(a)f(a)f(c) \\ &= yxxz \end{aligned}$$

Free monoid on a set

We will say that two words $\mathbf{a}$ and $\mathbf{b}$ are **congruent**, $\mathbf{a} \equiv^A \mathbf{b}$, if, for every $a \in A$, $|\mathbf{a}|_a = |\mathbf{b}|_a$.

EXAMPLE

The words $\mathbf{a} = baac$ and $\mathbf{b} = caba$ are congruent.

Free monoid on a set

For a word $\mathbf{a}$ and an index $i \in |\mathbf{a}|$, we will denote by $\mathrm{occ}_{\mathbf{a}}(i)$ the number of occurrence of $\mathbf{a}(i)$ in $\mathbf{a}$.

For a word $\mathbf{a}$, a letter a occurring in $\mathbf{a}$ and $j \in |\mathbf{a}|_a$, we will denote by $\mathrm{pos}_{\mathbf{a},a}(j)$ the position of the j-th occurrence of a in $\mathbf{a}$.

EXAMPLE

For the word $\mathbf{a} = baac$, $\mathrm{occ}_{\mathbf{a}}(2) = 1$ and $\mathrm{pos}_{\mathbf{a},c}(0) = 3$.

Free monoid on a set

For every pair of congruent words $\mathbf{a} \equiv^A \mathbf{b}$, the permutation

$$\begin{array}{rccl} \sigma_{\mathbf{a},\mathbf{b}}\colon & |\mathbf{a}| & \to & |\mathbf{b}| \\ & i & \mapsto & \mathsf{pos}_{\mathbf{b},\mathbf{a}(i)}(\mathsf{occ}_{\mathbf{a}}(i)) \end{array}$$

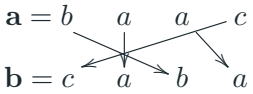
is called the **canonical permutation** for the pair $(\mathbf{a}, \mathbf{b})$

PROPOSITION

The canonical permutation is such that $\mathbf{a} = \mathbf{b} \circ \sigma_{\mathbf{a},\mathbf{b}}$.

Free monoid on a set

EXAMPLE



PROPOSITION

If $\mathbf{a} \equiv^A \mathbf{b}$ and $\mathbf{b} \equiv^A \mathbf{c}$, then

$$(1)\sigma_{\mathbf{a},\mathbf{a}} = \mathrm{id}_{|\mathbf{a}|} \qquad (2)\sigma_{\mathbf{b},\mathbf{c}} \circ \sigma_{\mathbf{a},\mathbf{b}} = \sigma_{\mathbf{a},\mathbf{c}} \qquad (3)\sigma_{\mathbf{a},\mathbf{b}}^{-1} = \sigma_{\mathbf{b},\mathbf{a}}$$

02

The category $\mathbf{C}(\mathbf{A}^\star)$

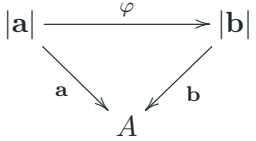
The first category equivalent to $\mathbf{Set}_{\mathrm{f}}^{A}$

The category $\mathsf{C}(\mathbf{A}^\star)$

Let A be a set. The category $\mathsf{C}(\mathbf{A}^\star)$ is defined as follows.

Objects. Words on A.

Morphisms. A morphism from $\mathbf{a}$ to $\mathbf{b}$ is a mapping $\varphi\colon |\mathbf{a}| \to |\mathbf{b}|$ such that the diagram



commutes.

If $|A| \geq 2$, the category $\mathsf{C}(\mathbf{A}^\star)$ is not skeletal.

The functor C from **Set** to **Cat**

$$\mathsf{Set} \xrightarrow{\ \mathrm{C}\ } \mathsf{Cat} \qquad \mathsf{C}(\mathbf{A}^\star) \xrightarrow{\ \mathrm{C}(f)\ } \mathsf{C}(\mathbf{B}^\star)$$

$$\begin{array}{c} A \\ {\scriptstyle f}\downarrow \\ B \end{array} \longmapsto \begin{array}{c} \mathsf{C}(\mathbf{A}^\star) \\ \downarrow{\scriptstyle \mathrm{C}(f)} \\ \mathsf{C}(\mathbf{B}^\star) \end{array} \quad \text{where} \quad \begin{array}{c} \mathbf{a} \\ {\scriptstyle \varphi}\downarrow \\ \mathbf{b} \end{array} \longmapsto \begin{array}{c} f^{@}(\mathbf{a}) \\ \downarrow{\scriptstyle \varphi} \\ f^{@}(\mathbf{b}) \end{array}$$

$\mathsf{C}(\mathbf{A}^\star) \simeq \mathsf{Set}_{\mathrm{f}}^A$

PROPOSITION

The categories $\mathsf{C}(\mathbf{A}^\star)$ and $\mathsf{Set}_{\mathrm{f}}^A$ are equivalent.

The equivalence is

$$\begin{array}{ccc} \mathsf{C}(\mathbf{A}^\star) & \xrightarrow{\downarrow^A(\bullet)} & \mathsf{Set}_{\mathrm{f}}^A \\ \mathbf{a} \\ \varphi \downarrow \\ \mathbf{b} & \longmapsto & \begin{array}{c} (\mathbf{a}^{-1}[\{a\}])_{a\in A} \\ \downarrow (\varphi)_{a\in A} \\ (\mathbf{b}^{-1}[\{a\}])_{a\in A} \end{array} \end{array}$$

03

The category $\mathbf{Q}(\mathbf{A}^\star)$

A Riguet quotient of $\mathbf{C}(\mathbf{A}^\star)$

Riguet congruence

Riguet 1960 [2]

A Riguet congruence on a category C is an ordered pair

$$\Phi = \left(\Phi^{\mathrm{ob}}, \left(\Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)} \right)_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right) \in \Phi^{\mathrm{ob}} \times \Phi^{\mathrm{ob}}} \right)$$

in which

- Φ^{ob} is an equivalence relation on $\mathrm{Ob}(\mathsf{C})$ and,
- for every matrix $\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right) \in \Phi^{\mathrm{ob}} \times \Phi^{\mathrm{ob}}$, in which we agree that $(a, a') \in \Phi^{\mathrm{ob}}$ and $(b, b') \in \Phi^{\mathrm{ob}}$, $\Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)}$ is a subset of $\mathrm{Hom}_{\mathsf{C}}(a, b) \times \mathrm{Hom}_{\mathsf{C}}(a', b')$ such that

Riguet congruence

for every $(a, a'), (a', a''), (b, b'), (b', b''), (c, c') \in \Phi^{\mathrm{ob}}$

1. $\Delta_{\mathrm{Hom_C}(a,b)} \subseteq \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right)}$;
2. $(\Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)})^{-1} = \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a' & b' \\ a & b \end{smallmatrix}\right)}$;
3. $\Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a' & b' \\ a'' & b'' \end{smallmatrix}\right)} \circ \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)} \subseteq \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a'' & b'' \end{smallmatrix}\right)}$;
4. if $(f, f') \in \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)}$ and $(g, g') \in \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} b & c \\ b' & c' \end{smallmatrix}\right)}$ then $(g \circ f, g' \circ f') \in \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & c \\ a' & c' \end{smallmatrix}\right)}$;
5. if $f \colon a \to b$, then there exists an $f' \colon a' \to b'$ such that $(f, f') \in \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)}$.

Riguet quotient

For a Riguet congruence Φ on a category C we define its quotient relative to Φ is defined as follows

Objects. $\mathrm{Ob}(\mathsf{C}/\Phi) = \mathrm{Ob}(\mathsf{C})/\Phi^{\mathrm{ob}}$

Morphisms. $\mathrm{Hom}_{\mathsf{C}/\Phi}([a],[b]) = \{[f] \mid f \in \mathrm{Hom}_{\mathsf{C}}(a,b)\}$ where

$$[f] = \bigcup_{a' \in [a], b' \in [b]} \left\{ f' \colon a' \to b' \mid (f, f') \in \Phi^{\mathrm{fl}}_{\binom{a\ \ b}{a'\ b'}} \right\}$$

PROPOSITION

The projection functor pr_Φ is a functor from C to C/Φ.

Riguet congruence on $\mathsf{C}(\mathbf{A}^\star)$

We consider the following binary relations on objects and morphisms of $\mathsf{C}(\mathbf{A}^\star)$:

Objects. $\mathbf{a} \equiv^A \mathbf{b}$.

Morphisms. If $\mathbf{a} \equiv^A \mathbf{a}'$ and $\mathbf{b} \equiv^A \mathbf{b}'$, then $\varphi\colon \mathbf{a} \to \mathbf{b}$ and $\varphi'\colon \mathbf{a}' \to \mathbf{b}'$ are such that $\varphi \equiv^A_{\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{a}' & \mathbf{b}'\end{smallmatrix}\right)} \varphi'$ if the following diagram commutes

$$\begin{array}{ccc} |\mathbf{a}| & \xrightarrow{\varphi} & |\mathbf{b}| \\ {\scriptstyle \sigma_{\mathbf{a},\mathbf{a}'}}\downarrow & & \downarrow{\scriptstyle \sigma_{\mathbf{b},\mathbf{b}'}} \\ |\mathbf{a}'| & \xrightarrow[\varphi']{} & |\mathbf{b}'| \end{array}$$

The category $\mathsf{Q}(\mathbf{A}^\star)$

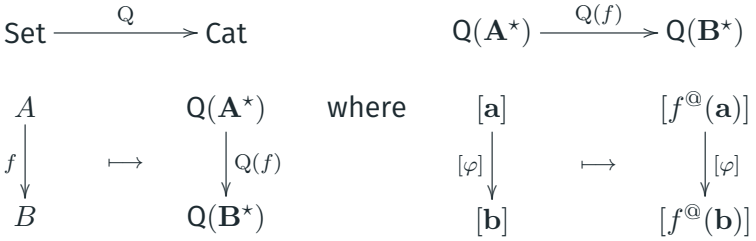
PROPOSITION

$$\left(\equiv^A, \left(\equiv^A_{\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{a}' & \mathbf{b}'\end{smallmatrix}\right)} \right)_{\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{a}' & \mathbf{b}'\end{smallmatrix}\right) \in \equiv^A \times \equiv^A} \right)$$ is a Riguet congruence on $\mathsf{C}(\mathbf{A}^\star)$.

We let $\mathsf{Q}(\mathbf{A}^\star)$ stand for the quotient category $\mathsf{C}(\mathbf{A}^\star)/\equiv^A$.

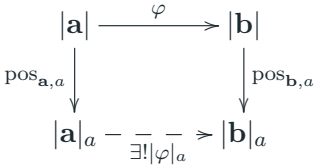
The category $\mathsf{Q}(\mathbf{A}^\star)$ is skeletal.

The functor Q from Set to Cat



The mapping $|\varphi|_a$

For every mapping $\varphi\colon \mathbf{a} \to \mathbf{b}$ in $\mathsf{C}(\mathbf{A}^\star)$ and every a in A,



PROPOSITION

If $\varphi \equiv^A_{\binom{\mathbf{a}\ \mathbf{b}}{\mathbf{a}'\ \mathbf{b}'}} \varphi'$, then, for every a in A, $|\varphi|_a = |\varphi'|_a$.

$\mathsf{Q}(\mathbf{A}^\star) \simeq \mathsf{Set}_{\mathrm{f}}^{A}$

PROPOSITION

The categories $\mathsf{Q}(\mathbf{A}^\star)$ and $\mathsf{Set}_{\mathrm{f}}^{A}$ are equivalent.

The equivalence is

$$\mathsf{Q}(\mathbf{A}^\star) \xrightarrow{(|\bullet|_a)_{a\in A}} \mathsf{Set}_{\mathrm{f}}^{A}$$

$$\begin{array}{ccc} [\mathbf{a}] & & (|\mathbf{a}|_a)_{a\in A} \\ {\scriptstyle [\varphi]}\downarrow & \longmapsto & \downarrow{\scriptstyle (|\varphi|_a)_{a\in A}} \\ [\mathbf{b}] & & (|\mathbf{b}|_a)_{a\in A} \end{array}$$

Conclusions

1. $\mathrm{Set}_{\mathrm{f}}^{A} \simeq \mathsf{C}(\mathbf{A}^{\star}) \simeq \mathsf{Q}(\mathbf{A}^{\star})$.
2. $\mathsf{Q}(\mathbf{A}^{\star})$ plays an analogous role for the category $\mathrm{Set}_{\mathrm{f}}^{A}$ as $\mathrm{Card}_{\mathrm{f}}$ does for $\mathrm{Set}_{\mathrm{f}}$.
3. To the best of our knowledge, it is the first example of a non-trivial Riguet congruence.

Bibliography

[1] Juan Climent Vidal, Enric Cosme Llópez and Raúl Ruiz Mora, **Free monoids and Riguet congruences**, 2025. arXiv:2505.15767

[2] Jacques Riguet, **Catégorisation de la notion de structures et de structures locales chez N. Bourbaki et C. Ehresmann**, I. S´eminaire Dubreil. Algèbre et théorie des nombres, tome 14, n°1 (1960-1961), exp. n°5, pp. 1–32.

XV PORTUGUESE CATEGORY SEMINAR
UNIVERSITY OF AVEIRO, SEPTEMBER 2025

Free monoids and Riguet congruences

Raúl Ruiz Mora
Departament de Matemàtiques
Universitat de València

