

Section 2: The z-Transform

In a linear discrete-time control system a linear difference equation characterises the dynamics of the system. In order to determine the system's response to a given input, such a difference equation must be solved. With the Z-Transform method, the solutions to linear difference equations become algebraic in nature. Just as the Laplace transformation transforms linear time-invariant differential equations into algebraic equations in s the Z-transformation transforms linear time-invariant difference equations into algebraic equations in z . The main objective of this section is to present definitions of the Z-Transform, basic theorems associated with the Z-Transform, and methods for finding the inverse Z-Transform.

2.1 The Sampling Process

In general, discrete-time signals arise in a system when a sampling operation is carried out on a continuous-time signal. The sampling process can be likened to the operation of a switch, see Figure 2.1(a), namely when the switch is open no information about the continuous-time signal is captured, but when the switch is closed the signal passes through and hence information is gathered. The switch closes periodically with period T . The operation of the switch is not instantaneous, hence if the input is the continuous signal $f(t)$ the output from the switch, denoted $f^*(t)$, consists of shaded pulses, as illustrated in Figure 2.1(b). This output pulse train $f(kT)$ may be approximated by an impulse train where each impulse function has an area equal to the value of the input signal at the sampling instant $t = nT$ which is the time of the impulse.

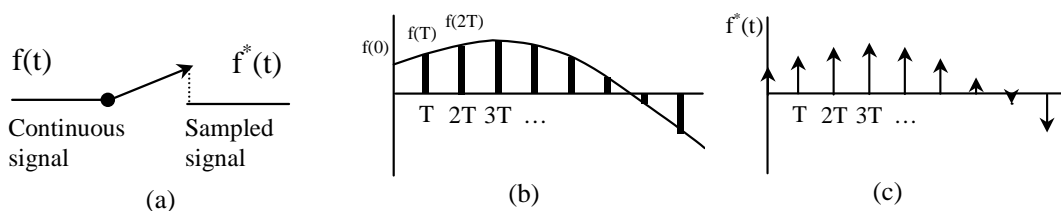


Figure 2.1: The sampling process: (a) The sampler as a switch; (b) Sampled output as pulses; (c) approximation to $f(kT)$ in terms of impulses.

Alternatively the sampling process may be seen as the modulation of an impulse train by the input signal $f(t)$, see Figure 2.2. The expression for the modulated train, $f(kT)$, is then

$$\begin{aligned}
 f(kT) &= f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + \dots & \text{Eqn. 2.1} \\
 &= \sum_{k=0}^{\infty} f(kT)\delta(t-kT)
 \end{aligned}$$

Taking the Laplace Transform of equation 2.1 and recalling that (a) $L[\delta(t)] = 1$, and (b) $L[f(t-T)] = F(s)e^{-sT}$, yields

$$L[f(kT)] = F^*(s)$$

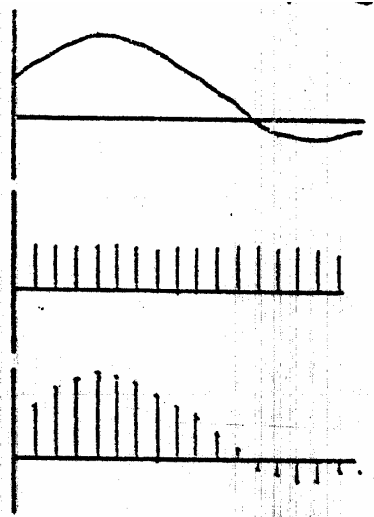


Figure 2.2. Modulation of the input signal by an impulse train to obtain a sampled signal

$$\begin{aligned}
 &= f(0) + f(T)e^{-Ts} + f(2T)e^{-2Ts} + \dots \\
 &= \sum_{k=0}^{\infty} f(kT)e^{-kTs}
 \end{aligned}
 \tag{Eqn. 2.2}$$

As an example consider the function $f(t) = e^{-at}$ and recall that

$$\mathcal{L} [f(t)] = F(s) = \frac{1}{s+a}
 \tag{Eqn. 2.3}$$

However, if we sample the function $f(t)$ to get the sampled or modulated signal $f(kT)$, one is then interested in the Laplace Transform of $f(kT)$. Equation 2.2 is used to obtain the Laplace Transform of the sampled signal $f(kT)$ i.e.

$$\begin{aligned}
 \mathcal{L} [f(kT)] &= F^*(s) = \sum_{k=0}^{\infty} e^{-akT} e^{-kTs} \\
 &= \sum_{k=0}^{\infty} e^{-k(s+a)T}
 \end{aligned}
 \tag{Eqn. 2.4}$$

which is an infinite series that cannot be easily summed. In general the Laplace Transform of the sampler output is in the form of an infinite series. Consequently the sampler output cannot be described in terms of a conventional transfer function relating output to input transforms and the standard techniques for control system analysis and design are not applicable. This difficulty can be overcome, and the analysis simplified, by defining the Z-Transform.

2.2 The z-Transform

The Z-Transform describes the behaviour of a signal at the sampling instants. The Z-Transform of an arbitrary signal may be obtained by applying the following procedure –

- 1) Determine the values of $f(t)$ at the sampling instants to obtain $f(kT)$; $f(kT)$ is now in the form of an impulse modulated train.
- 2) Take the Laplace Transform of the succession of impulses to obtain $F^*(s)$. $F^*(s)$ now contains terms of the form e^{Ts} .
- 3) Make the substitution

$$z = e^{Ts}
 \tag{Eqn. 2.5}$$

into $F^*(s)$ to give $F(z)$.

Note also that $F(z) \neq F(s)$ and $F(z) \neq F^*(s)$. $F(s)$ is the Laplace Transform of the signal $f(t)$ and as such is a continuous-time description of the signal $f(t)$ i.e. it contains information as to what is happening between sampling instants as well as at the sampling instants. The Z-Transform contains information about the signal at the sampling instants only, and hence the two descriptions cannot be equivalent. Furthermore $F(z)$ cannot be equal to $F^*(s)$ as this would imply that the Laplace variable s is equal to the discrete-time variable z . From equation 2.5 it is clear that this is not so, in fact the relationship is –

$$z = e^{Ts}$$

$$\Rightarrow s = \frac{1}{T} \ln(z)$$

and the equivalence is

$$F(z) = F^* \left(\frac{1}{T} \ln(z) \right)$$

From equation 2.2 we had that the Laplace Transform of the modulated train $f^*(t)$ is

$$\mathcal{L} [f(kT)] = F^*(s) = f(0) + f(T)e^{-Ts} + f(2T)e^{-2Ts} + \dots$$

Hence steps one and two of z-Transform procedure are complete. Note that $F^*(s)$ may be re-written as -

$$F^*(s) = f(0) + f(T) \frac{1}{e^{Ts}} + f(2T) \frac{1}{e^{2Ts}} + \dots$$

and making the substitution $z = e^{Ts}$ yields

$$F(z) = f(0) + f(T) \frac{1}{z} + f(2T) \frac{1}{z^2} + \dots$$

$$= f(0) + f(T)z^{-1} + f(2T)z^{-2} + \dots$$

$$= \sum_{k=0}^{\infty} f(kT)z^{-k}$$

Hence the Z-Transform of an arbitrary function $f(t)$ is defined by

$$\mathcal{Z} [f(t)] = \sum_{k=0}^{\infty} f(kT)z^{-k} \quad \text{Eqn. 2.6}$$

Recall

$$\mathcal{L} [f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

2.3 Examples

z-Transform of the Unit Step Function

The unit step function is defined as

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The unit step function is sampled every T seconds, yielding a discrete-time or sampled unit step signal denoted as $u(kT)$ and defined by

$$u(kT) = 1; \quad k = 0, 1, 2, 3, \dots$$

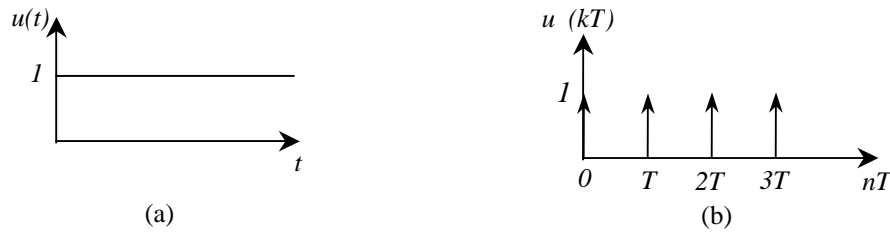


Figure 2.3: Unit step signal; (a) continuous-time representation and (b) discrete-time or sampled representation

Figure 2.3 depicts the relationship between the continuous-time function $u(t)$ and the sampled signal $u(kT)$. To obtain the Z-Transform of the unit step function equation 2.6 is applied to obtain -

$$\begin{aligned} Z [u(t)] &= Z [u(kT)] = 1 + (1)z^{-1} + (1)z^{-2} + \dots \\ &= 1 + \frac{1}{z} + \frac{1}{z^2} + \dots \end{aligned}$$

Recall that sum of an infinite geometric series is given by the formula

$$S_n = \frac{a}{1-r}$$

where a is the first term in the infinite geometric series and r is the ratio between any two terms of the series (provided r is less than one). In this case $a = 1$ and $r = 1/z$, therefore the Z-Transform of the unit step function is -

$$Z [u(t)] = \frac{z}{z-1} \quad \text{Eqn. 2.7}$$

and the series converges provided $z > 1$. Recall

$$L [u(t)] = \frac{1}{s}$$

z-Transform of the Exponential Function

The exponential function is defined as -

$$f(t) = e^{-at} \quad t \geq 0$$

Assuming that this function is sampled every T seconds, the value of the function at the sampling instants is simply $f(kT) = e^{-akT}$. The Z -Transform of $f(t)$ is therefore given by -

$$\begin{aligned} Z [f(t)] &= F(z) = 1 + \frac{e^{-aT}}{z} + \frac{e^{-a2T}}{z^2} + \dots \\ &= 1 + \frac{e^{-aT}}{z} + \left(\frac{e^{-aT}}{z}\right)^2 + \left(\frac{e^{-aT}}{z}\right)^3 + \dots \\ &= \frac{z}{z - e^{-aT}}. \end{aligned}$$

Therefore the Z-Transform of the exponential function is -

$$Z [f(t)] = \frac{z}{z - e^{-aT}} \quad \text{Eqn. 2.8}$$

Recall

$$L [e^{-at}] = \frac{1}{s+a}$$

Note that just as the continuous-time pole location is defined by $s = -a$, the location of the discrete-time pole is defined by $z = e^{-aT}$. Since both a and T are constants e^{-aT} is also a constant and the location of the discrete-time pole is fixed (obviously enough!!). However the pole position in the z -

plane depends on the position of the pole in the *s-plane* AND on the sampling interval, T . Hence by varying the sampling rate it is possible to vary the position of discrete-time pole.

z-Transform of the Unit Ramp Function

The unit ramp function is defined by –

$$f(t) = t \quad t \geq 0$$

with

$$f(kT) = kT \quad k = 0, 1, 2, \dots$$

i.e. the values of $f(t)$ at the sampling instants are $0, T, 2T, 3T, \dots$. By applying the definition of the Z-Transform, equation 2.6, we get that

$$F(z) = \frac{T}{z} + \frac{2T}{z^2} + \frac{3T}{z^3} + \dots$$

To obtain the summation in closed form, consider

$$\frac{F(z)}{zT} = \frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \dots \quad \text{Eqn. 2.9}$$

Multiplying through by dz and integrating yields

$$\int \frac{F(z)}{zT} dz = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \dots + K$$

where K is a constant of integration. The terms in z form a standard infinite geometric series and may easily be summed to yield –

$$\int \frac{F(z)}{zT} dz = -\frac{1}{z-1} + K$$

Differentiating the above with respect to z yields

$$\frac{F(z)}{zT} = \frac{1}{(z-1)^2}$$

Hence the Z-transform of the unit ramp function is given by –

$$Z [f(t)] = \frac{Tz}{(z-1)^2} \quad \text{Eq. 2.10}$$

while

$$L [t] = \frac{1}{s^2}$$

Second-Order Example

Find the Z-Transform of the function whose Laplace Transform is –

$$F(s) = \frac{1}{s(s+1)}$$

The simplest method of obtaining the Z-Transform of the above function is to split the second-order transfer function into first-order transfer functions (whose Z-Transforms we know) via *partial fraction expansion* i.e.

$$\frac{1}{s(s+1)} = \frac{a}{s} + \frac{b}{s+1}$$

where $a = 1, b = -1$. Hence

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

From equations 2.7 and 2.8 we have that

$$\mathcal{Z} \left[\frac{1}{s} \right] = \frac{z}{z-1}$$

$$\mathcal{Z} \left[\frac{1}{s+1} \right] = \frac{z}{z-e^{-T}}; \quad a=1$$

Hence

$$F(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$

$$= \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}$$

Note that the Laplace Transform of $f(t)$ and the Z-Transform of $f(t)$ have the same number of roots, hence the Z-Transform preserves the order of the equation.

z-Transform of a Sinusoid

The sinusoidal function

$$f(t) = \sin(\omega t) \quad t \geq 0$$

may alternately be expressed as

$$f(t) = \sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})$$

with

$$\mathcal{Z} [e^{j\omega t}] = \frac{z}{z-e^{j\omega T}}$$

Hence

$$\mathcal{Z} [f(t)] = \frac{1}{2j} \left(\frac{z}{z-e^{j\omega T}} - \frac{z}{z-e^{-j\omega T}} \right)$$

$$= \frac{1}{2j} \left(\frac{z(e^{j\omega T} - e^{-j\omega T})}{z^2 - (e^{j\omega T} + e^{-j\omega T})z + 1} \right)$$

$$= \frac{z \sin(\omega T)}{z^2 - 2z \cos(\omega T) + 1}$$

The Z-transform of the sinusoid –

$$\mathcal{Z} [\sin(\omega t)] = \frac{z \sin(\omega T)}{z^2 - 2z \cos(\omega T) + 1} \quad \text{Eq. 2.11}$$

while

$$\mathcal{L} [\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}$$

z-Transform of the Sequence $[a^n]$

The Z-Transform of the sequence a^n is given by

$$[a^n] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \dots$$

This infinite sum is

$$S_\infty = \frac{1}{1 - a/z} = \frac{z}{z - a}$$

Therefore the Z-transform of the sequence –

$$Z [a^n] = \frac{z}{z - a} \tag{Eq. 2.12}$$

In this section several examples have been presented which illustrate how to obtain the Z-Transform of a time function $f(t)$ by direct application of the one-sided Z-Transform, equation 2.6. It should be noted that alternative methods of obtaining the Z-Transform are also possible. A table of commonly encountered Z-Transforms is very useful when solving problems in the field of discrete-time control systems; Table 2.1 presents such a table.

Appendix A: Table of Laplace and z-transforms

Laplace transform $f(s)$	Time function $f(t) \quad t > 0$	z-transform $f(z)$	Modified z-transform $f(z,m)$
$\frac{1}{s}$	$u_s(t)$	$\frac{z}{z - 1}$	$\frac{1}{z - 1}$
$\frac{1}{s^2}$	t	$\frac{Tz}{(z - 1)^2}$	$\frac{mT}{(z - 1)} + \frac{T}{(z - 1)^2}$
$\frac{2}{s^3}$	t^2	$\frac{T^2z(z + 1)}{(z - 1)^3}$	$T^2 \left(\frac{m^2}{(z - 1)} - \frac{2m + 1}{(z - 1)^2} + \frac{2}{(z - 1)^3} \right)$
$\frac{1}{s + a}$	e^{-at}	$\frac{z}{z - e^{-aT}}$	$\frac{e^{-amT}}{z - e^{-aT}}$
$\frac{1}{(s + a)^2}$	te^{-at}	$\frac{Tze^{-aT}}{(z - e^{-aT})^2}$	$\frac{Te^{-amT}(e^{-aT} + m(z - e^{-aT}))}{(z - e^{-aT})^2}$
$\frac{a}{s(s + a)}$	$1 - e^{-at}$	$\frac{z(1 - e^{-aT})}{(z - 1)(z - e^{-aT})}$	$\frac{1}{(z - 1)} - \frac{e^{-amT}}{z - e^{-aT}}$
$\frac{1}{(s + a)(s + b)}$	$\frac{1}{(b - a)}(e^{-at} - e^{-bt})$	$\frac{1}{(b - a)} \left(\frac{z}{z - e^{-aT}} - \frac{z}{z - e^{-bT}} \right)$	$\frac{1}{b - a} \left(\frac{e^{-amT}}{z - e^{-aT}} - \frac{e^{-bmT}}{z - e^{-bT}} \right)$
$\frac{a}{s^2(s + a)}$	$t - \frac{1}{a}(1 - e^{-at})$	$\frac{Tz}{(z - 1)^2} - \frac{(1 - e^{-aT})z}{a(z - 1)(z - e^{-aT})}$	$\frac{T}{(z - 1)^2} + \frac{amT - 1}{a(z - 1)} + \frac{e^{-amT}}{a(z - e^{-aT})}$
$\frac{a}{s^2 + a^2}$	$\sin(at)$	$\frac{z \sin(aT)}{z^2 - 2z \cos(aT) + 1}$	$\frac{z \sin(amT) + \sin(1 - m)aT}{z^2 - 2z \cos(aT) + 1}$
$\frac{s}{s^2 + a^2}$	$\cos(at)$	$\frac{z(z - \cos(aT))}{z^2 - 2z \cos(aT) + 1}$	$\frac{z \cos(amT) - \cos(1 - m)aT}{z^2 - 2z \cos(aT) + 1}$
$\frac{b}{(s + a)^2 + b^2}$	$e^{-at} \sin bt$	$\frac{z \sin(bT)e^{-aT}}{z^2 - 2z \cos(bT)e^{-aT} + e^{-2aT}}$	$\frac{e^{-amT}(z \sin(bmT) + e^{-aT} \sin(1 - m)bT)}{z^2 - 2z \cos(bT)e^{-aT} + e^{-2aT}}$
$\frac{s + a}{(s + a)^2 + b^2}$	$e^{-at} \cos bt$	$\frac{z(z - \cos(bT)e^{-aT})}{z^2 - 2z \cos(bT)e^{-aT} + e^{-2aT}}$	$\frac{e^{-amT}(z \cos(bmT) + e^{-aT} \cos(1 - m)bT)}{z^2 - 2z \cos(bT)e^{-aT} + e^{-2aT}}$

Table 2.1: Table of commonly encountered Z-Transforms

2.4 Theorems & Properties

Multiplication by a Constant

If $F(z)$ is the Z-Transform of $f(t)$, then

$$\mathcal{Z} [af(t)] = a \mathcal{Z} [f(t)] = aF(z) \quad \text{Eq. 2.12}$$

where a is a constant. To prove this, note that by definition

$$\mathcal{Z} [af(t)] = \sum_{k=0}^{\infty} af(kT)z^{-k} = a \sum_{k=0}^{\infty} f(kT)z^{-k} = aF(z)$$

Linearity Property

The Z-Transform possesses an important property: linearity. This means that if it is possible to obtain the Z-Transform of two functions, $x(t)$ and $g(t)$, and α and β are scalars, then the function $f(t)$ formed by the linear combination -

$$f(t) = \alpha x(t) + \beta g(t)$$

has the Z-Transform

$$\mathcal{Z} [F(z)] = \alpha X(z) + \beta G(z) \quad \text{Eq. 2.13}$$

Exercise: Prove this.

Theorem No. 1: Multiplication by Exponential Function

Assuming that the function $f(t)$ has the Z-Transform $F(z)$, then the Z-Transform of the function $e^{-at} f(t)$ is

$$\mathcal{Z} [e^{-at} f(t)] = F(ze^{aT}) \quad \text{Eq. 2.14}$$

Proof:

$$\begin{aligned} \mathcal{Z} [e^{-at} f(t)] &= \sum_{k=0}^{\infty} f(kT)e^{-akT} z^{-k} \\ &= \sum_{k=0}^{\infty} f(kT) (ze^{aT})^{-k} \\ &= F(ze^{aT}) \end{aligned}$$

In words, to get the Z-Transform of the function $e^{-at} f(t)$, one obtains the Z-Transform of $f(t)$ i.e. $F(z)$ and replaces the variable z with ze^{aT} . As an example, consider the unit step function, $u(t)$, whose Z-Transform is

$$\mathcal{Z} [u(t)] = \frac{z}{z-1}$$

Therefore

$$\begin{aligned} \mathcal{Z} [e^{-at} u(t)] &= U(ze^{aT}) \\ &= \frac{ze^{aT}}{ze^{aT} - 1} \end{aligned}$$

$$= \frac{z}{z - e^{-aT}}$$

which is the same as that obtained previously, see equation 2.8 i.e. multiplication by the unit step function (or unity) will not alter the result.

Theorem No. 2: Multiplication by Ramp Function

If the function $f(t)$, with Z-Transform $F(z)$, is multiplied by the ramp function or t the Z-Transform of the combined function is given by

$$Z [tf(t)] = -zT \frac{dF(z)}{dz} \quad \text{Eq. 2.15}$$

Proof:

$$\begin{aligned} -zT \frac{dF(z)}{dz} &= -zT \frac{d}{dz} \left[f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \dots \right] \\ &= -zT \left[-\frac{f(T)}{z^2} - \frac{2f(2T)}{z^3} - \frac{3f(3T)}{z^4} \dots \right] \\ &= \frac{Tf(T)}{z} + \frac{2Tf(2T)}{z^2} + \frac{3Tf(3T)}{z^3} \dots \end{aligned}$$

which is the Z-Transform of $tf(t)$ [compare with equation 2.9].

Example: Find the Z-transform of

$$Z [tu(t)]$$

where $u(t)$ is the unit step function.

$$\begin{aligned} Z [u(t)] &= \frac{z}{z-1} \\ \Rightarrow \frac{d}{dz} \left(\frac{z}{z-1} \right) &= \frac{(z-1) - z}{(z-1)^2} \\ &= \frac{-1}{(z-1)^2} \\ Z [tu(t)] &= -zT \frac{dU(z)}{dz} = \frac{zT}{(z-1)^2} \end{aligned}$$

which again is the same as that obtained previously, equation 2.10.

Theorem No. 3: Initial Value Theorem

If $f(t)$ has the Z-Transform $F(z)$ and if $\lim_{z \rightarrow \infty} F(z)$ exists, then the initial value $f(0)$ of $f(t)$ is given by

$$f(0) = \lim_{z \rightarrow \infty} F(z) \quad \text{Eq. 2.16}$$

Proof:

$$F(z) = f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \dots$$

Clearly, taking the limit as $z \rightarrow \infty$ yields

$$\lim_{z \rightarrow \infty} F(z) = f(0)$$

The behaviour of the signal in the neighbourhood of $t = 0$ can thus be determined by the behaviour of $F(z)$ at $z = \infty$. The initial value theorem is convenient for checking Z-Transform calculations for possible errors. Since $f(0)$ is usually known, a check of the initial value by $\lim_{z \rightarrow \infty} F(z)$ can easily spot errors in $F(z)$, if any exist.

Theorem No. 4: Final Value Theorem

The final value of $f(t)$, that is, the value of $f(t)$ as $t \rightarrow \infty$ is given by

$$\lim_{t \rightarrow \infty} f(t) = \lim_{z \rightarrow 1} [(z-1)F(z)] \quad \text{Eq. 2.17}$$

Theorem No. 5: Real Translation (Backward Shift)

Consider Figure 2.4.

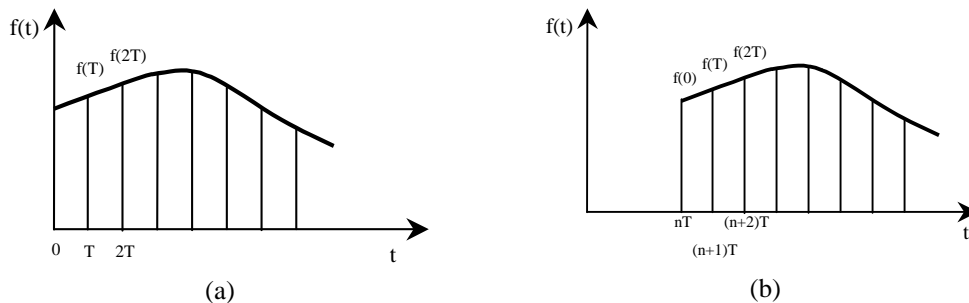


Figure 2.4: Continuous-time function delayed by nT samples; (a) Original signal; (b) Delayed version

The continuous-time function has values $f(0)$ at $t=0$, $f(T)$ at $t=T$, etc. When delayed by nT samples the resulting function $f(t-nT)$ has values $f(0)$ at $t=nT$, $f(T)$ at $t=(n+1)T$, etc. Now

$$\mathcal{Z} [f(t)] = f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \dots$$

Consequently

$$\begin{aligned} \mathcal{Z} [f(t-nT)] &= \frac{f(0)}{z^n} + \frac{f(T)}{z^{n+1}} + \frac{f(2T)}{z^{n+2}} + \dots \\ &= \frac{1}{z^n} \left[f(0) + \frac{f(T)}{z^1} + \frac{f(2T)}{z^2} + \dots \right] \\ &= z^{-n} F(z) \end{aligned}$$

Hence

$$\mathcal{Z} [f(t-nT)] = z^{-n} F(z) \quad \text{Eq. 2.18}$$

Theorem No. 6: Real Translation (Forward Shift)

The concept is illustrated by Figure 2.5.

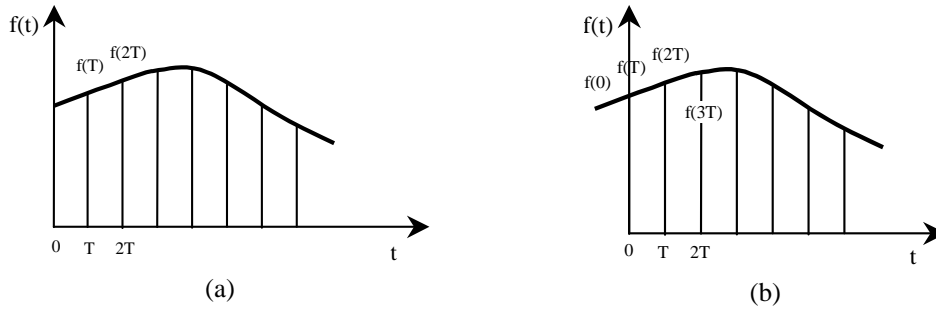


Figure 2.5: Continuous-time function advanced one sample; (a) Original signal; (b) Advanced version

In the forward shift case the signal is project forward in time (no real physical meaning) and hence the signal is shifted to the left. In Figure 2.4 the continuous-time signal is shifted to the left by one sampling interval T , to yield $f(t+T)$. $f(t+T)$ has the value $f(T)$ at $t = 0$.

$$\begin{aligned} Z [f(t+T)] &= f(T) + \frac{f(2T)}{z} + \frac{f(3T)}{z^2} + \dots \\ \frac{1}{z} Z [f(t+T)] &= \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \frac{f(3T)}{z^3} + \dots \\ \frac{1}{z} Z [f(t+T)] + f(0) &= f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \frac{f(3T)}{z^3} + \dots \\ \frac{1}{z} Z [f(t+T)] + f(0) &= F(z) \end{aligned}$$

And therefore

$$Z [f(t+T)] = zF(z) - zf(0) \tag{Eq. 2.19}$$

As a special case, if $f(0) = 0$, i.e. zero initial conditions apply, then

$$Z [f(t+T)] = zF(z) \tag{Eq. 2.20}$$

and multiplication of the Z-Transform of a signal $f(t)$ by z corresponds to a forward time shift of one sampling period. Equation 2.19 can easily be modified to obtain the following relationship –

$$Z [f(t+2T)] = z Z [f(t+T)] - zf(1) = z^2 F(z) - z^2 f(0) - zf(T) \tag{Eq. 2.21}$$

Similarly,

$$Z [f(t+n)] = z^n F(z) - z^n f(0) - z^{n-1} f(T) - z^{n-2} f(2T) - \dots - zf(n-1) \tag{Eq. 2.22}$$

2.5 Conclusion

In this section we have modelled sequences of samples as sums of impulse functions, the strength of each impulse corresponding to the numerical value of the sample it represented e.g. the sequence

$$3, 2, 1, 0, 0, 0, 0, \dots$$

is modelled by

$$3\delta(t) + 2\delta(t-T) + \delta(t-2T)$$

where the impulses represent the samples at times $T = 0, T$ and $2T$ respectively. To obtain the corresponding Z-Transform, table 2.1 may be utilised.

Description	$f(t)$	$F(s)$	$F(z)$
Unit Impulse at $t = 0$	$\delta(t)$	1	1
Unit Impulse at $t = T$	$\delta(t-T)$	e^{-sT}	z^{-1}
Impulse of strength w at $t = T$	$w\delta(t-T)$	$w e^{-sT}$	wz^{-1}

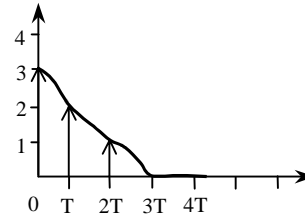


Table 2.1: Laplace and Z-Transform of Impulse Function

Thus any sampled signal may be modelled using the Z-Transform e.g. for the sequence of samples discussed above one gets

$$\begin{aligned} 3\delta(t) + 2\delta(t-T) + \delta(t-2T) &\Rightarrow 3 + 2e^{-sT} + e^{-2sT} \\ &\Rightarrow 3 + 2z^{-1} + z^{-2} \end{aligned}$$

Furthermore the Z-Transform may be used to describe delays e.g. if the original signal is delayed by one sampling period it is written as –

$$\begin{aligned} 0, 3, 2, 1, 0, 0, 0, 0, \dots \\ 3\delta(t-T) + 2\delta(t-2T) + \delta(t-3T) \\ \Rightarrow 3z^{-1} + 2z^{-2} + z^{-3} \end{aligned}$$

i.e. the original sequence multiplied by z^{-1} . For this reason z^{-1} is often referred to as the *delay operator* or the *backward shift operator*.

2.6 Exercises

1) Find the Z-Transform of –

- i) $\cos \omega t$
- ii) $3 + \cos \omega t$
- iii) b^n

2) If

$$G(s) = \frac{1}{s^2 + (a+b)s + ab}$$

show that

$$G(z) = \frac{1}{(b-a)} \frac{z(e^{-aT} - e^{-bT})}{(z - e^{-aT})(z - e^{-bT})}$$

3) Write down the Z-Transform of the sequence

$$u(k) = \{0, 0, 1, 1, 2, 0, 0 \quad \text{at } k = 0, T, 2T, \dots, 6T \text{ respectively}$$

4) Write down the Z-Transform of the decaying exponential function $f(t) = e^{-t}$, sampled at a frequency of 10Hz.

5) Find the Z-Transform of the following sampled signals

- (i) a ramp of slope 2, sampled every second
- (ii) a decaying exponential with a time constant of 0.1 seconds and an initial value of 5, sampled at 50Hz.
- (iii) The sampled step response of the system whose transfer function is $\frac{1}{1+2s}$ sampled at 3Hz.