Nonlocal p-laplacian type operators with Dirichlet boundary conditions & Applications

Julián Toledo

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Joint works with F. Andreu, J. M. Mazón and J. D. Rossi

Let $J:\mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial, continuous function, strictly positive in $B(0,1)$, vanishing in $\mathbb{R}^N \setminus B(0,1)$ and such that $\int_{\mathbb{R}^N} J(z)\,dz = 1$.
Let $u(x,t)$ a density of a population at position x at the time t. This population moves from location y to x using the probability distribution $J(x-y)$.

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We can think in non linear diffusion equations and put boundary conditions as follows:

$$\begin{cases} u_t(t,x) = \displaystyle\int_D J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))\,dy, \\ \hfill (t,x) \in (0,T)\times\Omega, \\ \text{B.C.} \\ u(0,x) = u_0(x), \ \ x \in \Omega. \end{cases}$$

B.C.:
Cauchy problem: $\Omega = D = \mathbb{R}^N$;
Homogeneous Neumann boundary conditions: $D = \Omega$ bounded domain;
Dirichlet boundary conditions: Ω bounded domain and
$D = \Omega_1 := \Omega + B(0,1)$,

$$u(t,x) = \psi(x), \quad (t,x) \in (0,T) \times (\Omega_1 \setminus \overline{\Omega}).$$

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When dealing with local problems usually the boundary datum is taken in the sense of traces. However, in the nonlocal formulation we do not impose any continuity between the values of u inside Ω and outside it, ψ.

Consider the operator

$$\mathrm{B}^J_{p,\psi}(u)(x) = -\int_{\Omega_1} J(x-y)|u_\psi(y)-u(x)|^{p-2}(u_\psi(y)-u(x))\,dy, \quad x \in \Omega,$$

where

$$u_\psi(x) := \begin{cases} u(x) & \text{if } \ x \in \Omega, \\ \psi(x) & \text{if } \ x \in \Omega_1 \setminus \overline{\Omega}. \end{cases}$$

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Then, we can rewrite the above problem as

$$P^J_{p,\psi}(u_0) \quad \begin{cases} u_t(x,t) + B^J_{p,\psi}(u)(x,t) = 0, & x \in \Omega,\ 0 < t < T, \\ u(0) = u_0. \end{cases}$$

Definition of solution

A *solution* in $[0, T]$ of the Dirichlet nonlocal problem $P^J_{p,\psi}(u_0)$ is a function

$$u \in W^{1,1}((0,T); L^1(\Omega))$$

which satisfies $u(0,x) = u_0(x)$ *a.e.* $x \in \Omega$ and

$$u_t(t,x) \quad = \int_{\Omega_1} J(x-y)|u_\psi(t,y) - u(t,x)|^{p-2}(u_\psi(t,y) - u(t,x))\, dy$$

a.e in $(0,T) \times \Omega$.

Existence and uniqueness. Contraction principle

Theorem

Let $u_0 \in L^p(\Omega)$ and $\psi \in L^p(\Omega_1 \setminus \overline{\Omega})$. Then, there exists a unique solution of $P^J_{p,\psi}(u_0)$.

Moreover, if $u_{i0} \in L^1(\Omega)$, $i = 1, 2$, and u_i is a solution in $[0, T]$ of $P^J_{p,\psi}(u_{i0})$. Then

$$\int_\Omega (u_1(t) - u_2(t))^+ \leq \int_\Omega (u_{10} - u_{20})^+, \qquad \text{for every } t \in (0, T).$$

If $u_{i0} \in L^p(\Omega)$, $i = 1, 2$, then

$$\|u_1(t) - u_2(t)\|_{L^p(\Omega)} \leq \|u_{10} - u_{20}\|_{L^p(\Omega)}, \qquad \text{for every } t \in (0, T).$$

Proof: We use Nonlinear Semigroup Theory

This means to discretize in time:

$$\frac{u(t_i) - u(t_{i-1})}{t_i - t_{i-1}} + B^J_{p,\psi}(u(t_i, .)) = 0,$$

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And for this we proof:

Theorem

• *A Poincaré's type inequality: Given p, Ω, ψ and J there exists $\lambda > 0$ such that*

$$\lambda \int_\Omega |u| \le \int_\Omega \int_{\Omega_1} J(x-y)|u_\psi(y) - u(x)|^p dydx + \int_{\Omega_1 \setminus \Omega} |\psi|^p, \ \forall u \in L^p(\Omega).$$

• *$B^J_{p,\psi}$ is completely accretive with is dense in $L^p(\Omega)$ and verifies the range condition*

$$L^p(\Omega) \subset Ran(I + B^J_{p,\psi}).$$

Rescaling

For Ω an smooth bounded domain and $\tilde{\psi} \in L^\infty(\partial\Omega) \cap W^{1/p',p}(\partial\Omega)$, the solutions of

$$D_{p,\tilde{\psi}}(u_0) \quad \begin{cases} u_t = \Delta_p u & \text{in } (0,T) \times \Omega, \\ u = \tilde{\psi} & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

can be approximated by solutions of a sequence of Dirichlet nonlocal *p-Laplacian* problems:

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can be approximated by solutions of a sequence of Dirichlet nonlocal *p-Laplacian* problems:

Consider $J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right)$, $C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p\, dz.$

Assume $J(x) \geq J(y)$ if $|x| \leq |y|$. Then:

Theorem

For $T > 0$, $u_0 \in L^p(\Omega)$, $\psi \in L^\infty(\Omega_1) \cap W^{1,p}(\Omega_1)$ such that $\psi_{|\partial\Omega} = \tilde{\psi}$, u_ε the unique solution of $P_{p,\psi}^{J_{p,\varepsilon}}(u_0)$ and u the unique solution of $D_{p,\tilde{\psi}}(u_0)$,

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_\varepsilon(t,.) - u(t,.)\|_{L^p(\Omega)} = 0.$$

For the proof we need the following result

Based on a result of Bourgain, Brezis, Mironescu: Another look at Sobolev spaces, 2001.

Theorem

Let $1 < q < +\infty$, D a smooth bounded domain in $\mathbb{R}^N$ and $\rho : \mathbb{R}^N \to \mathbb{R}$ a nonnegative continuous radial function with compact support, non identically zero, $\rho(x) \geq \rho(y)$ if $|x| \leq |y|$. Set $\rho_n(x) := n^N \rho(nx)$.
Let $\{f_n\}$ be a sequence of functions in $L^q(D)$ such that

$$\int_D \int_D |f_n(y) - f_n(x)|^q \rho_n(y-x)\, dx\, dy \leq M \frac{1}{n^q}.$$

Then, there exists a subsequence $\{f_{n_k}\}$ such that

$$f_{n_k} \to f \quad in \quad L^q(D)$$

with $f \in W^{1,q}(D)$; and moreover

$$(\rho(z))^{1/q} \chi_D\left(x + \frac{1}{n}z\right) \frac{f_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \rightharpoonup_{L^q \times L^q} (\rho(z))^{1/q}\, z \cdot \nabla f.$$

The Prigozhin model for sandpiles

Prigozhin [Euro. J. Applied Mathematics, 1996] interprets

$$\begin{cases} f(.,t) - (v_\infty)_t(.,t) \in \partial \mathbb{I}_{K(u_0)}(v_\infty(.,t)), & \text{a.e. } t \in (0,T), \\ v_\infty(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

$$K(u_0) = \left\{ v \in W^{1,\infty}(\Omega) : \|\nabla v\|_\infty \le 1, \ v_{|\partial\Omega} = u_{0|\partial\Omega} \right\},$$

to explain the movement of a sandpile ($v_\infty(x,t)$ describes the amount of the sand at the point x at time t), the main assumption being that the sandpile is stable when the slope is less or equal than one and unstable if not.

Fix Ω a convex domain in $\mathbb{R}^N$ and consider the solution u_p of the Dirichlet *nonlocal p-Laplacian evolution problem* with source f

$$\begin{cases} u_t(t,x) = \displaystyle\int_{\Omega_1} J(x-y)|u_\psi(t,y)-u(t,x)|^{p-2}(u_\psi(t,y)-u(t,x))dy + f(t,x), \\ \hfill (t,x)\in(0,T)\times\Omega, \\ u(0,x)=u_0(x), \quad x\in\Omega. \end{cases}$$

Letting $p\to+\infty$ we obtain a solution u_∞ of

$$\begin{cases} f(t,.)-u_t(t,.)\in\partial\mathbb{I}_{K^1_{\infty,\psi}}(u(t,.)), \quad t\in\Omega, \\ u(0,x)=u_0(x), \quad x\in\Omega, \end{cases}$$

$$K^1_{\infty,\psi} := \left\{u\in L^2(\Omega) \ : |u(x)-u(y)|\le 1 \text{ for } x\in\Omega, y\in\Omega_1, \ |x-y|\le 1\right\},$$

for $\psi\in L^\infty(\Omega_1\setminus\overline{\Omega})$ such that $K^1_{\infty,\psi}\neq\emptyset$, $T>0$, $f\in L^2(0,T;L^\infty(\Omega))$, and $u_0\in L^\infty(\Omega)$ such that $u_0\in K^1_{\infty,\psi}$:

$$\lim_{p\to\infty}\ \sup_{t\in[0,T]} \|u_p(t,\cdot)-u_\infty(t,\cdot)\|_{L^2(\Omega)} = 0.$$

Recovering the Prigozhin model

For $\varepsilon > 0$ we consider the following rescaled functionals: $\mathbb{I}_{K^{\varepsilon}_{\infty,\psi}}$ with $K^{\varepsilon}_{\infty,\psi} := \left\{ u \in L^2(\Omega) \ : |u(x) - u(y)| \leq \varepsilon, \text{ for } x \in \Omega, y \in \Omega_\varepsilon, \ |x-y| \leq \varepsilon \right\}.$
And we prove:

Theorem

Let Ω a convex bounded domain in $\mathbb{R}^N$ and $T > 0$. For $f \in L^2(0,T;L^\infty(\Omega))$, $\psi \in W^{1,\infty}(\Omega_1 \setminus \overline{\Omega})$, $\|\psi\|_\infty \leq 1$, and $u_0 \in W^{1,\infty}(\Omega)$ such that $\|\nabla u_0\| \leq 1$, ${u_0}_{|\partial\Omega} = \psi_{|\partial\Omega}$, the solutions of

$$\begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial \mathbb{I}_{K^{\varepsilon}_{\infty}}(u(t)), & t \in (0,T), \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases}$$

converges, in $C([0,T] : L^2(\Omega))$, to the solution of Prigozhin model.

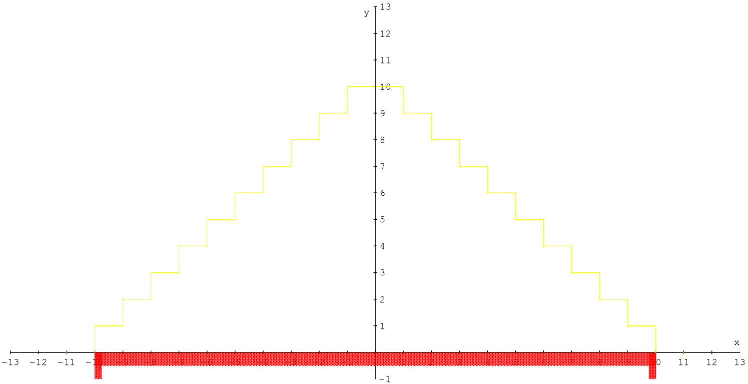
Nonlocal sand pile model

We are approximating the sandpile model of Prigozhin by a nonlocal model in which a configuration of sand is stable when its height u verifies $|u(x) - u(y)| \leq \varepsilon$ if $|x - y| \leq \varepsilon$. This is a sort of measure of how large is the size of irregularities of the sand; the sand can be completely irregular for sizes smaller than ε but it has to be arranged for sizes greater than ε.

Explicit solutions. Source

Let us assume that we are in an interval $\Omega = (-L, L)$, $\varepsilon = L/n$, $n \in \mathbf{N}$, $u_0 = 0$, homogeneous Dirichlet boundary conditions (a table!), and the source f is an approximation of a delta function,

$$f(t, x) = f_\eta(t, x) = \frac{1}{\eta}\chi_{[-\frac{\eta}{2}, \frac{\eta}{2}]}(x), \quad 0 < \eta \leq 2\varepsilon.$$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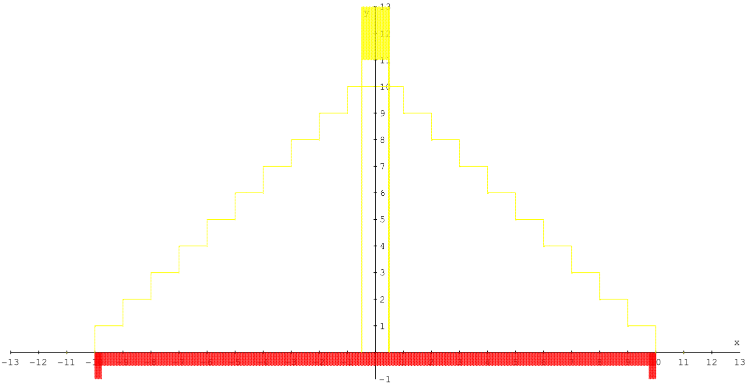
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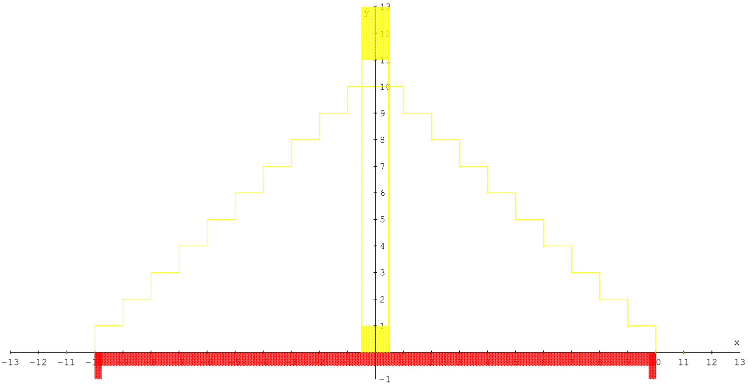
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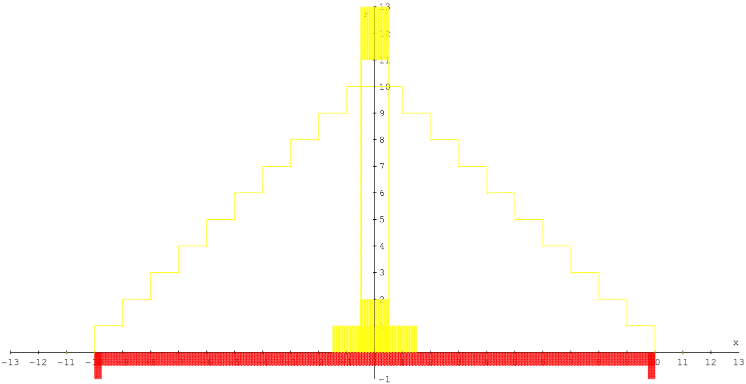
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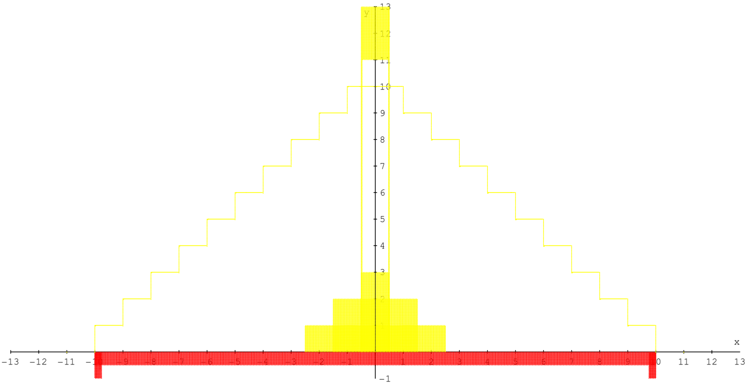
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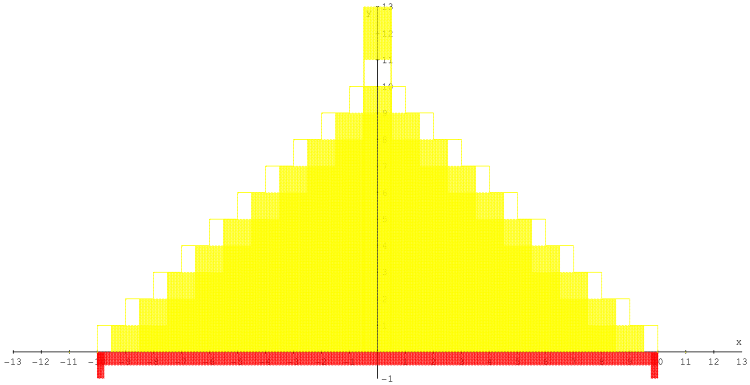
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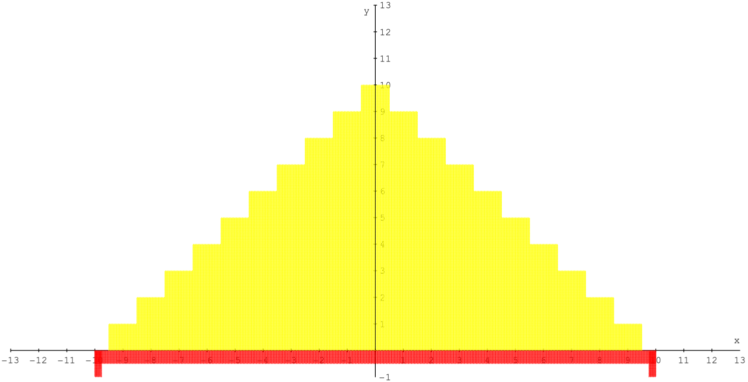
Limit as $\eta \to 0$

Taking limit as $\eta \to 0$, we get that the expected solution to $P^{\varepsilon}_{\infty,\psi}(u_0, \delta_0)$ is given by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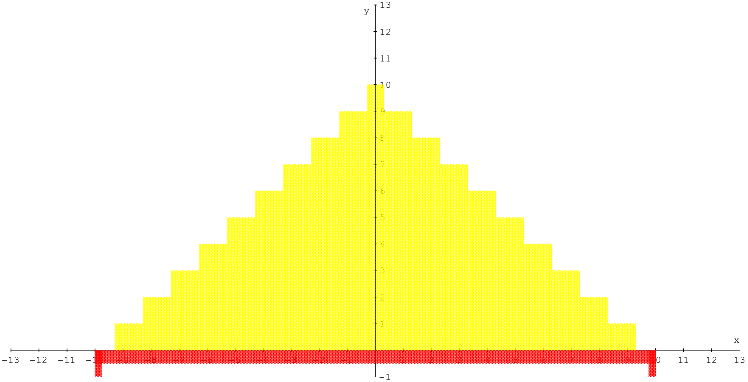
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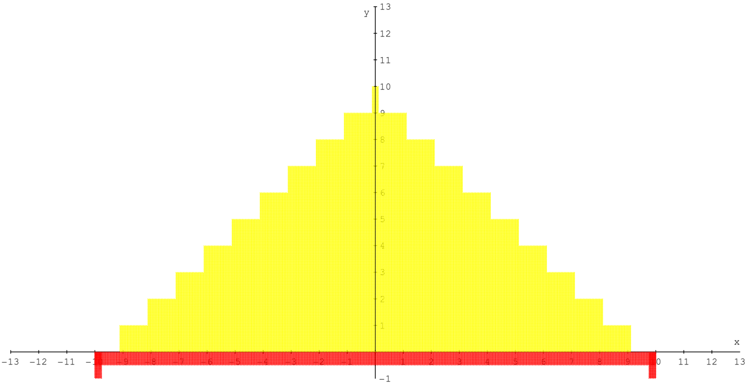
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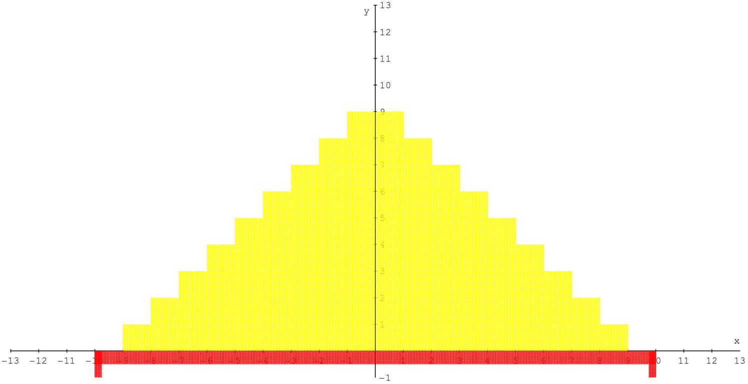
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Limit as $\eta \to 0$

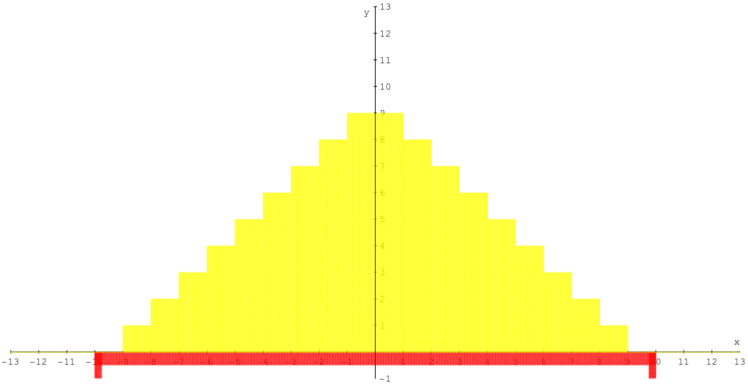
Remark that, since the space of functions $K^{\varepsilon}_{\infty,\psi}$ is not contained into $C(\mathbb{R})$, this solution has to be understood as a *generalized solution* to $P^{\varepsilon}_{\infty}(u_0, \delta_0)$ (it is obtained as a limit of solutions to approximating problems).

Recovering the sandpile model as $\varepsilon \to 0$

Taking limit as $\varepsilon \to 0$ in the previous example, we get that $u_\varepsilon(t,x) \to v(t,x)$, where

$$v(t,x) = (l - |x|)^+ \qquad \text{for } t = l^2,$$

(until the time at which $t = L^2$, and from that time the solution is stationary).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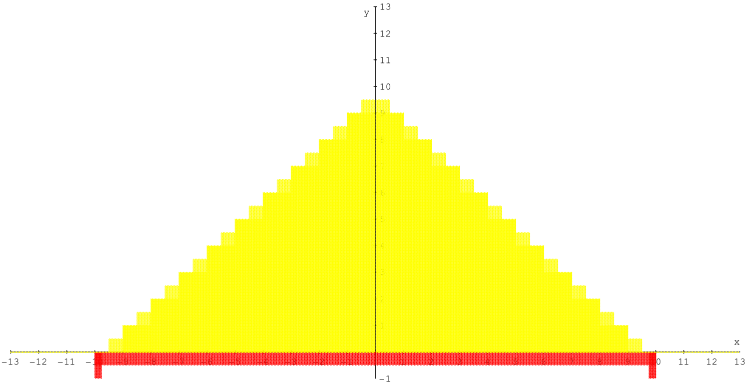
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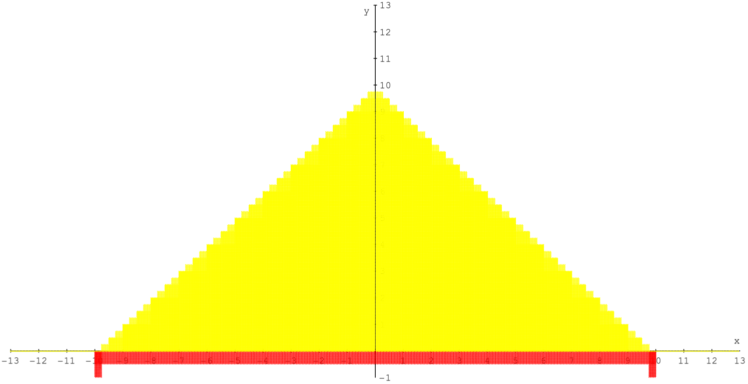
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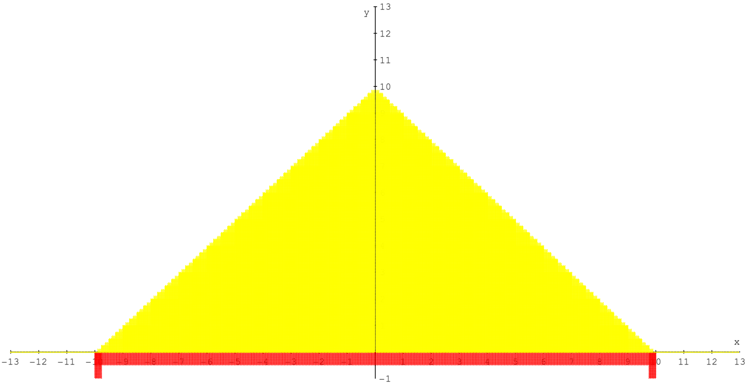
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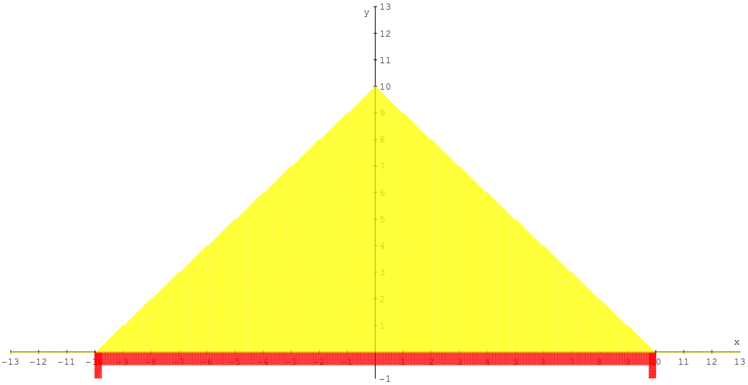
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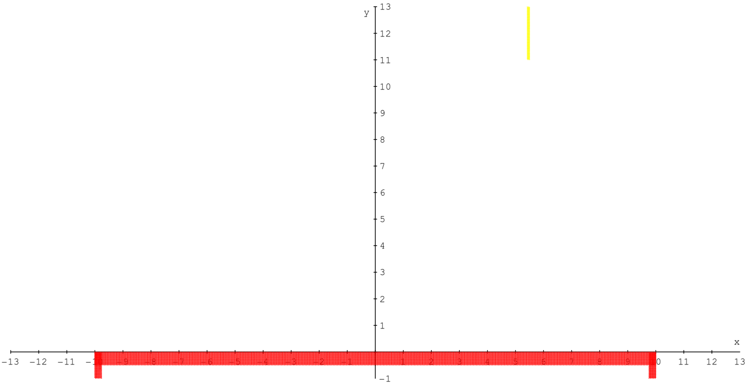
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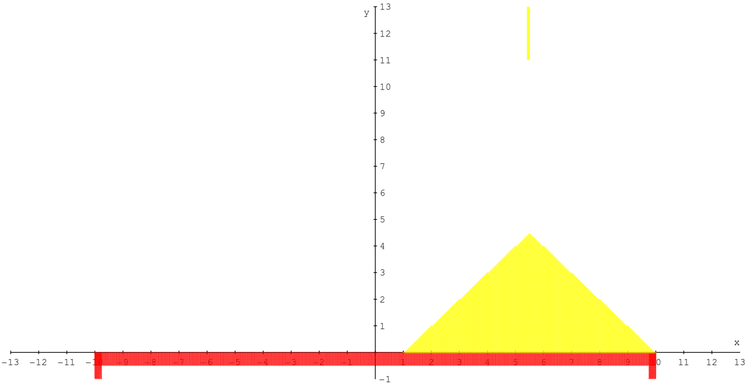
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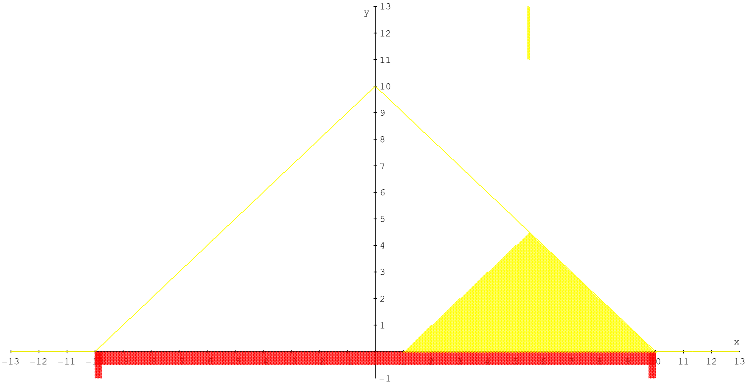
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A best Lipschitz extension problem

Theorem

- *For $p \geq 2$, there exists a unique $u_p^\varepsilon \in L^p(\Omega)$ such that*

$$\mathrm{B}^{J_\varepsilon}_{p,\psi}(u_p^\varepsilon) = 0.$$

This is, in fact, the asymptotic limit as t goes to $+\infty$ of the solutions of the Dirichlet nonlocal diffusion problems.

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This is, in fact, the asymptotic limit as t goes to $+\infty$ of the solutions of the Dirichlet nonlocal diffusion problems.

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A best Lipschitz extension problem

Theorem

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This is, in fact, the asymptotic limit as t goes to $+\infty$ of the solutions of the Dirichlet nonlocal diffusion problems.

- $u_p^\varepsilon \to u_\varepsilon \in L^\infty(\Omega)$ *strongly in any $L^q(\Omega)$ as $p \to +\infty$.*
- $(u_\varepsilon)_\psi$ *is the unique solution of*

$$\begin{cases} -\Delta_\infty^\varepsilon u = 0 & \text{in } \Omega, \\ u = \psi & \text{on } \Omega_\varepsilon \setminus \Omega, \end{cases}$$

where $\Delta_\infty^\varepsilon u(x) := \sup_{y \in \overline{B}_\varepsilon(x)} u(y) + \inf_{y \in \overline{B}_\varepsilon(x)} u(y) - 2u(x)$ is the discrete infinity Laplace operator. This is in fact the value function of a TUG-OF-WAR game.

Connection with the Lipschitz extensions

(Peres, Schramm, Sheffield and Wilson, 2006)

$$\lim_{\varepsilon \to 0} u_\varepsilon = h,$$

the absolutely minimizing Lispchitz extension (AMLE) of ψ to $\overline{\Omega}$,

that is, G. Aronsson (1967), $h : \overline{\Omega} \to \mathbb{R}$ such that:

- $h_{|\partial\Omega} = \psi$ and $L_d(h, \overline{\Omega}) = L_d(\psi, \partial\Omega)$ (this is, h is a minimal Lipschitz extension of ψ to $\overline{\Omega}$: $h \in \mathrm{MLE}_d(\psi, \overline{\Omega})$),

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• for every open set $D \subset\subset \Omega$,

$$L_d(h, D) \leq L_d(v, D) \qquad \forall v : \; h_{|\partial D} = v_{|\partial D}.$$

To obtain this AMLE extension of a datum ψ on the boundary, Aronsson proposed to take the limit as $p \to \infty$ in

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Are $(u_\varepsilon)_\psi$ *the best Lipschitz extensions with respect to some distance*?

The distance to be considered is the discrete distance

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \varepsilon & \text{if } 0 < |x-y| \leq \varepsilon, \\ 2\varepsilon & \text{if } \varepsilon < |x-y| \leq 2\varepsilon, \\ \vdots & \end{cases}$$

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We want to remark that $(\Omega_\varepsilon, d_\varepsilon)$ is not a separable length space and the boundary of any subset for this metric is empty. Then, even the results of Juutinen (2002), which extend those of Aronsson, does not apply here.

Given $u : \Omega_\varepsilon \to \mathbb{R}$ and $D \subset \Omega$,

$$L_\varepsilon(u, D) \quad := \sup_{\substack{x \,\in\, D,\ y \,\in\, D_\varepsilon \\ |x \,-\, y| \,\leq\, \varepsilon}} \frac{|u(x) - u(y)|}{\varepsilon}$$

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Definition

Let ψ defined on $\Omega_\varepsilon \setminus \Omega$. A function $h : \Omega_\varepsilon \to \mathbb{R}$ is an AMLE$_\varepsilon$ of ψ to Ω_ε if

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Theorem

Let $\psi : \Omega_\varepsilon \setminus \Omega \to \mathbb{R}$ be bounded. Then, u is a solution of

$$\begin{cases} -\Delta_\infty^\varepsilon u = 0 & \text{in } \Omega, \\ u = \psi & \text{on } \Omega_\varepsilon \setminus \Omega, \end{cases}$$

if and only if

$$u : \Omega_\varepsilon \to \mathbb{R} \ \textit{ is } \ \mathrm{AMLE}_\varepsilon(\psi, \Omega).$$

TUG-OF-WAR GAME

- There are two players moving a token inside Ω_ε. The token is placed at an initial position $x_0 \in \Omega$.

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The value of the game is the minimum (max.) amount that player I (II) expects to win (lose).

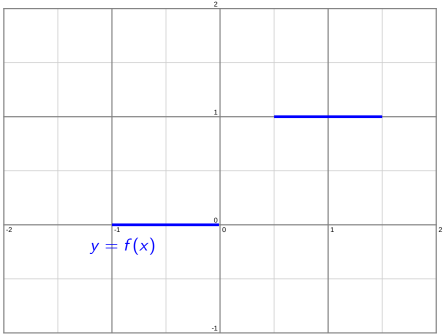
Dynamic Programming Principle

The value function u_ε of the above game satisfies:

$$u_\varepsilon(x) = \frac{1}{2} \sup_{y \in \overline{B}_\varepsilon(x)} u_\varepsilon(y) + \frac{1}{2} \inf_{y \in \overline{B}_\varepsilon(x)} u_\varepsilon(y).$$

Example

For $\varepsilon = 1$, $\Omega =]0, \frac{1}{2}[$ and $f = 0\chi_{]-1,0]} + 1\chi_{[\frac{1}{2},\frac{3}{2}[}$.



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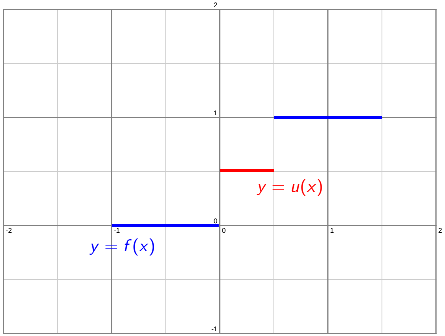

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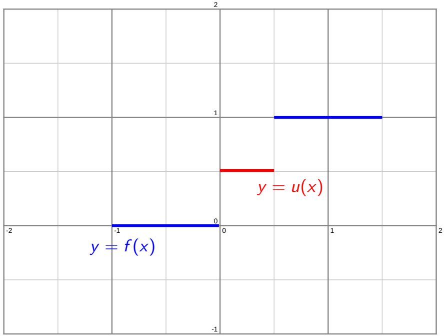
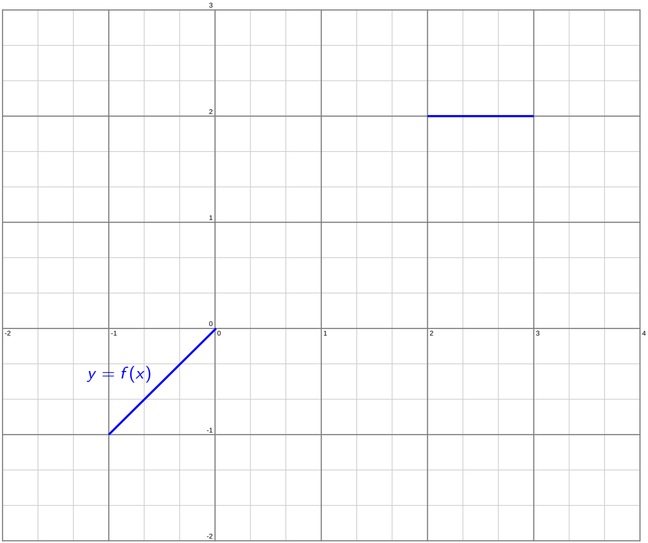


is the unique $\mathrm{AMLE}_1(f, \Omega)$.
There is not AMLE in the sense of Juutinen.

Extending $x\chi_{]-1,0]}(x) + 2\chi_{[2,3[}(x)$ to $]0,2[$ for $\varepsilon = 1$



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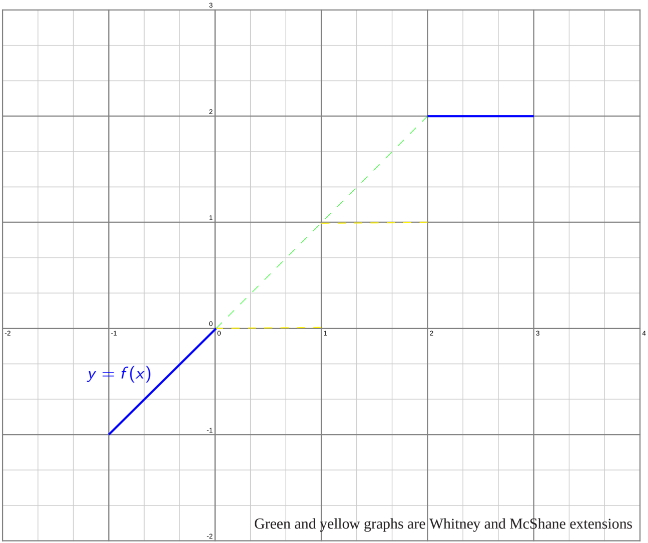

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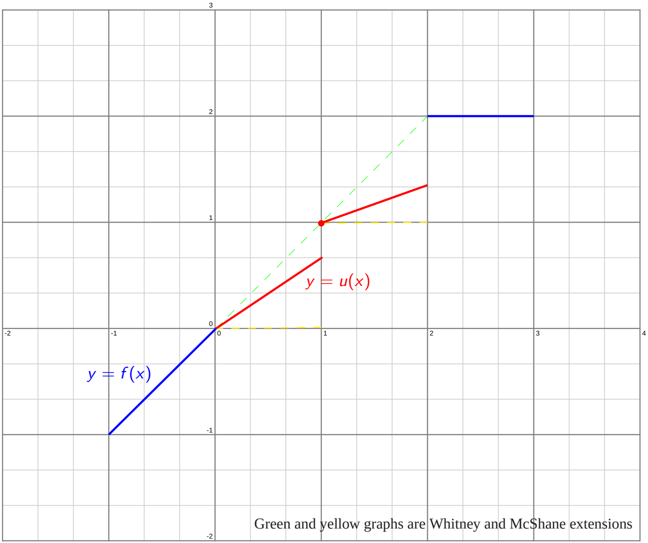
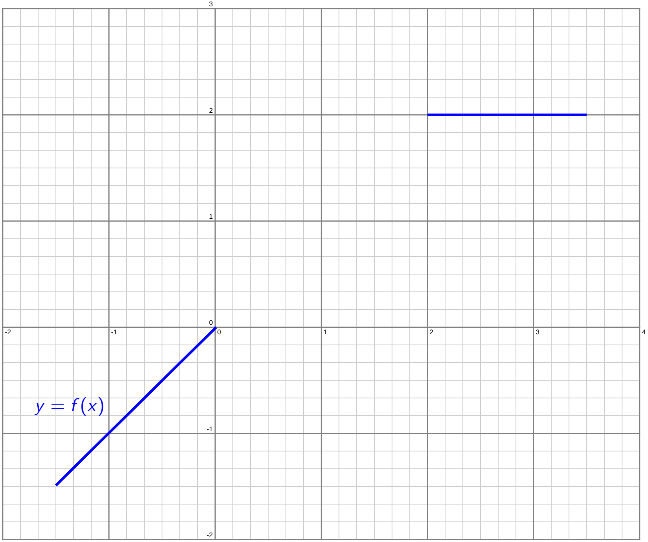


Extending $x\chi_{]-\frac{3}{2},0]}(x) + 2\chi_{[2,\frac{7}{2}[}(x)$ to $]0,2[$ for $\varepsilon = \frac{3}{2}$



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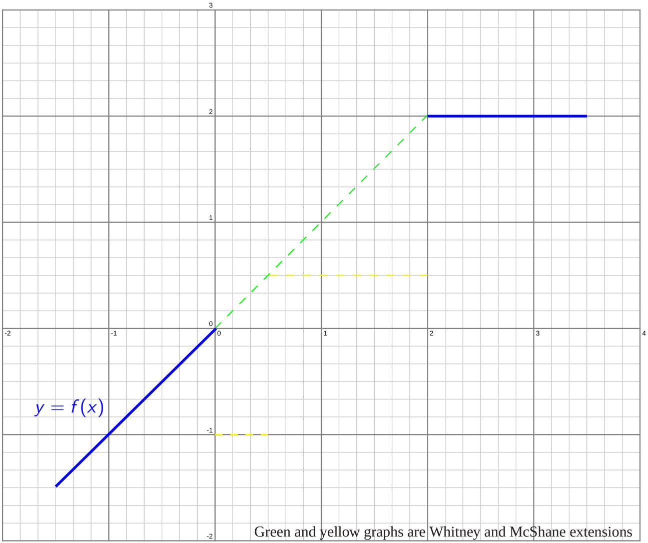

Green and yellow graphs are Whitney and McShane extensions

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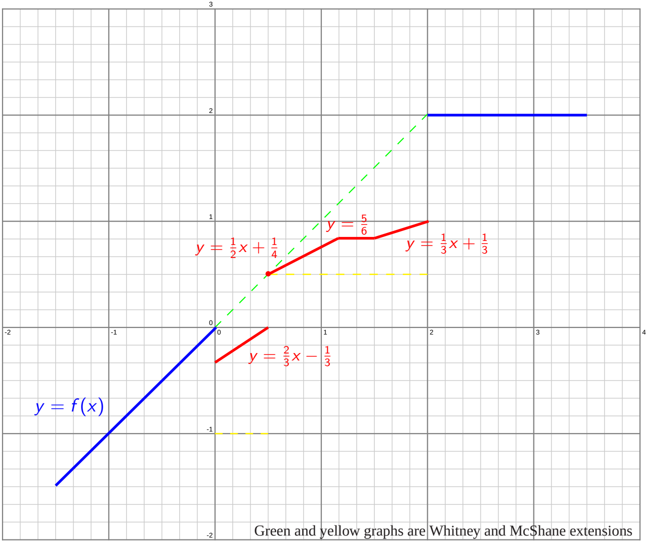
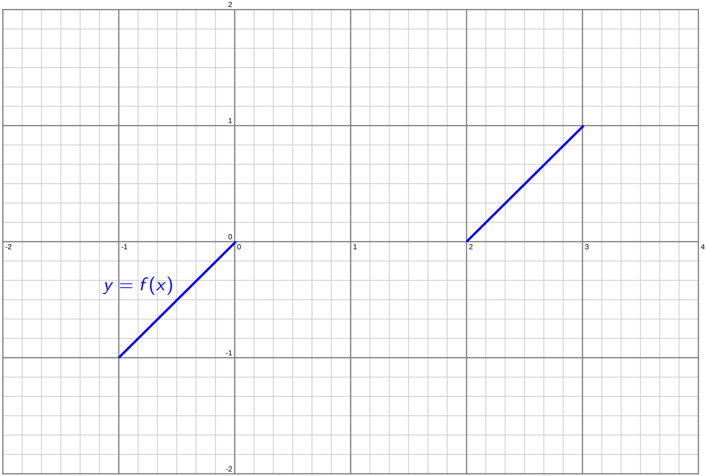


Extending $x\chi_{]-1,0]}(x) + (x-2)\chi_{[2,3[}(x)$ to $]0,2[$ for $\varepsilon = 1$



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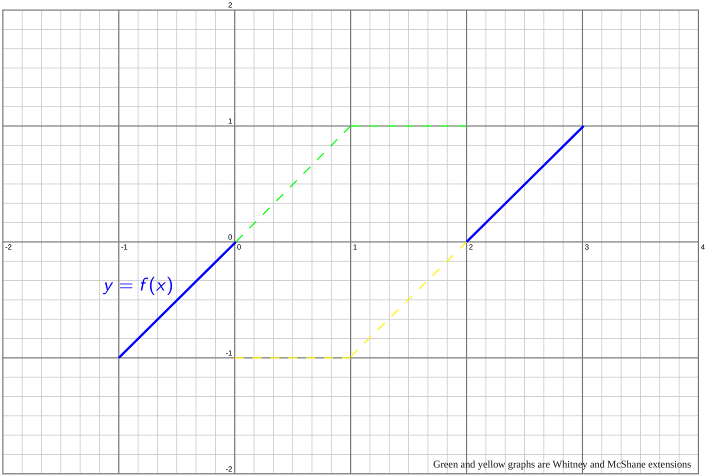

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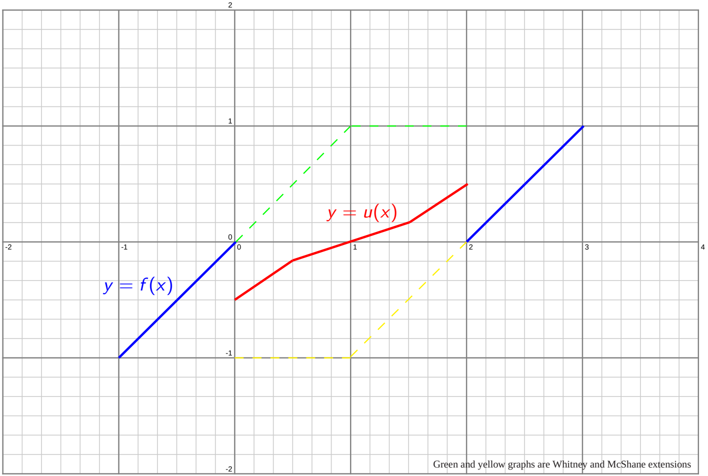

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