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Blow-up for a class of nonlinear parabolic problems [1]

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Abstract. For nonlinear parabolic equations of the form $u_t = \Delta u^m - u^\mu \|\nabla u^m\|^q + u^p$, we prove nonexistence of global admissible solutions for large initial data for some range of the parameters m, μ, q and p. To do so we use comparison with suitable blowing up self-similar subsolutions. We also prove that for the complementary range of the parameters for which we obtain blow-up, there exists globally bounded admissible solutions.

Keywords: nonlinear parabolic equations, admissible solution, finite time blow-up, gradient term

1. Introduction

In [5] it is proved the existence and uniqueness of a class of solutions, named admissible solutions, for the problem

$$\begin{cases} u_t = \Delta u^m + g(u, \nabla u^m) & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) \geqslant 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

$u_0 \in C(\Omega)$, $\lim_{x\to\partial\Omega} u_0(x) = 0$, where Ω is a bounded domain in $\mathbb{R}^N$ of class $C^{2+\alpha}$ for certain $\alpha > 0$, $N \geqslant 1$, $m \geqslant 1$, $g : \mathbb{R}^{N+1} \to \mathbb{R}$ is continuous in $[0,+\infty)\times\mathbb{R}^N$ and locally Lipschitz in $(0,+\infty)\times\mathbb{R}^N$ with $g(0,0) \geqslant 0$ and

$$\left|g(v,w)\right| \leqslant h(v)\left(1+|w|^2\right),$$

$h : [0,+\infty) \to [0,+\infty)$ a nondecreasing function. Moreover, it is proved that the admissible solution exists in some maximal interval $[0, T_{\max})$, and if $T_{\max} < +\infty$ then

$$\lim_{t\to T_{\max}} \left\|u(t,\cdot)\right\|_\infty = \infty.$$

Our aim is to show the existence of blowing up admissible solutions for problem (1.1) with

$$g\left(u, \nabla u^m\right) = u^p - u^\mu \left\|\nabla u^m\right\|^q, \quad p \geqslant 1,\ 1 \leqslant q \leqslant 2 \text{ and } \mu \geqslant 0. \tag{1.2}$$

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When

$$1 \leqslant p < \mu + mq \quad \text{or} \quad m < p = \mu + mq \tag{1.3}$$

we will prove that $T_{\max} = +\infty$ and the global L^∞-boundedness of the admissible solutions. And we will prove that $T_{\max} < +\infty$, i.e., the existence of blowing up admissible solutions, under the complementary condition

$$1 \leqslant \mu + mq < p. \tag{1.4}$$

In the limit case $p = m$, $\mu = 0$ and $q = 1$, problem (1.1) may have global or blowing up solutions.

Problem (1.1) with the g-term like

$$u^p - \left\|\nabla u^{m+\mu/q}\right\|^q, \tag{1.5}$$

$m \geqslant 1$, $m/2 + \mu/q > 0$, $1 \leqslant q < 2$, was studied in [2], where it was proved the existence of global weak solutions for initial data in $L^\infty(\Omega)$ (more generally in $L^{m+1}(\Omega)$) when

$$1 \leqslant p < \max\{m, \mu + mq\}.$$

Moreover, existence of weak solutions for data in $L^1(\Omega)$ when $m = 1$, $q = 2$ and $1 \leqslant p < \mu + 2$, is studied in [1].

We remark that admissible solutions are weak solutions, but in general, weak solutions of parabolic equations with a gradient term are not unique (see [3]).

Problem (1.1) with $g(u, \nabla u^m) = u^p$ has been extensively studied (see, for instance, [11] and the references therein). It is known that if $p < m$ there exists a global mild solution for initial data $u_0 \in L^1(\Omega)$ and if $p > m$ solutions may blow-up in finite time.

For the semilinear case ($m = 1$), problem (1.1), (1.2) with $\mu = 0$ was introduced by Chipot and Weissler [4] in order to investigate the effect of a damping term on existence or nonexistence of solutions. On the other hand, Souplet in [13] proposes a model in population dynamics where this type of equations describes the evolution of the population density of a biological species under the effect of certain natural mechanism, see also [2]. For this semilinear case, several authors have studied the existence of nonglobal positive solutions, giving conditions for blow-up under certain assumptions on p, q, N and Ω (see, for instance, [4,8,6,9,10,12–15]).

Now, in the degenerate case, the blow-up results of Souplet and Weissler [15] (see also [16,17]) imply that problem (1.1), (1.2) does not admit global *classical solutions* in the following case

$$p > \mu + mq \quad \text{and} \quad m \geqslant 2.$$

We shall use the method of comparison with suitable blowing up self-similar subsolutions introduced by Souplet and Weissler in [15]. Concerning this, we want to stress that the diffusion term Δu^m degenerates for $u = 0$, so that one can not expect in general an existence result for classical solutions, and it is in the framework of weak solutions, or admissible solutions, where this problem makes sense. Remark that this method of comparison may be used for admissible solutions but not in general for weak solutions, indeed weak solutions are not unique in general and the comparison principle does not apply.

2. Results

We start with the definition of admissible solution given in [5] for problem (1.1). For an open set $O \subset \mathbb{R}^{N+1}$ we define its parabolic boundary as

$$\partial_P O = \{(t,x) \in \partial O\text{: there exists } (\tau, y) \in O,\ \tau > t\}.$$

Definition 2.1. We say that u is an admissible solution of equation

$$u_t = \Delta u^m + g(u, \nabla u^m) \quad \text{in } (0,T) \times \Omega \tag{2.1}$$

if u is continuous on $[0,T) \times \overline{\Omega}$ and

(i) for any $v \in C^{1,2}(O) \cap C(\overline{O})$, $v > 0$ on O, where O is an open subset of $(0,T) \times \Omega$, satisfying

$$\inf_{(t,x)\in O} \{[v_t - \Delta v^m - g(v, \nabla v^m)](t,x)\} > 0 \quad \text{on } O \quad \text{and} \quad v > u \quad \text{on } \partial_P O,$$

we have

$$v \geqslant u \quad \text{on } O,$$

and

(ii) for any $v \in C^{1,2}(O) \cap C(\overline{O})$, $v > 0$ on O, satisfying

$$v_t - \Delta v^m - g(v, \nabla v^m) \leqslant 0 \quad \text{on } O \quad \text{and} \quad v \leqslant u \quad \text{on } \partial_P O,$$

we have

$$v \leqslant u \quad \text{on } O.$$

Consider the problem

$$\begin{cases} u_t = \Delta u^m - u^\mu \|\nabla u^m\|^q + u^p & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) \geqslant 0 & \text{in } \Omega, \end{cases} \tag{2.2)(u_0}$$

$u_0 \in C(\Omega)$, $\lim_{x\to\partial\Omega} u_0(x) = 0$, $m \geqslant 1$, $p \geqslant 1$, $1 \leqslant q \leqslant 2$ and $\mu \geqslant 0$. In [5] it is proved the existence and uniqueness of admissible solutions of problem (2.2)(u_0) (that is, an admissible solution of the equation $u_t = \Delta u^m - u^\mu \|\nabla u^m\|^q + u^p$ satisfying the boundary and the initial conditions) defined on a maximal interval $[0, T_{\max})$ such that if $T_{\max} < \infty$ then

$$\lim_{t\to T_{\max}} \|u(t,\cdot)\|_\infty = \infty. \tag{2.3}$$

Let see first the global existence result.

Theorem 2.2. *Under the assumptions*

$$1 \leqslant p < \mu + mq \quad \textit{or} \quad m < p = \mu + mq, \tag{2.4}$$

for every initial datum $u_0 \in C(\Omega)$, $u_0 \geqslant 0$, $\lim_{x \to \partial\Omega} u_0(x) = 0$, *the problem* (2.2)($u_0$) *admits a unique globally bounded admissible solution in* $[0, +\infty)$.

Proof. The case $p < m$ is a consequence of [11, Theorem 1.3] and the maximum principle for solutions of approached problems (see [5]). So consider the case $p \geqslant m$.

Let us define

$$Pw = w_t - \Delta w^m + w^\mu \|\nabla w^m\|^q - w^p$$

and consider the following positive C^2-function on $[0, T] \times \overline{\Omega}$

$$w(t, x) = C\,\mathrm{e}^{a \cdot x},$$

where C is a positive constant to be determined and a is a vector in $\mathbb{R}^N$ to be also determined.

Computing Pw we have

$$Pw = -m^2 C^m \|a\|^2\,\mathrm{e}^{ma \cdot x} + C^{\mu+mq} m^q \|a\|^q\,\mathrm{e}^{(\mu+mq)a \cdot x} - C^p\,\mathrm{e}^{pa \cdot x}.$$

Now, using condition (2.4), for $\|a\| > 1/m$ and C large enough it follows that

$$Pw > 0.$$

Taking C large enough, if necessary, it follows from Definition 2.1(i), using w as test function, that

$$0 \leqslant u \leqslant w.$$

And this finishes the proof. $\square$

Our blow-up result states that

$$T_{\max} < +\infty$$

for admissible solutions of problem (2.2) with exponents in the complementary range of (2.4) for initial data large enough.

Theorem 2.3. *Under the assumptions*

$$1 \leqslant \mu + mq < p, \tag{2.5}$$

given an initial datum $u_0 \in C(\Omega)$, $u_0 \geqslant 0$, $\lim_{x \to \partial\Omega} u_0(x) = 0$, *there exists* $\lambda_0 > 0$ (*depending on* u_0) *such that for all* $\lambda > \lambda_0$, *the admissible solution of problem* (2.2)(λu_0) *blows-up in finite time* $T_{\max}$. *Moreover,*

$$\lim_{\lambda \to +\infty} \lambda^{p-1} T_{\max}(\lambda u_0) = \frac{1}{(p-1)\|u_0\|_\infty^{p-1}}. \tag{2.6}$$

Proof. By translation, one can assume without loss of generality that $0 \in \Omega$ and $u_0(0) = \max_{x\in\Omega} u_0(x)$.

We seek an unbounded self-similar subsolution of problem (2.2)(λu_0) on $[t_0, 1/\varepsilon)\times\mathbb{R}^N$, $0 < t_0 < 1/\varepsilon$, of the form

$$v(t,x) = \frac{1}{(1-\varepsilon t)^{1/(p-1)}} V\left(\frac{\|x\|}{(1-\varepsilon t)^{\sigma}}\right), \tag{2.7}$$

where V is the function on $[0,\infty)$ given by

$$V(y) = \left((1-y^2)^+\right)^n$$

$n > 2$, $\sigma > 0$, ε and t_0 to be determined.

Let us consider

$$Pw = w_t - \Delta w^m + w^{\mu}\left\|\nabla w^m\right\|^q - w^p.$$

If we take $y = \|x\|/(1-\varepsilon t)^{\sigma}$, computing Pv, we get

$$\begin{aligned}
&Pv(t,x)\\
&\quad= \frac{1}{(1-\varepsilon t)^{p/(p-1)}}\varepsilon\left(\frac{1}{p-1}V(y) + \sigma y V'(y)\right) && (2.8)\\
&\qquad+ \frac{1}{(1-\varepsilon t)^{(\mu+mq)/(p-1)+\sigma q}} m^q V^{\mu+(m-1)q}(y)\left|V'(y)\right|^q && (2.9)\\
&\qquad+ \frac{1}{(1-\varepsilon t)^{m/(p-1)+2\sigma}} m V^{m-1}(y)\left(-V''(y)\right) && (2.10)\\
&\qquad+ \frac{1}{(1-\varepsilon t)^{m/(p-1)+2\sigma}} m V^{m-1}(y)\left(-\frac{V'(y)}{y}\right)(N-1) && (2.11)\\
&\qquad- \frac{1}{(1-\varepsilon t)^{m/(p-1)+2\sigma}} m(m-1) V^{m-1}(y)\frac{V'(y)^2}{V(y)} && (2.12)\\
&\qquad- \frac{1}{(1-\varepsilon t)^{p/(p-1)}} V^p(y). && (2.13)
\end{aligned}$$

Let $0 < r < 1/\sqrt{2}$,

$$0 < \sigma < \min\left\{\frac{1}{q}\left(\frac{p}{p-1} - \frac{\mu+mq}{p-1}\right), \frac{p-m}{2(p-1)}\right\},$$

and

$$n = 2 + \frac{1-r^2}{2\sigma r^2(p-1)} + \frac{q}{\mu+mq}.$$

Then, in the case $0 \leqslant y \leqslant r$, we have $Pv(t,x) \leqslant 0$, for

$$\varepsilon = (p-1)\big(1-2r^2\big)^{n(p-1)}$$

and t_0 sufficiently close to $1/\varepsilon$, taking into account the dominating negative term (2.13).

Considering now $r < y \leqslant 1$, we have

$$
\begin{aligned}
&Pv(t,x)\\
&\quad = -\frac{1}{(1-\varepsilon t)^{p/(p-1)}}\varepsilon\sigma 2y^2 n\big(1-y^2\big)^{n-1} &(2.14)\\
&\qquad + \frac{1}{(1-\varepsilon t)^{p/(p-1)}}\frac{\varepsilon}{p-1}\big(1-y^2\big)^{n} &(2.15)\\
&\qquad + \frac{1}{(1-\varepsilon t)^{(\mu+mq)/(p-1)+\sigma q}} m^q n^q 2^q y^q \big(1-y^2\big)^{(\mu+mq)n-q} &(2.16)\\
&\qquad - \frac{1}{(1-\varepsilon t)^{m/(p-1)+2\sigma}} mn(n-1)4y^2\big(1-y^2\big)^{nm-2} &(2.17)\\
&\qquad + \frac{1}{(1-\varepsilon t)^{m/(p-1)+2\sigma}} mNn2\big(1-y^2\big)^{nm-1} &(2.18)\\
&\qquad - \frac{1}{(1-\varepsilon t)^{m/(p-1)+2\sigma}} m(m-1)n^2 4y^2\big(1-y^2\big)^{nm-2} &(2.19)\\
&\qquad - \frac{1}{(1-\varepsilon t)^{p/(p-1)}}\big(1-y^2\big)^{np}. &(2.20)
\end{aligned}
$$

Now, obviously $(2.17) + (2.19) + (2.20) \leqslant 0$. Moreover, since

$$n > \frac{1-r^2}{2\sigma r^2(p-1)} + \frac{q}{\mu+mq},$$

if t_0 is sufficiently close to $1/\varepsilon$, taking into account the dominating negative term (2.14), we have $(2.14) + (2.15) + (2.16) + (2.18) \leqslant 0$. Consequently, we obtain that $Pv(t,x) \leqslant 0$ in the case $r < y \leqslant 1$. Finally, it is obvious that $Pv(t,x) \leqslant 0$ if $y > 1$. Therefore, we have obtained that $Pv(t,x) \leqslant 0$.

On the other hand, since u_0 is continuous, given $C = u_0(0)(1-r)$, there exists a ball $B(0,\rho) \subset \Omega$ such that

$$u_0(x) \geqslant C \quad \text{for all } x \in B(0,\rho).$$

Taking t_0 still closer to $1/\varepsilon$, if necessary, one can assume that

$$\operatorname{supp}\big(v(t,\cdot)\big) \subset B\big(0,(1-\varepsilon t_0)^{\sigma}\big) \subset B(0,\rho) \quad \text{for all } t \in \left[t_0, \frac{1}{\varepsilon}\right).$$

Then, for all $\lambda \geqslant \lambda_0 = 1/(C(1-\varepsilon t_0)^{1/(p-1)})$, it yields

$$\lambda u_0(x) \geqslant \frac{1}{(1-\varepsilon t_0)^{1/(p-1)}} \geqslant v(t_0,x) \quad \text{for all } x \in B(0,\rho),$$

and then

$$\lambda u_0(x) \geqslant v(t_0, x) \quad \text{for all } x \in \Omega. \tag{2.21}$$

Moreover, $v(t,x) = 0$ for $(t,x) \in [t_0, 1/\varepsilon) \times \partial\Omega$.

Let u be the admissible solution of problem (2.2)(λu_0). For $0 < T < \min\{T_{\max}(\lambda u_0), 1/\varepsilon - t_0\}$, consider the open set

$$O_T = \{(t,x) \in (0,T) \times \Omega\colon\ v(t+t_0, x) > u(t,x)\}.$$

Let us see that O_T is empty. By Definition 2.1(ii) it is enough to see

$$v(t+t_0, x) \leqslant u(t,x) \quad \text{in } \partial_P O_T. \tag{2.22}$$

To see (2.22), assume first

$$(t,x) \in (0,T) \times \Omega \cap \partial_P O_T.$$

In this case, as both u and v are continuous we have $v(t+t_0,x) = u(t,x)$, thus (2.22) holds.

It remains to examine the case

$$(t,x) \in \partial\big((0,T) \times \Omega\big) \cap \partial_P O_T.$$

But in this case, (2.22) holds since v and u are null in $(0,T) \times \partial\Omega$ and (2.21) is satisfied for the initial time.

Since O_T is empty for all $0 < T < \min\{T_{\max}(\lambda u_0), 1/\varepsilon - t_0\}$, we have

$$v(t+t_0, x) \leqslant u(t,x) \quad \text{for all } x \in \Omega,\ 0 < t < \min\left\{T_{\max}(\lambda u_0), \frac{1}{\varepsilon} - t_0\right\},$$

which implies

$$T_{\max}(\lambda u_0) \leqslant \frac{1}{\varepsilon} - t_0,$$

since $\lim_{t \to 1/\varepsilon - t_0} v(t+t_0, 0) = +\infty$.

Let t_1 be such that $t_0 < t_1 < 1/\varepsilon$. Proceeding as before, we have that

$$T_{\max}(\lambda u_0) \leqslant \frac{1}{\varepsilon} - t_1 \quad \text{for } \lambda \geqslant \frac{1}{C(1-\varepsilon t_1)^{1/(p-1)}}.$$

Consequently, if

$$\lambda \geqslant \frac{1}{C(1-\varepsilon t_0)^{1/(p-1)}}, \qquad \lambda = \frac{1}{C(1-\varepsilon t_1)^{1/(p-1)}}$$

for some t_1 such that, $t_0 < t_1 < 1/\varepsilon$. Hence

$$T_{\max}(\lambda u_0) \leqslant \frac{1}{\varepsilon}\frac{1}{(\lambda C)^{p-1}} \quad \text{for } \lambda \geqslant \frac{1}{C(1-\varepsilon t_0)^{1/(p-1)}}. \tag{2.23}$$

For $\delta > 0$, we consider now

$$w_\delta(t,x) = \frac{(\lambda+\delta)u_0(0)}{(1-\nu t)^{1/(p-1)}}, \quad 0 \leqslant t < 1/\nu,$$

which is a C^2-function such that

$$Pw_\delta > 0$$

if $\nu > (p-1)(\lambda+\delta)^{p-1}u_0(0)^{p-1}$. Then, since $w_\delta > u$ in $\partial_p(\Omega \times (0,T))$, by Definition 2.1(i), it follows that

$$u(t,x) \leqslant w_\delta(t,x) \quad \text{for } 0 \leqslant t < \min\left\{T_{\max}(\lambda u_0), \frac{1}{\nu}\right\},$$

and therefore, as u blows up in $T_{\max}(\lambda u_0)$, $1/\nu \leqslant T_{\max}(\lambda u_0)$. Then taking

$$\nu = (p-1)(\lambda+2\delta)^{p-1}u_0(0)^{p-1},$$

we get

$$T_{\max}(\lambda u_0) \geqslant \frac{1}{(p-1)(\lambda+2\delta)^{p-1}u_0(0)^{p-1}},$$

and letting $\delta \to 0^+$, we obtain

$$\frac{1}{(p-1)(\lambda u_0(0))^{p-1}} \leqslant T_{\max}(\lambda u_0). \tag{2.24}$$

Using now (2.23) and (2.24), letting r to 0 it follows that

$$\lim_{\lambda\to+\infty} \lambda^{p-1}T_{\max}(\lambda u_0) = \frac{1}{(p-1)\|u_0\|_\infty^{p-1}}. \qquad \square$$

Remark 2.4. A similar formula like (2.6) was proved using the eigenfunction method by Gui and Wang [7] for the Cauchy problem (i.e., $\Omega = \mathbb{R}^N$) for the semilinear heat equation. In the semilinear case for Ω bounded, Souplet and Weissler in [15] obtain the estimate

$$\frac{1}{(p-1)(\lambda\|u_0\|_\infty)^{p-1}} \leqslant T_{\max}(\lambda u_0) \leqslant \frac{C_1}{(\lambda\|u_0\|_\infty)^{p-1}},$$

where C_1 is a constant depending on the parameters of the equation, N and Ω.

In the critical case, that is for $\mu = 0$, $q = 1$ and $p = m \geqslant 1$, for some parameter $\alpha, \beta > 0$, we consider the problem

$$\begin{cases} u_t = \Delta u^m - \alpha \|\nabla u^m\| + \beta u^m & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) \geqslant 0 & \text{in } \Omega. \end{cases} \tag{2.25}$$

We are going to see now that in this critical case, depending on the relations between the parameters α and β, we get globally bounded admissible solutions or blowing up admissible solutions.

Theorem 2.5.

(i) *Under the assumption*

$$\alpha > 2\sqrt{\beta}, \tag{2.26}$$

for every initial datum $u_0 \in C(\Omega)$, $u_0 \geqslant 0$, $\lim_{x\to\partial\Omega} u_0(x) = 0$, *the problem* (2.25) *admits a unique globally bounded admissible solution in* $[0, +\infty)$.

(ii) *Assume that* $m > 1$. *Given* $\alpha_0 > 0$, *there exists* $\beta_0 > 0$, *depending on* α_0, N, m *and* Ω, *such that, if* $0 < \alpha \leqslant \alpha_0$ *and* $\beta \geqslant \beta_0$, *then for initial data* u_0 *large enough the admissible solution of problem* (2.25) *blows up in finite time.*

Proof. (i) Let P be the differential operator

$$Pw = w_t - \Delta w^m + \alpha\|\nabla w^m\| - \beta w^m$$

and consider the following positive C^2-function on $[0,T] \times \overline{\Omega}$

$$w(t,x) = C\,\mathrm{e}^{a\cdot x},$$

where C is a positive constant to be determined and a is a vector in $\mathbb{R}^N$ to be also determined. Computing Pw we have

$$Pw = -m^2 C^m \|a\|^2\,\mathrm{e}^{ma\cdot x} + \alpha C^m m\|a\|\,\mathrm{e}^{ma\cdot x} - \beta C^m\,\mathrm{e}^{ma\cdot x}.$$

Then, $Pw > 0$ if $\alpha > (\beta + m^2\|a\|^2)/(m\|a\|)$. Now, since $s = \sqrt{\beta}/m$ is a minimum of the function $s \mapsto (\beta + m^2 s^2)/(ms)$, choosing $\|a\| = \sqrt{\beta}/m$, we have that $Pw > 0$ if $\alpha > 2\sqrt{\beta}$. Taking C large enough, if necessary, it follows from Definition 2.1(i) that

$$0 \leqslant u \leqslant w.$$

(ii) Consider again the differential operator

$$Pw = w_t - \Delta w^m + \alpha\|\nabla w^m\| - \beta w^m.$$

Let B the greatest open ball contained in Ω. By translation, one can assume without loss of generality that $B = B(0, A)$, $A > 0$.

We seek an unbounded self-similar subsolution of problem (2.25) on $[0, 1/\varepsilon) \times \mathbb{R}^N$ of the form

$$v(t,x) = \frac{1}{(1-\varepsilon t)^{1/(m-1)}} \left(\left(A^2 - \|x\|^2\right)^+\right)^n, \tag{2.27}$$

where $1/m < n \leqslant 2/(m-1)$, and ε is a positive constant to be determined. If we take $y = \|x\|$, in the case $0 \leqslant y < A$, computing Pv, we get

$$Pv(t,x)$$
$$= \frac{1}{(1-\varepsilon t)^{m/(m-1)}} \left(\varepsilon \frac{1}{m-1} \left(A^2 - y^2\right)^n\right) \tag{2.28}$$
$$+ \frac{1}{(1-\varepsilon t)^{m/(m-1)}} \left(2mn(\alpha y + N)\left(A^2 - y^2\right)^{nm-1}\right) \tag{2.29}$$
$$- \frac{1}{(1-\varepsilon t)^{m/(m-1)}} \left(4nm(nm-1)y^2\left(A^2 - y^2\right)^{nm-2}\right) \tag{2.30}$$
$$- \frac{1}{(1-\varepsilon t)^{m/(m-1)}} \left(\beta\left(A^2 - y^2\right)^{nm}\right). \tag{2.31}$$

Let us see now that $Pv \leqslant 0$. Observe that the negative term (2.30) has not contribution when y is close to 0, thus we will use the dominating negative term (2.31) when y is close to 0 and the dominating negative term (2.30) when y is close to A. Fix some r, $0 < r < A$. Then, we have

$$(2.28) + \frac{1}{2}(2.31) \leqslant 0 \quad \text{if } \varepsilon \leqslant \frac{m-1}{2}\beta\left(A^2 - r^2\right)^{n(m-1)} \text{ and } 0 \leqslant y \leqslant r, \tag{2.32}$$

and, since $m \leqslant 2/n + 1$, it follows that

$$\begin{aligned} &(2.28) + \frac{1}{2}(2.30) \leqslant 0 \\ &\quad \text{if } \varepsilon \leqslant 2(m-1)nm(nm-1)r^2\left(A^2 - r^2\right)^{n(m-1)-2} \text{ and } r \leqslant y < A. \end{aligned} \tag{2.33}$$

On the other hand, to get $(2.29) + \frac{1}{2}(2.31) \leqslant 0$, we need that

$$2mn(N + \alpha y) \leqslant \frac{\beta}{2}\left(A^2 - y^2\right),$$

or equivalently, that

$$\beta y^2 + 4mn\alpha y + 4mnN - \beta A^2 \leqslant 0. \tag{2.34}$$

Let

$$y_{\alpha,\beta} := \frac{-2mn\alpha + \sqrt{4m^2n^2\alpha^2 - 4mnN\beta + \beta^2 A^2}}{\beta}.$$

If $\beta > 4mnN/A^2$, then $0 < y_{\alpha,\beta} < A$, and (2.34) holds for $0 \leqslant y \leqslant y_{\alpha,\beta}$. Hence, we obtain that

$$(2.29) + \frac{1}{2}(2.31) \leqslant 0 \quad \text{if } \beta > \frac{4mnN}{A^2} \text{ and } 0 \leqslant y \leqslant y_{\alpha,\beta}. \tag{2.35}$$

To get (2.29) $+ \frac{1}{2}$(2.30) $\leqslant 0$, we need that

$$(\alpha y + N)(A^2 - y^2) \leqslant (nm-1)y^2,$$

or equivalently, that

$$Q_\alpha(y) := \alpha y^3 + \big((mn-1)+N\big)y^2 - \alpha A^2 y - NA^2 \geqslant 0.$$

Since Q_α is a polynomial of degree three,

$$Q_\alpha(0) = -NA^2 < 0, \quad Q_\alpha(-A) = (nm-1)A^2 > 0 \quad \text{and} \quad Q_\alpha(A) = (nm-1)A^2 > 0,$$

there exists a root x_α of the polynomial Q_α, such that $0 < x_\alpha < A$ and $Q_\alpha(y) > 0$ for all $y > x_\alpha$. Now, since $\beta > 4mnN/A^2$, we have $y_{\alpha_1,\beta} \geqslant y_{\alpha_2,\beta}$ if $\alpha_1 \leqslant \alpha_2$. Moreover, if $\alpha_1 \leqslant \alpha_2$, and we consider the function

$$f(\alpha) := Q_\alpha(x_{\alpha_2}) = \alpha x_{\alpha_2}^3 + \big((mn-1)+N\big)x_{\alpha_2}^2 - \alpha A^2 x_{\alpha_2} - NA^2,$$

we have

$$f'(\alpha) = x_{\alpha_2}^3 - A^2 x_{\alpha_2} \leqslant 0.$$

Consequently, $f(\alpha_1) \geqslant f(\alpha_2) = 0$. Thus, $x_{\alpha_1} \leqslant x_{\alpha_2}$. On the other hand,

$$\lim_{\beta \to +\infty} y_{\alpha,\beta} = A.$$

Hence, if we fixe $\alpha_0 > 0$, there exists $\beta_0 > 4mnN/A^2$ such that $y_{\alpha,\beta} \geqslant x_\alpha$ for all $0 < \alpha \leqslant \alpha_0$ and all $\beta \geqslant \beta_0$. Thus, we obtain that

$$(2.29) + \frac{1}{2}(2.30) \leqslant 0 \quad \text{if } \beta \geqslant \beta_0,\ 0 < \alpha \leqslant \alpha_0 \text{ and } y_{\alpha,\beta} \leqslant y < A. \tag{2.36}$$

From (2.32), (2.33), taking $r = y_{\alpha,\beta}$, (2.35) and (2.36), it follows that $Pv \leqslant 0$ if

$$\beta \geqslant \beta_0, \qquad 0 < \alpha \leqslant \alpha_0$$

and

$$0 < \varepsilon \leqslant \min\left\{\frac{m-1}{2}\beta\big(A^2 - r^2\big)^{n(m-1)}, 2(m-1)nm(nm-1)r^2\big(A^2 - r^2\big)^{n(m-1)-2}\right\}.$$

Let $u_0 \in C(\Omega)$, $u_0 \geqslant 0$, $\lim_{x\to\partial\Omega} u_0(x) = 0$, such that

$$v(0,x) \leqslant u_0(x) \quad \text{for all } x \in \Omega, \tag{2.37}$$

and let u be the admissible solution of problem (2.25) for the initial datum u_0. Let us see that u blows-up in finite time. To do that we shall prove that $T_{\max}(u_0) < 1/\varepsilon$. In fact: for $0 < T < \min\{T_{\max}(u_0), 1/\varepsilon\}$, consider the open set

$$O_T = \{(t,x) \in (0,T) \times \Omega\colon\ v(t,x) > u(t,x)\}.$$

Since $v > 0$ on this open set, $v \in C^{1,2}(O_T)$ and $Pv \leqslant 0$. Let us see that O_T is empty. By Definition 2.1(ii) it is enough to see

$$v(t,x) \leqslant u(t,x) \quad \text{in } \partial_P O_T. \tag{2.38}$$

To see (2.38), assume first

$$(t,x) \in (0,T) \times \Omega \cap \partial_P O_T.$$

In this case, as both u and v are continuous we have $v(t,x) = u(t,x)$, thus (2.38) holds.

It remains to examine the case

$$(t,x) \in \partial\big((0,T) \times \Omega\big) \cap \partial_P O_T.$$

But in this case, (2.38) holds since v and u are null in $(0,T) \times \partial\Omega$ and (2.37) is satisfied for the initial time.

Since O_T is empty for all $0 < T < \min\{T_{\max}(u_0), 1/\varepsilon\}$, we have

$$v(t,x) \leqslant u(t,x) \quad \text{for all } x \in \Omega,\ 0 < t < \min\left\{T_{\max}(u_0), \frac{1}{\varepsilon}\right\},$$

which implies

$$T_{\max}(u_0) \leqslant \frac{1}{\varepsilon},$$

since $\lim_{t\to 1/\varepsilon} v(t,0) = +\infty$. $\square$

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