Quasicritical elliptic equation involving 1-Laplacian operator with Robin and Neumann Boundary conditions

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Abstract

In this work, we investigate the existence of solutions for quasicritical elliptic equations involving the 1-Laplacian operator with Robin and Neumann boundary conditions. Assuming that the nonlinearity satisfies suitable hypotheses, we establish the existence of solutions using approximation techniques. To overcome the challenges associated with the lack of compactness, we employ the Vitali Convergence Theorem.

Keywords: 1-Laplacian operator, quasicritical elliptic equation, functions of bounded variation, Robin boundary conditions, Neumann boundary conditions.

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1 Introduction

The study of problems involving the 1-Laplacian operator has gained significant attention in recent years due to its wide-ranging applications, particularly in image processing. As the formal limit of the $p-$Laplacian as $p \to 1$, the 1-Laplacian operator introduces analytical challenges due to its degenerate and non-smooth nature. Among the pioneering works on this operator, we highlight the contributions in [2, 8], where the authors characterize the highly singular quotient $\dfrac{Du}{|Du|}$ (when Du is just a Radon measure, rather than an L^1 function). Indeed, they introduce a vector field $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ that effectively serves as $\dfrac{Du}{|Du|}$ through the conditions $\|\mathbf{z}\|_\infty \leq 1$ and $(\mathbf{z}, Du) = |Du|$. Additionally, early contributions in [11] and [7] deserve to be mentioned for their foundational insights.

Our aim is to study two boundary value problems driven by the 1–Laplacian. The first problem exhibits a Robin condition (see (1.2) below), while the second one considers a Neumann condition (see (1.5) below).

To the best of our knowledge, problems involving the 1-Laplacian operator with Robin boundary conditions were first developed in [12] and very recently in [9], [10] and [19]. For Robin problems involving p–Laplacian operator, we refer to [17] .

In [10] the authors establish the existence of a solution to the following problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) &= f \quad \text{in } \Omega, \\ \dfrac{Du}{|Du|}\cdot\nu + \lambda \operatorname{sign} u &= g \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f belongs to the Marcinkiewicz space $L^{N,\infty}(\Omega)$ and g, λ are bounded functions defined on $\partial\Omega$ with $\lambda \geq 0$ and not identically zero.

In this context, in the first part of this paper, we generalize the results from [10], investigating the existence of solutions to the following Robin boundary problem involving the 1-Laplacian operator

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) &= f(x,u) \quad \text{in } \Omega, \\ \dfrac{Du}{|Du|}\cdot\nu + \lambda \operatorname{sign} u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with a $C^{1,1}$–boundary, $N \geq 2$, λ is a bounded and non-negative function defined on $\partial\Omega$ and not identically zero on the boundary of each connected component of Ω. The nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, non-decreasing in $s \in \mathbb{R}$, satisfying the following hypotheses

$$\lim_{s\to\infty} \frac{f(x,s)}{|s|^{1^*-2}s} = 0, \quad \text{uniformly in } x \in \Omega. \tag{1.3a}$$

$$|f(x,s)| \leq C_1|s|^\alpha + C_2|s|^{1^*-1}, \quad \text{for a.e. } x \in \Omega,\ s \in \mathbb{R}, \text{ for some } \alpha > 0 \text{ and } C_1, C_2 > 0. \tag{1.3b}$$

$$0 < \kappa F(x,s) \leq s f(x,s), \quad \text{for a.e. } x \in \Omega,\ |s| \geq s_0, \text{ for certain } \kappa > 1 \text{ and } s_0 > 0, \tag{1.3c}$$

where $F(x,s) = \int_0^s f(x,t)dt$.

The first main result of this work is stated as follows.

Theorem 1.1 *Assume that conditions* (1.3) *hold. Then, problem* (1.2) *admits at least two nontrivial bounded solutions $u, w \in BV(\Omega)$ such that $w \leq 0 \leq u$ for a.e. $x \in \Omega$.*

The proof employs approximation methods from [15], combined with ideas from [10] and [17]. We emphasize that, as a particular case, this theorem holds when f has subcritical growth as in [15]. This, in turn, implies that the result in [15] remains valid when replacing Dirichlet by Robin boundary conditions.

We want to point out that when using approximation methods we need to obtain a bound on the approximate solutions in the energy space $BV(\Omega)$. This fact allows us to obtain pointwise convergence to certain function, which we will check is the solution. Nevertheless, that boundedness does not imply pointwise convergence of traces on the boundary, for which we need strict convergence (see [1]). We point out that, in the 1-Laplacian setting, strict convergence cannot be expected. Hence, to pass to the limit in the approximating problems, a lower semicontinuity result involving the boundary term is needed. We apply [14, Proposition 1.2] that requires a smooth boundary. Recently it has been proved in [18] the precise regularity, which is $C^{1,1}$. This is the reason to assume that Ω have a $C^{1,1}$-boundary.

A Neumann problem for the 1-Laplacian was first considered in [3], where the authors deal with a homogeneous boundary value problem. A few years later, in [13], it was studied the limit as $p \to 1$ of solutions u_p to the following $p-$Laplacian with non-homogeneous Neumann boundary conditions.

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

They have shown that the family of solutions $(u_p)_{p>1}$ converges to a measurable function u, whose properties depend on the size of g with respect to some norm. Specifically, if g is small enough, then u is solution to a limit problem involving the $1-$Laplacian.

Following these articles [3, 13], the second part of this paper is devoted to establishing existence results for the following Neumann problem:

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) + V(x)\dfrac{u}{|u|} &= f(x,u) \quad \text{in } \Omega, \\ \dfrac{Du}{|Du|}\cdot \nu &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and open set, having a Lipschitz–continuous boundary, $N \geq 2$, $V \in L^N(\Omega)$ satisfies $V \geq 0$ and $V \not\equiv 0$ in each connected component of Ω. The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the same hypotheses as in (1.2).

The second main result of this paper is stated in the following theorem.

Theorem 1.2 *Assume that conditions* (1.3) *are satisfied. Then, problem* (1.5) *admits at least two nontrivial bounded solutions $u, w \in BV(\Omega)$ such that $w \leq 0 \leq u$ for a.e. $x \in \Omega$.*

It is important pointing out that, in spite of the similarity between the problems (1.2) and (1.5), the difference in their boundary condition is notable. As it will be presented in Section 2, the boundary conditions of both problems are translated into expressions of different natures. Whereas in the problem with the Robin boundary condition, the boundary condition is expressed through a global expression, in the case with the Neumann boundary condition, the equality holds almost everywhere with respect to the $(N-1)$-dimensional Hausdorff measure.

The work is organized as follows: In Section 2, we introduce the notations and auxiliary results necessary for this study. Section 3 is dedicated to proving Theorem 1.1. Finally, Section 4 is focused on the proof of Theorem 1.2.

2 Preliminaries

This section introduces the essential notation and auxiliary results required for the following sections. For additional details, we refer the reader to [1] and [5].

For a fixed $k > 0$, we define the truncation functions $T_k : \mathbb{R} \to \mathbb{R}$ as follows

$$T_k(s) = \max(-k, \min(s, k)).$$

Throughout this article, the sign function is understood to be multivalued. It is given by

$$\operatorname{sign}(s) = \begin{cases} 1 & \text{if } s > 0 \\ [-1,1] & \text{if } s = 0 \\ -1 & \text{if } s < 0. \end{cases}$$

Moreover, we assume $N \geq 2$. We use $\mathcal{H}^{N-1}(E)$ to denote the $(N-1)$-dimensional Hausdorff measure of a set $E \subset \mathbb{R}^N$. Additionally, for any set $A \subset \mathbb{R}^N$, $|A|$ represents its Lebesgue measure. Since our domain is smooth enough (its boundary is either $C^{1,1}$ or Lipschitz), we have an outward unit normal vector on its boundary which will be denoted by ν, it is defined at least $\mathcal{H}^{N-1}-$a.e.

We employ several functional spaces in this work. The space $L^q(\Omega)$ refers to the Lebesgue space of $q-$summable functions on Ω, while $L^q(\partial\Omega, \lambda)$ represents the weighted Lebesgue space on the boundary $\partial\Omega$, where the weight λ is specified. Furthermore, $W^{1,p}(\Omega)$ denotes the usual Sobolev space of functions whose weak derivatives belong to $L^p(\Omega)^N$.

We say that $u \in BV(\Omega)$, or that u is a function of bounded variation, if $u \in L^1(\Omega)$, and its distributional gradient Du is a vectorial Radon measure with finite total variation. It can be proved that $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and

$$\int_\Omega |Du| := \sup\left\{\int_\Omega u \operatorname{div} \phi dx;\ \phi \in C_c^1(\Omega, \mathbb{R}^N),\ \|\phi\|_\infty \leq 1\right\} < +\infty.$$

The space $BV(\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{BV} := \int_\Omega |Du| + \int_\Omega |u| dx.$$

In [1], a trace operator $BV(\Omega) \hookrightarrow L^1(\partial\Omega)$, is introduced, allowing the definition of the norm

$$\|u\| := \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1},$$

which is equivalent to $\|\cdot\|_{BV}$.

For a nonnegative function $\lambda \in L^\infty(\partial\Omega)$ that is not identically zero on the boundary of each connected component, we define the norm in $W^{1,1}(\Omega)$ as follows

$$\|w\|_\lambda := \int_\Omega |Dw| + \int_{\partial\Omega} \lambda(x)|w| d\mathcal{H}^{N-1}.$$

This norm is equivalent to the standard norm in $W^{1,1}(\Omega)$ (see [16, Theorem 7.1]). Furthermore, it also defines a norm in $BV(\Omega)$ that is equivalent to the usual one. In this work, we typically use this equivalent norm.

The space $BV(\Omega)$ is continuously embedded into $L^r(\Omega)$ for all $r \in [1, 1^*]$ and is also compactly embedded for all $r \in [1, 1^*)$, where $1^* = N/(N-1)$.

Next, we introduce the *pairing theory*, originally developed by Anzellotti in [4] and later extended by Chen and Frid in [6].

We denote

$$X_N(\Omega) = \left\{\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N) :\ \operatorname{div}(\mathbf{z}) \in L^N(\Omega)\right\}.$$

For $\mathbf{z} \in X_N(\Omega)$ and $w \in BV(\Omega)$, Anzellotti introduced the pairing $(\mathbf{z}, Dw) \in \mathcal{D}'(\Omega)$, which is the distribution defined as

$$\langle(\mathbf{z}, Dw), \varphi\rangle := -\int_\Omega w\varphi \operatorname{div}(\mathbf{z}) dx - \int_\Omega w\mathbf{z} \cdot \nabla\varphi dx,$$

for every $\varphi \in \mathcal{D}(\Omega)$. It can be further shown that $(\mathbf{z}, Dw)$ is, in fact, a Radon measure, satisfying

$$\left|\int_B (\mathbf{z}, Dw)\right| \leq \|\mathbf{z}\|_\infty \int_B |Dw|, \tag{2.1}$$

for every Borel set $B \subset \Omega$. To establish a version of Green's formula, it is also essential to develop a weak trace theory for the outward component of $\mathbf{z}$. There exists a trace operator $[\cdot, \nu] : X_N(\Omega) \to L^\infty(\partial\Omega)$, satisfying the following fact:

$$\| [\mathbf{z}, \nu] \|_{L^\infty(\partial\Omega)} \leq \|\mathbf{z}\|_\infty. \tag{2.2}$$

If $\mathbf{z} \in C^1(\overline{\Omega}_\delta, \mathbb{R}^N)$, the weak trace $[\mathbf{z}, \nu]$ coincides with the classical scalar product of $\mathbf{z}$ and the outward unit normal vector ν

$$[\mathbf{z}, \nu] = \mathbf{z} \cdot \nu,$$

where Ω_δ represents a δ-neighborhood of $\partial\Omega$. Using these definitions, Anzellotti proved that the following Green's formula holds for all $\mathbf{z} \in X_N(\Omega)$ and $w \in BV(\Omega)$:

$$\int_\Omega w \operatorname{div}(\mathbf{z}) dx + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu]\, w \, d\mathcal{H}^{N-1}. \tag{2.3}$$

Now, we present the following proposition (see [10, Proposition A.1] for details), which generalizes the usual Hölder inequality.

Proposition 2.1 *Let $1 < p, p' < \infty$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Assume that $f_1, f_2 : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ are measurable functions satisfying $f_1 \in L^p(\Omega)$, $f_2 \in L^{p'}(\Omega)$, $g_1 \in L^p(\partial\Omega, \lambda)$ and $g_2 \in L^{p'}(\partial\Omega, \lambda)$. Then $f_1 f_2 \in L^1(\Omega)$, $g_1 g_2 \in L^1(\partial\Omega, \lambda)$ and the following inequality holds*

$$\begin{aligned}\int_\Omega |f_1 f_2| dx + \int_{\partial\Omega} \lambda(x) |g_1 g_2| d\mathcal{H}^{N-1} \\ \leq \left[\int_\Omega |f_1|^p dx + \int_{\partial\Omega} \lambda(x) |g_1|^p d\mathcal{H}^{N-1} \right]^{\frac{1}{p}} \left[\int_\Omega |f_2|^{p'} dx + \int_{\partial\Omega} \lambda(x) |g_2|^{p'} d\mathcal{H}^{N-1} \right]^{\frac{1}{p'}}.\end{aligned}$$

We conclude this section by defining the respective notions of a solution to our problems. First we address (1.2).

Definition 2.1 *We say that a function $u \in BV(\Omega)$ is a solution to (1.2) if there exists $\boldsymbol{z} \in L^\infty(\Omega, \mathbb{R})$, with $\|\boldsymbol{z}\|_\infty \leq 1$ such that the following conditions hold*

$$-div\, \boldsymbol{z} = f(x, u) \ \text{in } \mathcal{D}'(\Omega), \tag{2.4}$$

$$(\boldsymbol{z}, Du) = |Du| \ \text{as measures in } \Omega, \tag{2.5}$$

$$[\boldsymbol{z}, \nu] + \lambda\beta = 0 \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega, \tag{2.6}$$

where β is a measurable function satisfying $\|\beta\chi_{\{\lambda>0\}}\|_\infty \leq 1$ and

$$\lambda\beta \in T_1(\lambda sign(u)) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega. \tag{2.7}$$

Now, for the problem (1.5), we have the following definition of a solution.

Definition 2.2 *We say that a function $u \in BV(\Omega)$ is a solution to (1.5) if there exist $\boldsymbol{z} \in L^\infty(\Omega, \mathbb{R}^N)$ and a measurable function β satisfying $\beta\chi_{\{V>0\}} \in L^\infty(\Omega)$, with $\|\boldsymbol{z}\|_\infty \leq 1$, $\|\beta\chi_{\{V>0\}}\|_\infty \leq 1$, such that the following conditions are satisfied*

$$-div\, \boldsymbol{z} + V\beta = f(x, u) \ \text{in } \mathcal{D}'(\Omega), \tag{2.8}$$

$$(\boldsymbol{z}, Du) = |Du| \ \text{as measures in } \Omega, \tag{2.9}$$

$$[\boldsymbol{z}, \nu] = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{2.10}$$

$$V\beta \in V sign(u) \ \text{a.e. in } \Omega. \tag{2.11}$$

It is worth spending some words to explain why Definitions 2.1 and 2.2 are suitable versions of (1.2) and (1.5), respectively. Requirement $\|\mathbf{z}\|_\infty \leq 1$ jointly with conditions (2.5) and (2.9) imply that the vector field $\mathbf{z}$ plays the role of $Du/|Du|$ in the original equations. On the other hand, (2.6) and (2.10) represent the boundary conditions in (1.2) and (1.5), respectively. The expression (2.7), in turn, implies that the function $\lambda\beta$ plays the role of $\lambda \operatorname{sign}(u)$, while (2.11) implies that $V\beta$ plays the role of $V \operatorname{sign}(u)$. We stress that λ may vanish on a subset of the boundary and so in that zone one cannot deduce how β is. Thus the boundedness of β is just guaranteed in the complementary zone $\{\lambda > 0\}$ and for that reason appear $\lambda\beta$ in (2.6) and (2.7). Analogously, the coefficient V can vanish, so that (as in the previous definition) we may only ensure that $\beta\chi_{\{V>0\}}$ is bounded and we must write $V\beta$ in Definition 2.2.

Observe that, owing to $\|\mathbf{z}\|_\infty \leq 1$, we get $[\mathbf{z}, \nu] \in L^\infty(\partial\Omega)$ and $\|[\mathbf{z}, \nu]\|_\infty \leq 1$. Hence, the Robin condition $[\mathbf{z}, \nu] + \lambda \operatorname{sign} u = 0$ only has sense when $\|\lambda\|_\infty \leq 1$. Nevertheless, a solution can also be found if the coefficient satisfies $\|\lambda\|_\infty > 1$. To give the right meaning to the Robin problem in this case, the condition $[\mathbf{z}, \nu] + \lambda \operatorname{sign} u = 0$ must be replaced with $[\mathbf{z}, \nu] + \lambda\beta = 0$ and $\lambda\beta \in T_1(\lambda \operatorname{sign}(u))$. This change of the Robin coefficient to its truncation comes from approximating the problem driven by the 1-Laplacian through problems governed by the p-Laplacian. Indeed, it is proven in [9, Theorem 4.1] that the Γ-convergence of Robin problems for the p-Laplacian leads to the truncate Robin problem for the 1-Laplace operator (see also [9, Proposition 5.2] and [19, Theorems 8 and 15]). This fact is the reason why, for the 1-Laplacian, the Robin problem having a coefficient bigger than 1 coincides with the Dirichlet problem.

All along this paper, it should be noted that in (1.3b), we can assume without loss of generality that $\alpha > 0$ is such that $1 + \alpha < 1^*$. Note that this is not a restriction, since if (1.3b) holds for some $\alpha > 0$, then it would still hold for any $0 < \alpha' < \alpha$.

3 Proof of Theorem 1.1

We will demonstrate the existence of a nontrivial solution $u \geq 0$ for (1.2). By a similar argument, the existence of a nontrivial solution $w \leq 0$ can also be established. As described in the introduction, this result is obtained through an approximation method.

Let $\tilde{p} = \min\{1 + \alpha, \kappa\}$. For each $1 < p < \tilde{p}$, we consider the following $p-$Laplacian problem

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega \\ |\nabla u|^{p-2}\nabla u \cdot \nu + \lambda |u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Based on our assumptions and the choice of p the following statements hold:

(a) $\displaystyle\lim_{s\to\infty} \frac{f(x,s)}{|s|^{p^*-2}s} = 0$, uniformly for all $x \in \Omega$.

(b) $\displaystyle\limsup_{s\to 0} \frac{f(x,s)}{|s|^{p-2}\, s} = 0$, uniformly for $x \in \Omega$.

(c) $0 < \kappa F(x,s) \leqslant s f(x,s)$, for $x \in \Omega$ and $s \in \mathbb{R}$, with $|s| \geqslant s_0$ and $\kappa > p$.

Definition 3.1 *We say that $u \in W^{1,p}(\Omega)$ is a weak solution to* (3.1) *if there holds*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx + \int_{\partial\Omega} \lambda |u|^{p-2} u v d\mathcal{H}^{N-1} = \int_\Omega f(x,u) v dx,$$

for every $v \in W^{1,p}(\Omega)$.

In order to obtain a positive solution of (3.1), we consider the associated energy functional $J_p^+ : W^{1,p}(\Omega) \to \mathbb{R}$, defined by

$$J_p^+(u) = \frac{1}{p}\|u\|_{\lambda,p}^p - \int_\Omega F_+(x,u)dx, \tag{3.2}$$

where the norm $\|u\|_{\lambda,p} = \left(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} \lambda |u|^p d\mathcal{H}^{N-1}\right)^{\frac{1}{p}}$ is equivalent to the usual norm in $W^{1,p}(\Omega)$ (see [16, Theorem 7.1]). The function $F_+(x,u)$ is defined as

$$F_+(x,s) = \int_0^s f_+(x,t)dt,$$

where $f_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$f_+(x,s) = \begin{cases} 0 & \text{if } s \leq 0, \\ f(x,s) & \text{if } s > 0. \end{cases} \tag{3.3}$$

Notice that $F_+(x,s) \geq 0$ for all $s \in \mathbb{R}$ and for a.e. $x \in \Omega$. Moreover, since assumption (1.3b) implies $f(x,0) = 0$, then $f_+(x,\cdot)$ is continuous.

It is worth pointing out that, by following an analogous reasoning, we could find negative solutions by dealing with the functional

$$J_p^-(u) = \frac{1}{p}\|u\|_{\lambda,p}^p - \int_\Omega F_-(x,u)dx,$$

where $F_-(x,u) = \displaystyle\int_0^s f_-(x,t)dt$, and

$$f_-(x,s) = \begin{cases} f(x,s) & \text{if } s \leq 0, \\ 0 & \text{if } s > 0. \end{cases}$$

For $u \in W^{1,p}(\Omega)$, we define the functional

$$I_p(u) = J_p^+(u) + \frac{p-1}{p}\left(|\Omega| + \int_{\partial\Omega} \lambda(x) d\mathcal{H}^{N-1}\right).$$

Lemma 3.1 *The family $(I_p(u))_{1<p}$ is nondecreasing with respect to p, for all u such that $I_p(u)$ is well defined.*

Proof. Let $1 < p_1 < p_2 < N$ and consider $u \in W^{1,p_2}(\Omega)$. By Young's inequality, we have

$$\int_\Omega |\nabla u|^{p_1} dx \leq \frac{p_1}{p_2}\int_\Omega |\nabla u|^{p_2} dx + \frac{p_2 - p_1}{p_2}|\Omega| \tag{3.4}$$

and

$$\int_{\partial\Omega} \lambda |u|^{p_1} d\mathcal{H}^{N-1} \leq \frac{p_1}{p_2}\int_{\partial\Omega} \lambda |u|^{p_2} d\mathcal{H}^{N-1} + \frac{p_2 - p_1}{p_2}\int_{\partial\Omega} \lambda d\mathcal{H}^{N-1}. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} I_{p_1}(u) &= \frac{1}{p_1}\int_\Omega |\nabla u|^{p_1} dx + \frac{1}{p_1}\int_{\partial\Omega} \lambda |u|^{p_1} d\mathcal{H}^{N-1} - \int_\Omega F_+(x,u)dx + \frac{p_1 - 1}{p_1}\left(|\Omega| + \int_{\partial\Omega} \lambda(x) d\mathcal{H}^{N-1}\right) \\ &\leq \frac{1}{p_2}\int_\Omega |\nabla u|^{p_2} dx + \frac{p_2 - p_1}{p_1 p_2}|\Omega| + \frac{1}{p_2}\int_{\partial\Omega} \lambda |u|^{p_2} d\mathcal{H}^{N-1} + \frac{p_2 - p_1}{p_1 p_2}\int_{\partial\Omega} \lambda d\mathcal{H}^{N-1} \\ &\quad - \int_\Omega F_+(x,u)dx + \frac{p_1 - 1}{p_1}\left(|\Omega| + \int_{\partial\Omega} \lambda(x) d\mathcal{H}^{N-1}\right) \\ &= I_{p_2}(u). \end{aligned}$$

$\square$

Now, let $\phi \in \mathcal{D}(\Omega)$ be fixed, $\phi \not\equiv 0$. Note that from (1.3b) and (1.3c), there exists $C_3, C_4 > 0$, such that $F(x,s) \geq C_3 s^\kappa - C_4$, for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Then, for $t > 0$,

$$\begin{aligned} I_{\tilde{p}}(t\phi) &\leq \frac{t^{\tilde{p}}}{\tilde{p}} \|\phi\|_{\lambda,\tilde{p}}^{\tilde{p}} - \int_\Omega (C_3 t^\kappa |\phi|^\kappa - C_4) dx + \frac{\tilde{p}-1}{\tilde{p}} \left(|\Omega| + \int_{\partial\Omega} \lambda(x) dx \right) \\ &\leq \frac{t^{\tilde{p}}}{\tilde{p}} \|\phi\|_{\lambda,\tilde{p}}^{\tilde{p}} - C_3 t^\kappa \int_\Omega |\phi|^\kappa dx + C_4 |\Omega| + \frac{\tilde{p}-1}{\tilde{p}} \left(|\Omega| + \int_{\partial\Omega} \lambda(x) dx \right). \end{aligned}$$

Then, as $t \to \infty$, we find that

$$I_{\tilde{p}}(t\phi) \to -\infty.$$

This implies the existence of some

$$e = T\phi \tag{3.6}$$

such that $I_{\tilde{p}}(e) < 0$. Consequently, by Lemma 3.1, we conclude that

$$I_p(e) < 0, \quad \text{for all } p \in (1, \tilde{p}).$$

Furthermore, standard arguments imply that for $1 < p < \tilde{p}$, I_p satisfies the first geometrical assumption of the Mountain Pass Theorem. Hence, for each $1 < p < \tilde{p}$, the energy functional I_p satisfies the geometric conditions of the Mountain Pass Theorem and the Palais-Smale condition (since f has a subcritical growth with respect to $W^{1,p}(\Omega)$). Therefore, by Mountain Pass Theorem, for each $1 < p < \tilde{p}$, there exists a function u_p solution of (3.1) such that

$$I_p(u_p) = \inf_{\gamma \in \Gamma_p} \max_{t \in [0,1]} I_p(\gamma(t)) =: c_p, \tag{3.7}$$

where

$$\Gamma_p = \left\{ \gamma \in C\left([0,1], W^{1,p}(\Omega)\right) : \gamma(0) = 0, \gamma(1) = e \right\}.$$

Moreover, by taking u_p^- as a test function in the weak form of (3.1), one can see that $u_p^- = 0$, in such a way that u_p is a nonnegative solution.

Lemma 3.2 *The family $(I_p(u_p))_{1<p<\tilde{p}}$ is nondecreasing with respect to p.*

Proof. Let $1 < p_1 < p_2 < \tilde{p}$. Notice that, $\Gamma_{p_2} \subset \Gamma_{p_1}$. From Lemma 3.1, it follows that

$$\begin{aligned} I_{p_1}(u_{p_1}) &= \inf_{\gamma \in \Gamma_{p_1}} \max_{t \in [0,1]} I_{p_1}(\gamma(t)) \\ &\leq \inf_{\gamma \in \Gamma_{p_2}} \max_{t \in [0,1]} I_{p_1}(\gamma(t)) \\ &\leq \inf_{\gamma \in \Gamma_{p_2}} \max_{t \in [0,1]} I_{p_2}(\gamma(t)) \\ &= I_{p_2}(u_{p_2}). \end{aligned}$$

$\square$

From now on, we fix $p_0 \in (1, \tilde{p})$.

Lemma 3.3 *The family $(u_p)_{1<p<p_0}$ is bounded in $BV(\Omega)$. As a consequence of the continuous embedding, the family $(u_p)_{1<p<p_0}$ is bounded in $L^{1^*}(\Omega)$.*

Proof. From Lemma 3.2 for a fixed $p_0 \in (1, \tilde{p})$, it follows that

$$I_p(u_p) \le I_{p_0}(u_{p_0}) = c_{p_0}$$

for all $p \in (1, p_0)$.

Define $\Omega_p = \{x \in \Omega : u_p(x) \le s_0\}$, for any $p \in (1, p_0)$.

By condition (1.3b), we deduce

$$\int_{\Omega_p} F(x, u_p) \le C\left(s_0^{\alpha+1} + s_0^{1^*}\right) =: \overline{C}.$$

On the other hand, using condition (1.3c) and the fact that u_p solves (3.1), we obtain

$$\int_{\Omega\setminus\Omega_p} F(x, u_p) \le \frac{1}{\kappa}\int_\Omega u_p f(x, u_p)dx = \frac{1}{\kappa}\left[\int_\Omega |\nabla u_p|^p dx + \int_{\partial\Omega} \lambda |u_p|^p d\mathcal{H}^{N-1}\right].$$

From these three inequalities, it follows that

$$\begin{aligned}
c_{p_0} &\ge \frac{1}{p}\left[\int_\Omega |\nabla u_p|^p dx + \int_{\partial\Omega} \lambda|u_p|^p d\mathcal{H}^{N-1}\right] - \left(\int_{\Omega\setminus\Omega_p} F(x, u_p) + \int_{\Omega_p} F(x, u_p)\right) \\
&\quad + \frac{p-1}{p}\left(|\Omega| + \int_{\partial\Omega} \lambda(x) d\mathcal{H}^{N-1}\right) \\
&\ge \left(\frac{1}{p} - \frac{1}{\kappa}\right)\left[\int_\Omega |\nabla u_p|^p dx + \int_{\partial\Omega} \lambda|u_p|^p d\mathcal{H}^{N-1}\right] - C_0 \\
&\ge \left(\frac{1}{p_0} - \frac{1}{\kappa}\right)\left[\int_\Omega |\nabla u_p|^p dx + \int_{\partial\Omega} \lambda|u_p|^p d\mathcal{H}^{N-1}\right] - C_0.
\end{aligned}$$

Thus, there exists a constant $\tilde{C}$, independent of p, such that

$$\int_\Omega |\nabla u_p|^p dx + \int_{\partial\Omega} \lambda|u_p|^p d\mathcal{H}^{N-1} \le \tilde{C}, \quad \forall\, p \in (1, p_0). \tag{3.8}$$

Using Young's inequality, we now estimate

$$\begin{aligned}
\|u_p\|_\lambda &= \int_\Omega |\nabla u_p| dx + \int_{\partial\Omega} \lambda|u_p| d\mathcal{H}^{N-1} \\
&\le \frac{1}{p}\int_\Omega |\nabla u_p|^p dx + \frac{1}{p}\int_{\partial\Omega} \lambda|u_p|^p d\mathcal{H}^{N-1} + \left(\frac{p-1}{p}\right)|\Omega| + \left(\frac{p-1}{p}\right)\int_{\partial\Omega} \lambda d\mathcal{H}^{N-1} \\
&\le \tilde{C}_1.
\end{aligned}$$

This implies that $(u_p)_{1<p<p_0}$ is bounded in $BV(\Omega)$. ☐

As an immediate consequence of the previous lemma and the compact embeddings, there exists a nonnegative $u \in BV(\Omega)$ such that, up to subsequences,

(A) $u_p \to u$ in $L^m(\Omega)$, for $1 \le m < \frac{N}{N-1}$, as $p \to 1^+$.

(B) $u_p \to u$ a.e. in Ω, as $p \to 1^+$.

Lemma 3.4 *Let u_p be a solution to problem* (3.1)*. Then, there exist a vector field $\boldsymbol{z} \in L^\infty(\Omega, \mathbb{R}^N)$ and a function $\beta \in L^s(\partial\Omega, \lambda)$ for every $1 \le s < \infty$, such that $\beta\chi_{\{\lambda>0\}} \in L^\infty(\partial\Omega)$ and satisfying, up to subsequences, the following convergences*

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \boldsymbol{z}, \quad \textit{in } L^s(\Omega, \mathbb{R}^N) \textit{ for all } 1 \le s < \infty \tag{3.9}$$

and

$$|u_p|^{p-2}u_p \rightharpoonup \beta, \quad \textit{in } L^s(\partial\Omega,\lambda) \textit{ for all } 1 \le s < \infty. \tag{3.10}$$

Moreover, the following identities hold

$$\|\boldsymbol{z}\|_\infty, \|\beta\chi_{\{\lambda>0\}}\|_\infty \le 1, \tag{3.11}$$

$$-div\, \boldsymbol{z} = f(x,u) \quad \textit{in } \mathcal{D}'(\Omega), \tag{3.12}$$

$$[z,\nu] + \lambda\beta = 0 \quad \mathcal{H}^{N-1} - \textit{a.e. on } \partial\Omega. \tag{3.13}$$

Proof. Let us fix $1 < s < \infty$ such that $\frac{s}{s-1} < p_0$ and consider $1 < p < \dfrac{s}{s-1}$. By applying the Proposition 2.1 and (3.8), we obtain

$$\begin{aligned}
&\left[\int_\Omega |\nabla u_p|^{s(p-1)}\, dx + \int_{\partial\Omega} \lambda |u_p|^{s(p-1)} d\mathcal{H}^{N-1}\right]^{\frac{1}{s}} \\
\le\ & \left[\int_\Omega |\nabla u_p|^p dx + \int_{\partial\Omega} \lambda |u_p|^p d\mathcal{H}^{N-1}\right]^{\frac{p-1}{p}} \left[|\Omega| + \int_{\partial\Omega} \lambda d\mathcal{H}^{N-1}\right]^{\frac{1}{s}-\frac{p-1}{p}} \\
\le\ & \tilde{C}^{\frac{p-1}{p}} \left[|\Omega| + \int_{\partial\Omega} \lambda d\mathcal{H}^{N-1}\right]^{\frac{1}{s}-\frac{p-1}{p}}.
\end{aligned} \tag{3.14}$$

This shows that, for any fixed $s > 1$, the families $\left(|\nabla u_p|^{p-2}\nabla u_p\right)_{1<p<p_0}$ and $\left(|u_p|^{p-2}u_p\right)_{1<p<p_0}$ are bounded in $L^s(\Omega,\mathbb{R}^N)$ and $L^s(\partial\Omega,\lambda)$ respectively. Consequently, up to subsequences, there exist $\mathbf{z}_s \in L^s(\Omega,\mathbb{R}^N)$ and $\beta_s \in L^s(\partial\Omega,\lambda)$ satisfying

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z}_s\,, \quad \text{in } L^s(\Omega,\mathbb{R}^N) \quad \text{as } p \to 1^+\,.$$

and

$$|u_p|^{p-2}u_p \rightharpoonup \beta_s\,, \quad \text{in } L^s(\partial\Omega,\lambda) \quad \text{as } p \to 1^+\,.$$

Since these results hold for all $s > 1$, two diagonal procedures allow us to find $\mathbf{z} \in L^s(\Omega,\mathbb{R}^N)$ and $\beta \in L^s(\partial\Omega,\lambda)$ for all $s \in (1,\infty)$, satisfying (3.9) and (3.10).

On the other hand, taking into account the lower semicontinuity of the norm in $L^s(\Omega,\mathbb{R}^N)$ with respect to the weak convergence, and by passing to the limit as $p \to 1^+$, in (3.14), we have

$$\left[\int_\Omega |\mathbf{z}|^s dx + \int_{\partial\Omega} \lambda|\beta|^s d\mathcal{H}^{N-1}\right]^{\frac{1}{s}} \le \left[|\Omega| + \int_{\partial\Omega} \lambda d\mathcal{H}^{N-1}\right]^{\frac{1}{s}}$$

Letting $s \to \infty$, by applying [10, Proposition A2], we conclude that $\mathbf{z} \in L^\infty(\Omega,\mathbb{R}^N)$, $\beta\chi_{\{\lambda>0\}} \in L^\infty(\partial\Omega)$ and $\|\mathbf{z}\|_\infty, \|\beta\chi_{\{\lambda>0\}}\|_\infty \le 1$.

To establish (3.12), take $\varphi \in \mathcal{D}(\Omega)$ as a test function in (3.1). This yields

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi dx = \int_\Omega f(x,u_p)\varphi dx.$$

Recalling (1.3a) and (1.3b), we may find two constant $C_1, C_2 > 0$ such that

$$|f(x,u_p)| \le C_1|u_p|^\alpha + C_2|u_p|^{1^*-1}\,.$$

It follows from (A) that $f(x,u_p)$ is equi–integrable. Now by (B) and Vitali's Theorem, we have

$$\int_\Omega f(x,u_p)\varphi dx \to \int_\Omega f(x,u)\varphi dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Moreover, from (3.9), it follows that

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi dx \to \int_\Omega \mathbf{z}\cdot\nabla\varphi dx \quad \text{for all } \varphi\in\mathcal{D}(\Omega).$$

Combining these convergences, we conclude

$$-\operatorname{div}\mathbf{z} = f(x,u) \quad \text{in } \mathcal{D}'(\Omega). \tag{3.15}$$

Observe that condition (1.3a) and the embedding $BV(\Omega)\hookrightarrow L^{1^*}(\Omega)$ yields $f(x,u)\in L^N(\Omega)$. So, we get $\operatorname{div}\mathbf{z}\in L^N(\Omega)$ and consequently Anzellotti's theory can be applied.

To derive (3.13), for $1<p<p_0$, consider $v\in W^{1,p_0}(\Omega)$ as a test function in (3.1). This leads to

$$\int_\Omega f(x,u_p)v dx = \int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla v dx + \int_{\partial\Omega}\lambda|u_p|^{p-2}u_p v d\mathcal{H}^{N-1}. \tag{3.16}$$

To pass to the limit in the above identity, we apply (1.3a) and Young's inequality (recall we are assuming $1+\alpha<1^*$) to get

$$|f(x,u_p)v| \le C_1|u_p|^\alpha|v| + C_2|u_p|^{\frac{1}{N-1}}|v| \le C|u_p|^{\frac{1}{N-1}}|v| + C|v| \le \overline{C}|u_p|^{\frac{Np_0}{(N-1)(Np_0-N+p_0)}} + \overline{C}|v|^{\frac{Np_0}{N-p_0}} + C|v|\,.$$

Since $\frac{Np_0}{(N-1)(Np_0-N+p_0)} < \frac{N}{N-1}$, it follows that the right-hand side converges in $L^1(\Omega)$, so it is equi-integrable. Consequently, the left-hand side is also equi-integrable. The pointwise convergence and Vitali's Theorem imply that $f(x,u_p)v\to f(x,u)v$ in $L^1(\Omega)$. Observe that then $f(x,u)v\in L^1(\Omega)$. So letting $p\to1^+$ in (3.16), we obtain

$$\int_\Omega f(x,u)v dx = \int_\Omega \mathbf{z}\cdot\nabla v dx + \int_{\partial\Omega}\lambda\beta v d\mathcal{H}^{N-1}$$

for every $v\in W^{1,p_0}(\Omega)$. By density, this equality extends to all $v\in W^{1,1}(\Omega)$.

Using (3.15), we rewrite the left-hand side as

$$\int_\Omega v\operatorname{div}\mathbf{z} dx + \int_\Omega \mathbf{z}\cdot\nabla v dx + \int_{\partial\Omega}\lambda\beta v d\mathcal{H}^{N-1} = 0.$$

Applying the Green formula (2.3), this becomes

$$\int_{\partial\Omega}[\mathbf{z},\nu]v d\mathcal{H}^{N-1} + \int_{\partial\Omega}\lambda\beta v d\mathcal{H}^{N-1} = 0,$$

for all $v\in W^{1,1}(\Omega)$. Consequently,

$$0 = \int_{\partial\Omega}[\mathbf{z},\nu]h d\mathcal{H}^{N-1} + \int_{\partial\Omega}\lambda\beta h d\mathcal{H}^{N-1},$$

for every $h\in L^1(\partial\Omega)$. Hence, we conclude that

$$[\mathbf{z},\nu]+\lambda\beta = 0 \quad \mathcal{H}^{N-1}-\text{a.e. on } \partial\Omega.$$

□

Since the nonlinearity does not have subcritical growth, we have an additional difficulty due to the lack of compactness. To overcome this difficulty, we employ once more Vitali's Theorem.

Lemma 3.5 *Let u_p be a solution to problem* (3.1)*. Then, as p goes to 1,*

$$f(x,u_p)u_p \to f(x,u)u \qquad \textit{strongly in } L^1(\Omega).$$

As a consequence,

$$\int_\Omega f(x,u_p)u_p dx \to \int_\Omega f(x,u)u dx \tag{3.17}$$

and, for every $\varphi \in C_c^1(\Omega)$,

$$\int_\Omega f(x,u_p)u_p\varphi dx \to \int_\Omega f(x,u)u\varphi dx. \tag{3.18}$$

Proof. On account of Lemma 3.3, define $c_* = \sup\limits_{1<p<p_0} \displaystyle\int_\Omega |u_p|^{1^*} dx < +\infty$. By hypotheses (1.3a) and (1.3b) given $\epsilon > 0$, there exists c_ϵ such that

$$|f(x,s)| \le \frac{\epsilon}{2c_*}|s|^{1^*-1} + c_\epsilon \quad \text{ for almost all } x \in \Omega \text{ and all } s \in \mathbb{R}. \tag{3.19}$$

Using this inequality, for any measurable set $E \subseteq \Omega$ such that $|E| \leqslant \left(\frac{\epsilon}{2c_\epsilon c_*^{1/1^*}}\right)^N$, we obtain

$$\int_E |f(x,u_p)u_p|dx \le \frac{\epsilon}{2c_*}\int_E |u_p|^{1^*}dx + c_\epsilon \int_E |u_p|. \tag{3.20}$$

By Hölder's inequality, we get

$$c_\epsilon \int_E |u_p|dx \le c_\epsilon |E|^{\frac{1}{N}} \left(\int_\Omega |u_p|^{1^*}dx\right)^{\frac{N-1}{N}} \le \frac{\epsilon}{2}.$$

Substituting this inequality into (3.20), we have

$$\int_E |f(x,u_p)u_p|dx \le \epsilon \quad \text{ for all } 1 < p < p_0.$$

This shows that the family $(f(\cdot,u_p(\cdot))u_p(\cdot))_{1<p<p_0} \subset L^1(\Omega)$ is equi-integrable. Since

$$f(x,u_p)u_p \to f(x,u)u, \quad \text{ a.e. in } \Omega,$$

the result follows by applying Vitali's Theorem. □

Lemma 3.6 *Let $\mathbf{z}$ be the vector field established in Lemma 3.4. Then, the following equality holds:*

$$|Du| = (\mathbf{z}, Du) \qquad \textit{as measures in } \Omega\,.$$

Proof. Let $0 \le \varphi \in C_c^1(\Omega)$. Choosing $u_p\varphi$ as a test function in (3.1), we obtain

$$\int_\Omega |\nabla u_p|^p \varphi dx + \int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi dx = \int_\Omega f(x,u_p)u_p\varphi dx. \tag{3.21}$$

On the other hand, applying Young's inequality,

$$\int_\Omega \varphi|\nabla u_p|dx \le \frac{1}{p}\int_\Omega \varphi|\nabla u_p|^p dx + \frac{p-1}{p}\int_\Omega \varphi dx,$$

so, by using (3.21), we obtain

$$\int_\Omega \varphi|\nabla u_p|dx + \frac{1}{p}\int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi dx \le \frac{1}{p}\int_\Omega f(x,u_p)u_p\varphi dx + \frac{p-1}{p}\int_\Omega \varphi dx. \tag{3.22}$$

It is easy to pass to the limit on the right hand side using (3.18). Regarding the left hand side, observe that the first term can be handled considering the lower semicontinuity of the map $u \mapsto \int_\Omega \varphi|Du|$ with respect to the L^1–convergence. We deal with the remaining term by noting that $|\nabla u_p|^{p-2}\nabla u_p$ weakly converges in $L^s(\Omega;\mathbb{R}^N)$ for all $1 \le s < \infty$ and u_p strongly converges in $L^r(\Omega)$ for all $1 \le r < 1^*$. So, letting $p \to 1^+$ in (3.22), it follows that

$$\int_\Omega \varphi|Du| + \int_\Omega u\mathbf{z}\cdot\nabla\varphi dx \le \int_\Omega f(x,u)u\varphi dx.$$

Finally, using (3.15) we obtain

$$\int_\Omega \varphi|Du| \le -\int_\Omega u\mathbf{z}\cdot\nabla\varphi dx - \int_\Omega \operatorname{div}\mathbf{z}\, u\varphi dx = \int_\Omega (\mathbf{z}, Du)\varphi.$$

The reverse inequality follows straightforwardly from $\|\mathbf{z}\|_\infty \le 1$. □

Lemma 3.7 *The vector field* $\boldsymbol{z}$ *and the functions* u *and* λ *verify the following equality:*

$$u([\boldsymbol{z},\nu] + T_1(\lambda)) = 0 \quad \mathcal{H}^{N-1} - a.e \text{ on } \partial\Omega.$$

In particular (2.7) *holds.*

Proof. Taking u_p as a test function in (3.1), we obtain

$$\int_\Omega |\nabla u_p|^p dx + \int_{\partial\Omega} \lambda|u_p|^p d\mathcal{H}^{N-1} = \int_\Omega f(x,u_p)u_p dx.$$

Applying Young's inequality, this implies

$$\begin{aligned}\int_\Omega |\nabla u_p| dx + \int_{\partial\Omega} \lambda|u_p| d\mathcal{H}^{N-1} &\le \int_\Omega |\nabla u_p|^p dx + \int_{\partial\Omega} \lambda|u_p|^p d\mathcal{H}^{N-1} \\ &\quad + \left(\frac{p-1}{p}\right)|\Omega| + \left(\frac{p-1}{p}\right)\int_{\partial\Omega} \lambda d\mathcal{H}^{N-1} \\ &\le \int_\Omega f(x,u_p)u_p dx + \left(\frac{p-1}{p}\right)|\Omega| + \left(\frac{p-1}{p}\right)\int_{\partial\Omega} \lambda d\mathcal{H}^{N-1}.\end{aligned}$$

On the other hand, it is immediate that $T_1(\lambda) \le \lambda$. From the last two inequalities, we have

$$\int_\Omega |\nabla u_p| dx + \int_{\partial\Omega} T_1(\lambda)|u_p| d\mathcal{H}^{N-1} \le \int_\Omega f(x,u_p)u_p dx + \left(\frac{p-1}{p}\right)\left[|\Omega| + \int_{\partial\Omega} \lambda d\mathcal{H}^{N-1}\right]. \tag{3.23}$$

Observe that the left-hand side of (3.23) is lower semicontinuous with respect to the L^1-convergence as $p \to 1^+$ (see [14, Proposition 1.2] and [18]). Therefore, by letting $p \to 1^+$ in (3.23) and using (3.17), we obtain

$$\int_\Omega |Du| + \int_{\partial\Omega} T_1(\lambda) u d\mathcal{H}^{N-1} \le \int_\Omega f(x,u)u dx. \tag{3.24}$$

Using the fact that $-\operatorname{div}\mathbf{z} = f(x,u)$ in (3.24) and applying Green's formula (2.3), we get

$$\begin{aligned}\int_\Omega |Du| + \int_{\partial\Omega} T_1(\lambda) u d\mathcal{H}^{N-1} &\le -\int_\Omega \operatorname{div}\mathbf{z}\, u dx \\ &= \int_\Omega (\mathbf{z}, Du) - \int_{\partial\Omega} u[\mathbf{z},\nu] d\mathcal{H}^{N-1}.\end{aligned}$$

Since Lemma 3.6 ensures $(\mathbf{z}, Du) = |Du|$, this leads to

$$\int_{\partial\Omega} (T_1(\lambda) + [\mathbf{z}, \nu]) u d\mathcal{H}^{N-1} \leq 0. \tag{3.25}$$

Now note that $(T_1(\lambda) + [\mathbf{z}, \nu])u \geq 0$ for $\mathcal{H}^{N-1}-$ almost every point on $\partial\Omega$. This fact is verified as follows: Let $x \in \partial\Omega$ such that $u(x) > 0$.

1. If $\lambda(x) \leq 1$, then by (3.13), we have $T_1(\lambda) + [\mathbf{z}, \nu] = \lambda - \lambda\beta \geq 0$.
2. If $\lambda(x) > 1$, then $T_1(\lambda) + [\mathbf{z}, \nu] = 1 + [\mathbf{z}, \nu] \geq 0$ since $|[\mathbf{z}, \nu]| \leq 1$.

Thus, (3.25) implies that $(T_1(\lambda) + [\mathbf{z}, \nu])u = 0$ $\mathcal{H}^{N-1}-$ almost every point on $\partial\Omega$.

Finally, using $[\mathbf{z}, \nu] = -\lambda\beta$, we conclude that (2.7) holds. □

Now, let us show that u is nontrivial. For that aim, let us introduce $J : BV(\Omega) \to \mathbb{R}$, the energy functional corresponding to (1.2), defined as

$$J(u) = \|u\|_\lambda - \int_\Omega F_+(x, u) dx.$$

From (1.3b), there exist $C_1, C_2 > 0$, such that

$$F_+(x, s) \leq C_1 |s|^{1+\alpha} + C_2 |s|^{\frac{N}{N-1}} \qquad \forall s \in \mathbb{R} \text{ and for a.e. } x \in \Omega.$$

Hence, for $u \in BV(\Omega)$, we deduce from the above inequalities and the Sobolev's imbedding that

$$\begin{aligned} J(u) &= \|u\|_\lambda - \int_\Omega F_+(x, u) dx \\ &\geq \|u\|_\lambda - C_1 \int_\Omega |u|^{1+\alpha} dx - C_2 \int_\Omega |u|^{\frac{N}{N-1}} dx \\ &\geq \|u\|_\lambda - C_3 \|u\|_\lambda^{1+\alpha} - C_4 \|u\|_\lambda^{\frac{N}{N-1}} \\ &= \|u\|_\lambda \left(1 - C_3 \|u\|_\lambda^\alpha - C_4 \|u\|_\lambda^{\frac{1}{N-1}}\right). \end{aligned}$$

Choosing $0 < \rho < \min\left\{ \left(\frac{1}{4C_3}\right)^{\frac{1}{\alpha}}, \left(\frac{1}{4C_4}\right)^{N-1} \right\}$, we obtain

$$J(u) \geq \frac{\|u\|_\lambda}{2}, \quad \text{for } \|u\|_\lambda \leq \rho.$$

Let us fix $p \in (1, p_0)$. By applying Young's inequality, it yields $I_p(u) \geq J(u)$ for all $u \in W^{1,p}(\Omega)$. Since $e \neq 0$ (where e is given in (3.6)), we may take ρ small enough to get $\|e\|_\lambda > \rho$. Let us now consider any path $\gamma \in \Gamma_p$. By the continuity of the map $t \mapsto I_p(\gamma(t))$, there exists $t_0 > 0$ such that $\|\gamma(t_0)\|_\lambda = \rho$. Hence, we obtain

$$I_p(u_p) = \inf_{\gamma \in \Gamma_p} \max_{t \in [0,1]} I_p(\gamma(t)) \geq \frac{\rho}{2}. \tag{3.26}$$

On the other hand, from (2.5), (2.6) and (3.17) we have

$$
\begin{aligned}
\lim_{p\to 1^+}\frac{1}{p}\left[\int_\Omega |\nabla u_p|^p dx+\int_{\partial\Omega}\lambda|u_p|^p d\mathcal{H}^{N-1}\right] &= \lim_{p\to 1^+}\frac{1}{p}\int_\Omega f(x,u_p)u_p dx\\
&= \int_\Omega f(x,u)u dx\\
&= -\int_\Omega u\,\mathrm{div}\,\mathbf{z} dx\\
&= \int_\Omega (\mathbf{z},Du)-\int_{\partial\Omega} u\,[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}\\
&= \int_\Omega |Du|+\int_{\partial\Omega}\lambda\beta u d\mathcal{H}^{N-1}\\
&\leq \int_\Omega |Du|+\int_{\partial\Omega}\lambda|u| d\mathcal{H}^{N-1}\\
&= \|u\|_\lambda. \qquad (3.27)
\end{aligned}
$$

Now, let again $c_* = \sup_{p>1}\int_\Omega |u_p|^{1^*}dx<\infty$. From hypothesis (1.3a), given $\epsilon>0$, there exists a_ϵ such that

$$
|F(x,s)|\leq \frac{\epsilon}{2c_*}|s|^{1^*}+a_\epsilon \quad \text{ for a.e. } x\in\Omega \text{ and all } s\in\mathbb{R}. \qquad (3.28)
$$

Using this inequality, for any measurable set $E\subset\Omega$ such that $|E|<\frac{\epsilon}{2a_\epsilon}$, we obtain

$$
\left|\int_E F(x,u_p)dx\right|\leq \int_E |F(x,u_p)|dx\leq \frac{\epsilon}{2c_*}\int_\Omega |u_p|^{1^*}dx+a_\epsilon|E|\leq\epsilon.
$$

This implies that $\left(F(.,u_p(.)\right)_{p>1}$ is equi-integrable. Moreover, since

$$
F(x,u_p)\to F(x,u), \text{ a.e } x\in\Omega,
$$

it follows from Vitali's Theorem that

$$
\lim_{p\to 1^+}\int_\Omega F(x,u_p)dx=\int_\Omega F(x,u)dx.
$$

From the above equality and (3.27), it follows that

$$
J(u)\geq \lim_{p\to 1^+} I_p(u_p).
$$

Hence, from (3.26), we conclude that $J(u)\geq\dfrac{\rho}{2}$, which implies that u is nontrivial.

3.1 Boundedness of the solutions

In this short section, we prove an L^∞ estimate to the nonnegative solution $u\in BV(\Omega)$. To do that, we use standard Stampacchia's argument, which we detail here for the sake of completeness. It is worth to highlight that by an analogous argument, we could prove that the nonpositive solution w is also bounded.

All along this section, u_p denotes the nonnegative solution of (3.1), that satisfies (3.7) and we denote by S_1 the best constant of the Sobolev embedding $W^{1,1}(\Omega)\hookrightarrow L^{1^*}(\Omega)$.

For $k \geq 0$ and $p \in (1, p_0)$, we define the set

$$A_{k,p} = \{x \in \Omega : u_p(x) > k\}.$$

By the definition of $A_{k,p}$, we have

$$|A_{k,p}| \leq \frac{1}{k^{\frac{N}{N-1}}} \int_{A_{k,p}} u_p^{\frac{N}{N-1}} dx \leq \frac{1}{k^{\frac{N}{N-1}}} \int_{\Omega} u_p^{\frac{N}{N-1}} dx.$$

Hence, for every $\epsilon > 0$ there exists $k_0 > 0$ such that

$$|A_{k,p}| < \epsilon, \tag{3.29}$$

for every $k \geq k_0$ and for all $p \in (1, p_0)$, owing to Lemma 3.3.

We now proceed to prove that the nonnegative solution u belongs to $L^\infty(\Omega)$. For each $k \geq 0$, we define the auxiliary function $G_k : \mathbb{R} \to \mathbb{R}$ as follows

$$G_k(s) = \begin{cases} s - k, & s > k, \\ 0, & |s| \leq k, \\ s + k, & s < -k. \end{cases}$$

Taking $G_k(u_p)$ as a test function in (3.1), we obtain

$$\int_\Omega |\nabla G_k(u_p)|^p dx + \int_{\partial\Omega} \lambda |u_p|^{p-1} |G_k(u_p)| d\mathcal{H}^{N-1} = \int_\Omega f(x, u_p) G_k(u_p) dx$$

wherewith

$$\int_\Omega |\nabla G_k(u_p)|^p dx + \int_{\partial\Omega} \lambda |G_k(u_p)|^p d\mathcal{H}^{N-1} \leq \int_\Omega f(x, u_p) G_k(u_p) dx. \tag{3.30}$$

On the other hand, by assumptions (1.3a) and (1.3b), for any $\rho > 0$, there exists c_ρ such that

$$|f(x, s)| \leq \rho |s|^{1^*-1} + c_\rho, \qquad \text{for almost all } x \in \Omega \text{ and all } s \in \mathbb{R}. \tag{3.31}$$

Using (3.30), (3.31), Young's inequality and Sobolev's embedding, it follows that

$$\begin{aligned}
\left(\int_\Omega G_k(u_p)^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} &\leq S_1 \int_\Omega |\nabla G_k(u_p)| dx + S_1 \int_{\partial\Omega} \lambda |G_k(u_p)| d\mathcal{H}^{N-1} \\
&\leq \frac{S_1}{p} \int_\Omega |\nabla G_k(u_p)|^p dx + \frac{S_1}{p} \int_\Omega \lambda |G_k(u_p)|^p d\mathcal{H}^{N-1} + \frac{S_1(p-1)}{p} \left[|\Omega| + \int_{\partial\Omega} \lambda d\mathcal{H}^{N-1} \right] \\
&\leq S_1 \int_\Omega |f(x, u_p)| G_k(u_p) dx + \frac{S_1(p-1)}{p} \left[|\Omega| + \int_{\partial\Omega} \lambda d\mathcal{H}^{N-1} \right] \\
&\leq S_1 \int_{A_{k,p}} (\rho\, u_p^{1^*-1} + c_\rho) G_k(u_p) dx + \frac{S_1(p-1)}{p} \left[|\Omega| + \int_{\partial\Omega} \lambda d\mathcal{H}^{N-1} \right] \\
&= S_1 \rho \int_{A_{k,p}} u_p^{1^*-1} G_k(u_p) dx + S_1 c_\rho \int_{A_{k,p}} G_k(u_p) dx + \frac{S_1(p-1)}{p} \left[|\Omega| + \int_{\partial\Omega} \lambda d\mathcal{H}^{N-1} \right]
\end{aligned}$$

Finally Hölder's inequality leads to

$$\begin{aligned}
&\left(\int_\Omega G_k(u_p)^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} \\
&\quad \leq S_1 \rho \left(\int_\Omega u_p^{1^*} dx \right)^{\frac{1}{N}} \left(\int_\Omega G_k(u_p)^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} + S_1 c_\rho \left(|A_{k,p}| \right)^{\frac{1}{N}} \left(\int_\Omega G_k(u_p)^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} \\
&\qquad + \frac{S_1(p-1)}{p} \left[|\Omega| + \int_{\partial\Omega} \lambda d\mathcal{H}^{N-1} \right].
\end{aligned}$$

From (3.29), we choose $k_0 > 0$ such that

$$|A_{k,p}| < \frac{1}{(4S_1 c_\rho)^N}, \text{ for all } k \geq k_0.$$

Since ρ is arbitrary, we choose

$$\rho < \frac{1}{4S_1 c_N^*},$$

where $c_N^* = \sup_{1<p<p_0} \left(\int_\Omega |u_p|^{1^*} dx \right)^{\frac{1}{N}}$.

Consequently, we obtain

$$\int_\Omega G_k(u_p)^{\frac{N}{N-1}} dx \leq \left(\frac{2S_1(p-1)}{p} \left[|\Omega| + \int_{\partial\Omega} \lambda d\mathcal{H}^{N-1} \right] \right)^{\frac{N}{N-1}}, \text{ for all } k \geq k_0.$$

Finally, taking into account that $u_p \to u$ a.e. in Ω, we pass to the limit as $p \to 1^+$ and, applying Fatou's Lemma, we conclude that

$$\int_\Omega ((u-k)^+)^{\frac{N}{N-1}} dx = 0, \text{ for all } k \geq k_0.$$

Hence, $\|u\|_\infty \leq k_0$.

4 Proof of Theorem 1.2

We will prove that (1.5) admits a nontrivial solution $u \geq 0$. By an analogous argument, the existence of a nontrivial solution $w \leq 0$ also follows.

First we recall that, for $p \geq 1$, since $V \in L^N(\Omega)$ satisfies $V \geq 0$ and $V \neq 0$ in each connected component, the expression $\|u\|_{V,p} = \left(\int_\Omega |\nabla u|^p dx + \int_\Omega V(x)|u|^p dx\right)^{\frac{1}{p}}$ defines a norm in $W^{1,p}(\Omega)$ which is equivalent to usual one (see [16, Theorem 7.1]). As a consequence of the case $p = 1$, we obtain that $\|u\|_V = \int_\Omega |Du| + \int_\Omega V(x)|u|dx$ defines a norm in $BV(\Omega)$ equivalent to the usual one.

Now, let $\tilde{p} = \min\{1+\alpha, \kappa, 1^*\}$. For each $1 < p < \tilde{p}$, we consider the following problem

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = f(x,u) & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

From our hypotheses and the choice of p, the conditions $(a) - (c)$ of the proof of Theorem 1.1 are satisfied.

In order to obtain a nonnegative solution of (4.1), as before, we consider the associated energy functional $J_p^+ : W^{1,p}(\Omega) \to \mathbb{R}$, defined by

$$J_p^+(u) = \frac{1}{p}\|u\|_{V,p}^p - \int_\Omega F_+(x,u)dx, \tag{4.2}$$

where $F_+(x,u) = \int_0^s f_+(x,t)dt$, with $f_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$f_+(x,s) = \begin{cases} 0 & \text{if } s \leq 0, \\ f(x,s) & \text{if } s > 0. \end{cases} \tag{4.3}$$

We define the functional

$$I_p(u) = J_p^+(u) + \left(\frac{p-1}{p}\right)\left(|\Omega| + \int_\Omega V(x)dx\right).$$

Proceeding analogously to Lemma 3.1 we can establish that the sequence $(I_p(u))_{p>1}$ is nondecreasing with respect to p.

It is straightforward to verify that the energy functional I_p satisfies the geometry of the Mountain Pass Theorem. Moreover there exists $e \in \mathcal{D}(\Omega)$, such that

$$I_p(e) < 0, \quad \text{for all } p \in (1, \tilde{p}).$$

Consequently, by Mountain Pass Theorem, for each $1 < p < \tilde{p}$, there exists u_p solution of (4.1) such that

$$I_p(u_p) = \inf_{\gamma \in \Gamma_p} \max_{t\in[0,1]} I_p(\gamma(t)) =: c_p,$$

where

$$\Gamma_p = \left\{\gamma \in C\left([0,1], W^{1,p}(\Omega)\right) : \gamma(0) = 0, \gamma(1) = e\right\}.$$

Furthermore, following the arguments in Lemma 3.2, we also conclude that the family $(I_p(u_p))_{1<p<\tilde{p}}$ is nondecreasing with respect to p. Henceforth, we fix $1 < p_0 < \tilde{p}$.

Lemma 4.1 *The family $(u_p)_{1<p<p_0}$ is bounded in $BV(\Omega)$.*

Proof. Since the family $(I_p(u_p))_{1<p<\tilde{p}}$ is nondecreasing, given $p_0 \in (1, \tilde{p})$, it follows that

$$I_p(u_p) \le I_{p_0}(u_{p_0}) = c_{p_0}$$

for all $p \in (1, p_0)$.

We denote $\Omega_p = \{x \in \Omega : u_p \le s_0\}$, for any $p \in (1, p_0)$.

Then, by condition (1.3b), we obtain

$$\int_{\Omega_p} F(x, u_p)dx \le C\left(s_0^{\alpha+1} + s_0^{1^*}\right) = \overline{C}.$$

On the other hand, using condition (1.3c) and the fact that u_p is a solution of (4.1), we get

$$\int_{\Omega\setminus\Omega_p} F(x, u_p)dx \le \frac{1}{\kappa}\int_\Omega u_p f(x, u_p)dx = \frac{1}{\kappa}\left[\int_\Omega |\nabla u_p|^p dx + \int_\Omega V(x)|u_p|^p dx\right].$$

From the above inequalities, we deduce that

$$\begin{aligned} c_{p_0} &\ge \frac{1}{p}\left[\int_\Omega |\nabla u_p|^p dx + \int_\Omega V(x)|u_p|^p dx\right] - \int_{\Omega\setminus\Omega_p} F(x, u_p)dx - \int_{\Omega_p} F(x, u_p)dx \\ &\ge \left(\frac{1}{p} - \frac{1}{\kappa}\right)\left[\int_\Omega |\nabla u_p|^p dx + \int_\Omega V(x)|u_p|^p dx\right] - \overline{C} \\ &\ge \left(\frac{1}{p_0} - \frac{1}{\kappa}\right)\left[\int_\Omega |\nabla u_p|^p dx + \int_\Omega V(x)|u_p|^p dx\right] - \overline{C}. \end{aligned}$$

Hence, there exists $\tilde{C}$, independent of p such that

$$\int_\Omega |\nabla u_p|^p dx + \int_\Omega V(x)|u_p|^p dx \le \tilde{C}, \quad \forall\, p \in (1, p_0). \tag{4.4}$$

Now, applying Young's inequality, we obtain

$$\begin{aligned}
\|u_p\|_V &= \int_\Omega |\nabla u_p| dx + \int_\Omega V(x)|u_p| dx \\
&\leq \frac{1}{p}\int_\Omega |\nabla u_p|^p dx + \frac{1}{p}\int_\Omega V(x)|u_p|^p dx + \left(\frac{p-1}{p}\right)|\Omega| + \left(\frac{p-1}{p}\right)\int_\Omega V(x) dx \\
&\leq \tilde{C}_1,
\end{aligned}$$

so that $(u_p)_{1<p<p_0}$ is bounded in $BV(\Omega)$. □

As a direct consequence of the previous lemma and the compact embeddings, there exist $u \in BV(\Omega)$ such that, up to subsequences

(C) $u_p \to u$ in $L^m(\Omega)$, for $1 \leq m < \frac{N}{N-1}$, as $p \to 1^+$.

(D) $u_p \to u$ a.e. $x \in \Omega$, as $p \to 1^+$.

Moreover, as in Lemma 3.4, it is possible to show that there exist $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ and a function $\beta \in L^s(\Omega, V)$ for all $1 \leq s < \infty$, such that $\beta\chi_{\{V>0\}} \in L^\infty(\Omega)$, $\|\mathbf{z}\|_\infty \leq 1$, $\|\beta\chi_{\{V>0\}}\|_\infty \leq 1$,

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z}, \quad \text{in } L^s(\Omega, \mathbb{R}^N) \text{ for all } 1 \leq s < \infty \tag{4.5}$$

and

$$|u_p|^{p-2}u_p \rightharpoonup \beta, \quad \text{in } L^s(\Omega, V) \text{ for all } 1 \leq s < \infty. \tag{4.6}$$

Again, as in Section 3, we can readily conclude that

$$-\operatorname{div} \mathbf{z} + V\beta = f(x,u), \quad \text{in } \mathcal{D}'(\Omega) \tag{4.7}$$

Notice that (4.7) implies $\operatorname{div}(\mathbf{z}) \in L^N(\Omega)$. On the other hand, the proof of Lemma 3.5 still holds.

Lemma 4.2 *It is fulfilled that*

$$\lim_{p\to 1^+} \int_\Omega V(x)|u_p|\, dx = \int_\Omega V(x)|u|\, dx$$

and, for any $\varphi \in C_c^1(\Omega)$,

$$\lim_{p\to 1^+} \int_\Omega \varphi V(x)|u_p|\, dx = \int_\Omega \varphi V(x)|u|\, dx.$$

Proof. Since $\{|u_p|\}_p$ is bounded in $BV(\Omega)$, it follows that it is also bounded in $L^{\frac{N}{N-1}}(\Omega)$. This fact and the pointwise convergence of $|u_p|$ to $|u|$ imply that

$$|u_p| \rightharpoonup |u| \qquad \text{weakly in } L^{\frac{N}{N-1}}(\Omega).$$

The result now is a consequence of $V \in L^N(\Omega)$. □

Lemma 4.3 *The following equality holds in the sense of measures in* Ω*:*

$$(\boldsymbol{z}, Du) = |Du|. \tag{4.8}$$

Proof. Let $\varphi \in C_c^1(\Omega)$ be nonnegative and take $u_p\varphi$ as test function in (4.1). We get

$$\int_\Omega \varphi|\nabla u_p|^p dx + \int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi\, dx + \int_\Omega \varphi V(x)|u_p|^p dx = \int_\Omega f(x,u_p)u_p\varphi\, dx.$$

By Young's inequality,

$$\begin{aligned}\int_\Omega \varphi|\nabla u_p|\, dx &+ \int_\Omega \varphi V(x)|u_p|\, dx \\ &\le \frac{1}{p}\int_\Omega \varphi|\nabla u_p|^p dx + \frac{1}{p}\int_\Omega \varphi V(x)|u_p|^p dx + \frac{p-1}{p}\left[|\Omega| + \int_\Omega V(x)\, dx\right] \\ &= -\frac{1}{p}\int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\, dx + \frac{1}{p}\int_\Omega f(x,u_p)u_p\varphi\, dx + \frac{p-1}{p}\left[|\Omega| + \int_\Omega V(x)\, dx\right].\end{aligned}$$

To pass to the limit on the left hand side, we may use the lower semicontinuity in the first term and Lemma 4.2 in the second one. The limit on the right hand side is consequence of (C), 4.5 and Lemma 3.5. It yields

$$\int_\Omega \varphi|Du| + \int_\Omega \varphi V(x)|u|\, dx \le -\int_\Omega u\,\mathbf{z}\cdot\nabla\varphi\, dx + \int_\Omega f(x,u)u\varphi\, dx,$$

so that

$$\int_\Omega \varphi|Du| + \int_\Omega \varphi V(x)\beta u\, dx \le -\int_\Omega u\,\mathbf{z}\cdot\nabla\varphi\, dx + \int_\Omega f(x,u)u\varphi\, dx.$$

Owing to 4.7, we deduce

$$\int_\Omega \varphi|Du| \le -\int_\Omega u\,\mathbf{z}\cdot\nabla\varphi\, dx - \int_\Omega u\varphi\,\mathrm{div}\,\mathbf{z}\, dx = \int_\Omega \varphi(\mathbf{z}, Du)$$

from where the result follows. □

Regarding the boundary condition, we get the following result.

Lemma 4.4 *The vector field $\boldsymbol{z}$ satisfies the following identity:*

$$[\boldsymbol{z},\nu] = 0 \quad \mathcal{H}^{N-1} - a.e.\ on\ \partial\Omega. \tag{4.9}$$

Proof. Let $1 < p < p_0$, and consider $v \in W^{1,p_0}(\Omega)$ as a test function in (4.1). Then, we obtain

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla v dx + \int_\Omega V(x)|u_p|^{p-2}u_p v dx = \int_\Omega f(x,u_p)v dx. \tag{4.10}$$

By applying the same arguments as in Lemma 3.4, we can directly pass to the limit in (4.10) as $p \to 1^+$, obtaining

$$\int_\Omega \mathbf{z}\cdot\nabla v dx + \int_\Omega V\beta v dx = \int_\Omega f(x,u) v dx$$

for every $v \in W^{1,p_0}(\Omega)$. By density, this equality extends to all $v \in W^{1,1}(\Omega)$.

Applying Green's formula and (4.7), we deduce

$$\begin{aligned}\int_\Omega f(x,u)v dx &= -\int_\Omega v\mathrm{div}\mathbf{z} dx + \int_\Omega V\beta v dx \\ &= \int_\Omega \mathbf{z}\cdot\nabla v dx - \int_{\partial\Omega}[\mathbf{z},\nu]v d\mathcal{H}^{N-1} + \int_\Omega V\beta v dx.\end{aligned}$$

This, in turn, implies that

$$\int_{\partial\Omega}[\mathbf{z},\nu]v d\mathcal{H}^{N-1} = 0, \quad \forall v \in W^{1,1}(\Omega)$$

and consequently

$$\int_{\partial\Omega} h[\mathbf{z},\nu]d\mathcal{H}^{N-1}=0,$$

holds for every $h\in L^1(\partial\Omega)$. Therefore,

$$[\mathbf{z},\nu]=0 \quad \mathcal{H}^{N-1}-\text{a.e. on } \partial\Omega.$$

□

We now turn to check the condition (2.11).

Lemma 4.5 *The function β satisfies $V\beta\in V\,sign(u)$.*

Proof. Consider u_p as a test function in (4.1). Then, by Young's inequality, we obtain

$$\begin{aligned}\int_\Omega|\nabla u_p|\,dx+\int_\Omega V(x)|u_p|\,dx &\leq \frac{1}{p}\int_\Omega|\nabla u_p|^p dx+\frac{1}{p}\int_\Omega V(x)|u_p|^p dx+\frac{p-1}{p}\Big[|\Omega|+\int_\Omega V(x)\,dx\Big]\\ &\leq \int_\Omega f(x,u_p)u_p dx+\frac{p-1}{p}\Big[|\Omega|+\int_\Omega V(x)\,dx\Big].\end{aligned}$$

We may apply the lower semicontinuity of the total variation in the first term of the left hand side. In the other terms we use lemmas 3.5 and 4.2. Thus, it yields

$$\int_\Omega|Du|+\int_\Omega V(x)|u|\,dx\leq\int_\Omega f(x,u)u\,dx=-\int_\Omega u\operatorname{div}\mathbf{z}\,dx+\int_\Omega V(x)\beta u\,dx. \tag{4.11}$$

Observe that Green's formula, identity (4.8) and Lemma 4.4 lead to

$$-\int_\Omega u\operatorname{div}\mathbf{z}\,dx=\int_\Omega(\mathbf{z},Du)-\int_{\partial\Omega}u[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}=\int_\Omega|Du|$$

Hence, inequality (4.11) becomes

$$\int_\Omega\big(V(x)|u|-V(x)\beta u\big)\,dx\leq 0.$$

Taking into account that $\|\beta\chi_{\{V>0\}}\|_\infty\leq 1$, the result follows. □

Finally, following the same reasoning as in Section 3, we conclude that u is nontrivial and belongs to $L^\infty(\Omega)$.

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