NONLOCAL NONLINEAR PROBLEMS AT ESSAOUIRA

J. JULIÁN TOLEDO

Dpto. Análisis Matemático, Univ. València

This course is based on a series of works co-authored by

F. Andreu

N. Igbida

J. Mazón

M. Pérez

J. Toledo

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Introduction

The goal of this course is to present recent results on nonlocal problems with different boundary conditions. One of the main tools used is the Nonlinear Semigroup Theory. We also give some results concerning limits of solutions to nonlocal problems when a rescaling parameter goes to zero, recovering local problems.

The prototype of nonlocal problems that will be considered is the following:

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t)\ ,$$

where $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative, radial, continuous function with total mass equal to 1.

Prototype equation:

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))\,dy\,. \tag{0.1}$$

If $u(x,t)$ is thought of as a density at a point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x,

then

$\int_{\mathbb{R}^N} J(y-x)u(y,t)\,dy = (J*u)(x,t)$ is the rate at which individuals are arriving at position x from all other places and

$u(x,t) = \int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites.

This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (0.1).

Equation (0.1) is said to be of nonlocal diffusion since the diffusion of the density u at a point x and time t depends not only on u at x, but on all the values of u in a neighborhood of x through the convolution term $J * u$.

Let us now fix a bounded domain Ω in $\mathbb{R}^N$. For local problems the two most common boundary conditions are Neumann's and Dirichlet's. When looking at boundary conditions for nonlocal problems, one has to modify the usual formulations for local problems. For Neumann boundary conditions we propose

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)(u(y,t)-u(x,t))\,dy, & x \in \Omega,\ t>0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

In this model, the integral term takes only into account the diffusion inside Ω. The individuals may not enter or leave the domain. This is analogous to what is called homogeneous Neumann boundary conditions in the literature.

For Dirichlet boundary conditions we consider

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \notin \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

In this model, diffusion takes place in the whole $\mathbb{R}^N$, but we assume that u vanishes outside Ω. Think as we had a hostile environment outside Ω, and any individual that jumps outside dies instantaneously. This is the analog of what is called homogeneous Dirichlet boundary conditions for the heat equation. However, the boundary datum is not understood in the usual sense of traces considered for local problems.

Nonlocal problems have been used to model very different applied situations, for example in biology, image processing, particle systems, coagulation models, nonlocal anisotropic models for phase transition, mathematical finances using optimal control theory, etc. They share many properties with the classical local evolution problems, however, there is no regularizing effect in general.

Our interest in this course is concerned with nonlinear problems. We study nonlocal analogs of the p–Laplacian evolution problems for $1 < p < \infty$:

$$u_t(t,x) = \int_A J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy \qquad \text{for } x \in \Omega,\ t > 0,$$

for $J : \mathbb{R}^N \to \mathbb{R}$ a nonnegative continuous radial function with compact support, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$.

In a bounded domain Ω, the above problem is a Dirichlet type problem by taking $A = \mathbb{R}^N$ and $u = 0$ in $\mathbb{R}^N \setminus \Omega$, a Neumann type problem by taking $A = \Omega$. And a Cauchy problem in the whole $\mathbb{R}^N$ if $A = \Omega = \mathbb{R}^N$.

We also present a nonlocal versions of the Total Variation Flow, for a non-degenerate kernel and for a singular kernel.

Finally we present a sandpile model, an optimal mass transport problem, a median value problem and a best Lipschitz extension problem, obtained as limit problems of nonlocal p–Laplacian problems.

This course is based on:

[**10**] F. Andreu, J. M. Mazón, J. D. Rossi and J. Toledo, *A nonlocal p–Laplacian evolution equation with Neumann boundary conditions*. J. Math. Pures Appl. (9) **90(2)** (2008), 201–227.

[**11**] F. Andreu, J. M. Mazón, J. D. Rossi and J. Toledo, *The limit as $p \to \infty$ in a nonlocal p–Laplacian evolution equation. A nonlocal approximation of a model for sandpiles*. Calc. Var. Partial Differential Equations **35** (2009), 279–316.

[**12**] F. Andreu, J. M. Mazón, J. D. Rossi and J. Toledo, *A nonlocal p–Laplacian evolution equation with non homogeneous Dirichlet boundary conditions.* SIAM J. Math. Anal. **40** (2009), 1815–1851.

[**13**] F. Andreu, J.M. Mazón, J. Rossi and J. Toledo, *Nonlocal Diffusion Problems*. Mathematical Surveys and Monographs, vol. 165. AMS, 2010.

[**40**] N. Igbida, J. M. Mazón, J. D. Rossi and J. Toledo, *A Monge–Kantorovich mass transport problem for a discrete distance*. J. Funct. Anal. 260 (2011), 3494–3534.

[**47**] J. M. Mazón, J. D. Rossi and J. Toledo. *On the best Lipschitz extension problem for a discrete distance and the discrete infinity–Laplacian*. J. Math. Pures Appl. (9) 97 (2012), no. 2, 98–119.

[**45**] J. M. Mazón, M. Pérez–Llanos, J. D. Rossi and J. Toledo. *A nonlocal* 1*–Laplacian and median values.* Publ. Mat.

[**48**] J. M. Mazón, J. D. Rossi and J. Toledo, *Fractional p–Laplacian evolution equations*. Preprint.

THEME 1

Nonlinear semigroups: an overview

1.1. Abstract Cauchy problems. Mild solutions

We outline some of the main points of the theory of nonlinear semigroups and evolution equations governed by accretive operators. We refer to [**17**], [**18**], [**19**], [**20**], [**26**], [**27**], [**28**], [**29**] and [**30**].

One of our main objectives will be the study of evolution problems of the form

$$(\mathrm{CP})_{x,f} \qquad \begin{cases} u'(t) + Au(t) = f(t) & \text{on } (0,T), \\ u(0) = x, \end{cases}$$

where X is a real Banach space with norm denoted by $\|\cdot\|$, $f : (0,T) \to X$ and $A : D(A) \to 2^X$ is a (multivalued) operator. The use of multivalued nonlinear operators permits to obtain a coherent theory but also it is quite useful in applications.

A problem of the form $(\mathrm{CP})_{x,f}$ is called an *abstract Cauchy problem.*

Let A be an operator in X and $f \in L^1(0,T;X)$.

Definition 1.1. A function u is called a *strong solution* of $(\mathrm{CP})_{x,f}$ if

$$u \in C([0,T];X) \cap W^{1,1}_{loc}(0,T;X) \text{ and}$$

$$\begin{cases} u' + Au(t) \ni f(t) \quad \text{a.e. } t \in (0,T), \\ u(0) = x. \end{cases}$$

Let us now introduce a more general concept of solution for $(\mathrm{CP})_{x,f}$, mild solution, introduced by M. G. Crandall and T. M. Liggett in [**30**] and Ph. Bénilan in [**18**]. Roughly speaking, a mild solution of the problem

$$u' + Au \ni f \qquad \text{on } [a,b]$$

is a continuous function $u \in C([0,T];X)$ which is the uniform limit of solutions of time-discretized problems given by the following implicit Euler scheme:

$$\frac{v(t_i) - v(t_{i-1})}{t_i - t_{i-1}} + Av(t_i) \ni f_i,$$

where f_i are approximations of f as $|t_i - t_{i-1}| \to 0$.

Definition 1.2.

1. Let $\varepsilon > 0$. An ε-*discretization* of $u' + Au \ni f$ on $[a,b]$ consists of a partition $t_0 < t_1 < \cdots < t_N$ and a finite sequence $f_1, f_2, \ldots, f_N$ of elements of X such that

$$a \leq t_0 < t_1 < \cdots < t_N \leq b, \quad \text{with}$$

$$t_i - t_{i-1} \leq \varepsilon,\ i = 1, \ldots, N, \quad t_0 - a \leq \varepsilon \ \text{ and } \ b - t_N \leq \varepsilon.$$

and

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|f(s) - f_i\| \, ds \leq \varepsilon.$$

We will denote this discretization by $D_A(t_0, \ldots, t_N; f_1, \ldots, f_N)$.

2. A *solution of the discretization* $D_A(t_0, \ldots, t_N; f_1, \ldots, f_N)$ is a piecewise constant function $v : [t_0, t_N] \to X$ whose values $v(t_0) = v_0$, $v(t) = v_i$ for $t \in]t_{i-1}, t_i]$, $i = 1, \ldots, N$ satisfy

$$\frac{v_i - v_{i-1}}{t_i - t_{i-1}} + Av_i \ni f_i, \qquad i = 1, \ldots, N.$$

3. An ε-*approximate solution* of $(\mathrm{CP})_{x_0,f}$ is a solution v of an ε-discretization $D_A(0 = t_0, \ldots, t_N, f_1, \ldots, f_N)$ of $u' + Au \ni f$ on $[0,T]$ with $\|v(0) - x_0\| < \varepsilon$.

4. And u is a mild solution of $(\mathrm{CP})_{x_0,f}$ on $[0,T]$ if and only if $u \in C([0,T];X)$ and for each $\varepsilon > 0$ there is an ε-approximate solution v of $(\mathrm{CP})_{x_0,f}$ such that $\|u(t) - v(t)\| < \varepsilon$ on the domain of v.

THEOREM 1.3. *Let A be an operator in X and $f \in L^1_{loc}(0,T;X)$. Then*

(i) If u is a strong solution of $(\mathrm{CP})_{x_0,f}$ on $[0,T]$ then u is a mild solution.

(ii) If u is a mild solution of $(\mathrm{CP})_{x_0,f}$ on $[0,T]$, then $u(t) \in \overline{D(A)}$ for all $t \in [0,T]$.

(iii) *Let $\overline{A}$ be the closure of the operator A. Then u is a mild solution of $u' + Au \ni f$ on $[0,T]$, $u(0) = x_0$, if and only if u is a mild solution of $u' + \overline{A}u \ni f$ on $[0,T]$, $u(0) = x_0$.*

1.2. Accretive operators. Uniqueness of mild solutions

The existence of mild solutions requires, as we just pointed out before, the existence of solutions of discretized equations of the form

$$\frac{x_i - x_{i-1}}{t_i - t_{i-1}} + Ax_i \ni f_i, \qquad i = 1, \ldots, N$$

or equivalently

$$x_i + (t_i - t_{i-1})Ax_i \ni (t_i - t_{i-1})f_i + x_{i-1}, \qquad i = 1, \ldots, N,$$

Accretive operators guarantees uniqueness of such solutions:

DEFINITION 1.4. An operator A in X is *accretive* if

$$\|x - \hat{x}\| \leq \|x - \hat{x} + \lambda(y - \hat{y})\| \quad \text{whenever} \quad \lambda > 0 \quad \text{and} \quad (x,y), (\hat{x},\hat{y}) \in A.$$

Note that A is accretive if and only if, for $\lambda > 0$,

$$\left\|(I+\lambda A)^{-1}z - (I+\lambda A)^{-1}\hat{z}\right\| \leq \|z - \hat{z}\|,$$

that is, A is accretive if and only if $J_\lambda^A := (I+\lambda A)^{-1}$ (called the *resolvent* of A) is a single-valued nonexpansive map for $\lambda > 0$.

An operator in a Hilbert space is accretive iff it is *monotone,* that is,

$$(x - \hat{x} | y - \hat{y}) \geq 0 \qquad \text{for all } (x,y),(\hat{x},\hat{y}) \in A.$$

Accretivity implies uniqueness of mild solutions:

THEOREM 1.5. *Let A accretive, $f, \hat{f} \in L^1(0,T;X)$, and let $u, \hat{u}$ be mild solutions on $[0,T]$ of $(CP)_{x_0,f}$ and $(CP)_{\hat{x}_0,\hat{f}}$ respectively. Then*

$$\|u(t) - \hat{u}(t)\| \leq \|x_0 - \hat{x}_0\| + \int_0^t \|f(s) - \hat{f}(s)\|\, ds, \quad \textit{for } t \in [0,T].$$

1.3. Range conditions. Existence of mild solutions

But apart from accretivity one should expect a range condition to get the existence of solution as well. One could ask for $\mathrm{R}(I+\lambda A) = X$ for all $\lambda > 0$:

An operator A is said to be *m-accretive* in X if A is accretive and $\mathrm{R}(I+\lambda A) = X$ for all $\lambda > 0$; if and only if there exists one $\lambda > 0$ such that $\mathrm{R}(I+\lambda A) = X$.

It is easy to see that each m-accretive operator A in X is *maximal accretive* in the sense that every accretive extension of A coincides with A. In general, the converse is not true, but it is in Hilbert spaces:

THEOREM 1.6 (Minty's Theorem). *Let H be a Hilbert space and A an accretive operator in H. Then A is m-accretive if and only if A is maximal monotone.*

One of the most important examples of maximal monotone operators in Hilbert spaces comes from optimization theory: subdifferentials of convex functions.

Let $(H, (\ |\ ))$ be a Hilbert space and $\varphi : H \to (-\infty, +\infty]$ convex. Its *subdifferential* $\partial\varphi$ is the operator defined by

$$w \in \partial\varphi(z) \iff \varphi(x) \geq \varphi(z) + (w|x-z) \quad \forall\, x \in H.$$

If $(\hat{z}, \hat{w}), (z, w) \in \partial\varphi$, then $\varphi(z) \geq \varphi(\hat{z}) + (\hat{w}|z-\hat{z})$ and $\varphi(\hat{z}) \geq \varphi(z) + (w|\hat{z}-z)$. Adding this inequalities we get $(w-\hat{w}|z-\hat{z}) \geq 0$. Thus, $\partial\varphi$ is a monotone operator.

Now, if φ is convex, lower semicontinuous and proper, then $\partial\varphi$ is maximal monotone and $\overline{D(\partial\varphi)} = \overline{D(\varphi)}$.

Observe that $0 \in \partial\varphi(z)$ if and only if $\varphi(x) \geq \varphi(z)$ for all $x \in H$, if and only if $\varphi(z) = \min_{x\in D(\varphi)} \varphi(x)$. Therefore,
$0 \in \partial\varphi(z)$ is the *Euler equation* of the variational problem

$$\varphi(z) = \min_{x\in D(\varphi)} \varphi(x).$$

Given a closed convex subset K of H, the *indicator function* of K is defined by

$$\mathbb{I}_K(u) = \begin{cases} 0 & \text{if } \ u \in K, \\ +\infty & \text{if } \ u \notin K. \end{cases}$$

Its subdifferential is characterized as follows:

$$v \in \partial \mathbb{I}_K(u) \iff u \in K \ \text{ and } \ (v, w-u) \leq 0 \quad \forall\, w \in K.$$

THEOREM 1.7 (Crandall-Liggett, Bénilan). *Suppose that A is m–accretive in X, $f \in L^1(0,T;X)$ and $x \in \overline{D(A)}$. Then*

$$u' + Au \ni f \quad \text{on} \quad [0,T], \qquad u(0) = x,$$

has a unique mild solution u on $[0,T]$.

If we set $e^{-tA}x$ to be the mild solution of $u' + Au \ni 0$ on $(0,+\infty)$ with initial data x, then $(e^{-tA})_{t\geq 0}$ is a contraction semigroup: the *semigroup generated* by $-A$.

In the homogeneous case we can debilitate the m-accretivity of the operator and get an explicit representation of the mild solution.

DEFINITION 1.8. We say that an accretive operator A satisfies the *range condition* when

$$\overline{D(A)} \subset \mathrm{R}(I + \lambda A) \quad \text{for all } \lambda > 0.$$

THEOREM 1.9 (Crandall-Liggett Theorem). *If A is accretive and satisfies the range condition, then $(e^{-tA})_{t\geq 0}$ is a semigroup of contractions on $\overline{D(A)}$ and*

$$e^{-tA}x = \lim_{n\to\infty}\left(I+\frac{t}{n}A\right)^{-n}x \qquad \text{for } x\in\overline{D(A)}.$$

1.4. Regularity of mild solutions

In general mild solutions are not strong solutions. When they are?

THEOREM 1.10. *Assume X is reflexive. Let A be an accretive operator in X, $f\in BV(0,T;X)$ and $x\in D(A)$. If u is a mild solution of $(\mathrm{CP})_{x,f}$ on $[0,T]$, then $u\in W^{1,1}(0,T;X)$ and u is a strong solution.*

Also:

THEOREM 1.11. *Let H be a Hilbert space and $\varphi : H\to(-\infty,+\infty]$ a proper, convex and lower semicontinuous function such that $\operatorname{Min}\varphi = 0$. Suppose $f\in L^2(0,T;H)$ and $x_0\in\overline{D(\partial\varphi)}$; then the mild solution $u(t)$ of*

$$\begin{cases} u'+\partial\varphi(u)\ni f \quad \textit{on} \;\; [0,T],\\ u(0)=x_0,\end{cases}$$

is a strong solution.

1.5. Dependence

THEOREM 1.12 (Brezis-Pazy Theorem). *Let A_n be m-accretive in X, $x_n \in \overline{D(A_n)}$ and $f_n \in L^1(0,T;X)$ for $n = 1, 2, \ldots, \infty$. Let u_n be the mild solution of*

$$u_n' + A_n u_n \ni f_n \quad \text{in } [0,T], \quad u_n(0) = x_n.$$

If $f_n \to f_\infty$ in $L^1(0,T;X)$ and $x_n \to x_\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} (I + \lambda A_n)^{-1} z = (I + \lambda A_\infty)^{-1} z,$$

for some $\lambda > 0$ and all $z \in D$, with D dense in X, then

$$\lim_{n\to\infty} u_n(t) = u_\infty(t) \quad \text{uniformly on } [0,T].$$

For subdifferentials of convex lower semicontinuous functionals in Hilbert spaces, to prove the convergence of the resolvent it is enough to show the convergence of the functionals in the sense of Mosco ([**49**]).

Mosco convergence.

Given a sequence $\Psi_n, \Psi : H \to (-\infty, +\infty]$ of convex lower semicontinuous functionals, we say that Ψ_n converges to Ψ in the sense of Mosco if

(1) $\forall\, u \in D(\Psi) \;\; \exists u_n \in D(\Psi_n) \,:\, u_n \to u$ and $\Psi(u) \geq \limsup_{n\to\infty} \Psi_n(u_n)$;

(2) for every subsequence n_k, as $u_k \rightharpoonup u$, we have $\Psi(u) \leq \liminf_k \Psi_{n_k}(u_k)$.

Or equivalently:

$$\text{w-}\limsup_{n\to\infty} \mathrm{Epi}(\Psi_n) \subset \mathrm{Epi}(\Psi) \subset \text{s-}\liminf_{n\to\infty} \mathrm{Epi}(\Psi_n),$$

where

$$\text{s-}\liminf_{n\to\infty} A_n = \{x \in H :\, \exists x_n \in A_n,\; x_n \to x\}$$

$$\text{w-}\limsup_{n\to\infty} A_n = \{x \in H :\, \exists x_{n_k} \in A_{n_k},\; x_{n_k} \rightharpoonup x\}.$$

From Theorem 1.12 and using the results of H. Attouch ([**16**]) we have:

THEOREM 1.13. *The following statements are equivalent:*
(i) Ψ_n *converges to* Ψ *in the sense of Mosco.*
(ii) $(I+\lambda\partial\Psi_n)^{-1}x \to (I+\lambda\partial\Psi)^{-1}x, \quad \forall\,\lambda>0,\ x\in H.$

Moreover, any of these two conditions implies that

(iii) for every $x_0 \in \overline{D(\partial\Psi)}$ *and* $x_{0,n} \in \overline{D(\partial\Psi_n)}$ *such that* $x_{0,n} \to x_0$*, and every* $f_n, f \in L^2(0,T;H)$ *with* $f_n \to f$*, if* $u_n(t)$*,* $u(t)$ *are the strong solutions of*

$$\begin{cases} u_n'(t) + \partial\Psi_n(u_n(t)) \ni f_n, & a.e. \ \ t\in(0,T), \\ u_n(0) = x_{0,n}, \end{cases}$$

and

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni f, & a.e. \ \ t\in(0,T), \\ u(0) = x_0, \end{cases}$$

respectively, then

$$u_n \to u \ \ in \ \ C([0,T];H).$$

1.6. Completely accretive operators

Many nonlinear semigroups that appear in the applications are also order-preserving and contractions in every L^p. Ph. Bénilan and M. G Crandall ([**19**]) introduced a class of operators, named completely accretive, for which the semigroup generated by the Crandall-Liggett exponential formula enjoys these properties. In this section we outline some of the main points given in [**19**].

Let Ω be an open set in $\mathbb{R}^N$ and let $M(\Omega)$ be the space of measurable functions from Ω into $\mathbb{R}$.

For $u, v \in M(\Omega)$, we write

$$u \ll v \quad \text{if and only if} \int_\Omega j(u)dx \leq \int_\Omega j(v)dx$$

for all $j \in J_0 := \{j : \mathbb{R} \to [0, \infty] \text{ convex, l.s.c., } j(0) = 0\}$.

Definition 1.14. Let A be an operator in $M(\Omega)$. We say that A is *completely accretive* if

$$u - \hat{u} \ll u - \hat{u} + \lambda(v - \hat{v}) \quad \text{for all } \lambda > 0 \text{ and all } (u, v), (\hat{u}, \hat{v}) \in A.$$

The definition of completely accretive operators does not refer explicitly to topologies or norms. However, if A is completely accretive in $M(\Omega)$ and $A \subset L^p(\Omega) \times L^p(\Omega)$ $(1 \leq p \leq \infty)$ then A is accretive in $L^p(\Omega)$.

Let $P_0 = \{q \in C^\infty(\mathbb{R}) : 0 \leq q' \leq 1,\ \mathrm{supp}(q') \text{ is compact and } 0 \notin \mathrm{supp}(q)\}$. The following result provides a very useful characterization of complete accretivity.

PROPOSITION 1.15. *If $A \subseteq L^p(\Omega) \times L^p(\Omega)$, $1 \leq p < \infty$, then A is completely accretive if and only if*

$$\int_\Omega q(u - \hat{u})(v - \hat{v}) \geq 0 \quad \textit{for any} \quad q \in P_0,\ (u, v), (\hat{u}, \hat{v}) \in A.$$

PROPOSITION 1.16. *Let $1 \leq p < +\infty$.*

(i) *Let $u \in L^p(\Omega)$. Then $\{v \in M(\Omega) \ : v \ll u\}$ is a weakly sequentially compact subset of $L^p(\Omega)$.*

(ii) *If $\{u_n\}$ is a sequence satisfying $u_n \ll u \in L^p(\Omega)$ for all n, and $u_n \rightharpoonup u$ weakly in $L_p(\Omega)$, then $\|u_n - u\|_p \to 0$.*

DEFINITION 1.17. Let X be a linear subspace of $M(\Omega)$. An operator A in X is *m-completely accretive* in X if A is completely accretive and $\mathrm{R}(I + \lambda A) = X$ for $\lambda > 0$

PROPOSITION 1.18. *Let $1 \leq p < +\infty$. Let A be a completely accretive operator in $L^p(\Omega)$. Suppose there exists $\lambda > 0$ for which $\mathrm{R}(I + \lambda A)$ is dense in $L^p(\Omega)$. Then the operator $\overline{A}^{L^p(\Omega)}$ is the unique m-completely accretive extension of A in $L^p(\Omega)$.*

Let $1 \leq p \leq +\infty$. If A is m-completely accretive in $L^p(\Omega)$, by Crandall-Liggett's theorem, A generates a contraction semigroup in $L^p(\Omega)$ given by the exponential formula

$$e^{-tA}u_0 = \| \cdot \|_p\text{-} \lim_{n\to\infty} \left(I + \frac{t}{n}A \right)^{-n} u_0 \quad \text{for any } u_0 \in \overline{D(A)}^{L^p(\Omega)}.$$

Moreover, for any $t > 0$, e^{-tA} is a $\|.\|_q$–T–contraction for any $1 \leq q \leq \infty$.

Proposition 1.19. *Let $|\Omega| < \infty$. If $A \subseteq L^1(\Omega) \times L^1(\Omega)$ is an m-completely accretive operator in $L^1(\Omega)$, then for every $u_0 \in D(A)$, the mild solution of the problem*

$$\begin{cases} u' + Au \ni 0, \\ u(0) = u_0 \end{cases}$$

is in $W^{1,1}(0,T;X)$ and is a strong solution.

Moreover the following regularizing effect holds:

Theorem 1.20. *Let $1 \leq p < +\infty$. Let A an m-completely accretive operator in $L^p(\Omega)$, positively homogeneous of degree $0 < m \neq 1$, i.e, $A(\lambda u) = \lambda^m Au$ for $u \in D(A)$. Then for $u \in \overline{D(A)} \cap L^p(\Omega)$ and $t > 0$, we have $e^{-tA}u \in D(A)$.*

THEME 2

Nonlocal p–Laplacian problems

Model evolution equations for nonlinear local diffusion:

- the porous medium equation, $v_t = \Delta \left(|v|^{m-1}v\right)$,

- the p-Laplacian evolution, $v_t = \mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right)$.

Here we will study a nonlocal analog of the p-Laplacian evolution with Neumann boundary conditions and the Neumann problem for the nonlocal total variational flow, both for non-degenerate kernels, and the Dirichlet problem for fractional 1–Laplacian evolution.

We will study:

- existence and uniqueness,

- if the kernel is rescaled in an appropriate way, the corresponding solutions of the nonlocal evolution problems converge to the solution of the corresponding local evolution problems.

The asymptotic behaviour is also studied in [**10**], [**12**], [**48**]. A nonlocal analogous problem to the porous medium equation for non–degenerate kernel can be found in [**9**].

2.1. The Neumann problem for nonlocal p–Laplacian evolution with non–degenerate kernels

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with compact support,

$$J(0) > 0 \quad \text{and} \quad \int_{\mathbb{R}^N} J(x)dx = 1,$$

and $\Omega \subset \mathbb{R}^N$ a bounded domain.

We begin with the study of the nonlocal p–Laplacian evolution problem with Neumann boundary conditions:

$$\left\{ \begin{array}{ll} u_t(x,t) = \displaystyle\int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy, & \\ u(x,0) = u_0(x), & x \in \Omega,\ t > 0. \end{array} \right. \tag{2.1}$$

Definition 2.1. A *solution* of (2.1) in $[0,T]$ is a function $u \in W^{1,1}(0,T;L^1(\Omega))$ that satisfies $u(x,0) = u_0(x)$ a.e. $x \in \Omega$ and

$$u_t(x,t) = \int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy$$

a.e. in $\Omega \times (0,T)$.

2.1.1. Existence and uniqueness.

Tools: Nonlinear Semigroup Theory.

We introduce the following operator in $L^1(\Omega)$ associated with problem (2.1):

$$B_p^J u(x) = -\int_\Omega J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, \qquad x \in \Omega.$$

Observe that

- B_p^J is positively homogeneous of degree $p-1$;
- for $p > 2$, $L^{p-1}(\Omega) \subset D(B_p^J)$;
- and for $1 < p \leq 2$, $D(B_p^J) = L^1(\Omega)$ and B_p^J is closed in $L^1(\Omega) \times L^1(\Omega)$.

Using the following integration by parts formula:

LEMMA 2.2. *For every $u, v \in L^p(\Omega)$,*

$$-\int_\Omega \int_\Omega J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy\, v(x)\,dx$$

$$= \frac{1}{2}\int_\Omega \int_\Omega J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))(v(y)-v(x))\,dy\,dx.$$

we have the following monotonicity result for B_p^J:

LEMMA 2.3. *Let $\mathsf{T} : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function. Then, for every $u, v \in L^p(\Omega)$ such that $\mathsf{T}(u-v) \in L^p(\Omega)$, we have*

$$\int_\Omega (B_p^J u(x) - B_p^J v(x))\,\mathsf{T}(u(x) - v(x))dx$$

$$= \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\,(\mathsf{T}(u(y)-v(y)) - \mathsf{T}(u(x)-v(x)))$$

$$\times \left(|u(y)-u(x)|^{p-2}(u(y)-u(x)) - |v(y)-v(x)|^{p-2}(v(y)-v(x))\right)\,dy\,dx \geq 0.$$

From it we obtain easily that B_p^J is completely accretive. Now we will also prove that:

THEOREM 2.4. *The operator B_p^J satisfies the range condition*

$$L^p(\Omega) \subset \mathrm{R}(I + B_p^J). \tag{2.2}$$

Therefore, for any $\phi \in L^p(\Omega)$ there is a unique solution of the problem $u + B_p^J u = \phi$ and the resolvent $(I + B_p^J)^{-1}$ is a contraction in $L^q(\Omega)$ for all $1 \leq q \leq +\infty$.

Proof. We want to prove that for any $\phi \in L^p(\Omega)$ there exists $u \in D(B_p^J)$ such that $u = (I + B_p^J)^{-1}\phi$.

Let us first take $\phi \in L^\infty(\Omega)$. Let $A_{n,m} : L^p(\Omega) \to L^{p'}(\Omega)$ be the continuous monotone operator defined by

$$A_{n,m}(u) := T_c(u) + B_p^J u + \frac{1}{n}|u|^{p-2}u^+ - \frac{1}{m}|u|^{p-2}u^-,$$

where $T_c(r) = c \wedge (r \vee (-c))$, $c \geq 0$, $r \in \mathbb{R}$.

$A_{n,m}$ is coercive in $L^p(\Omega)$:

$$\lim_{\|u\|_{L^p(\Omega)} \to +\infty} \frac{\displaystyle\int_\Omega A_{n,m}(u)u}{\|u\|_{L^p(\Omega)}} = +\infty.$$

Then ([**25**, Corollary 30]) there exists $u_{n,m} \in L^p(\Omega)$ such that

$$T_c(u_{n,m}) + B_p^J u_{n,m} + \frac{1}{n}|u_{n,m}|^{p-2}u_{n,m}^+ - \frac{1}{m}|u_{n,m}|^{p-2}u_{n,m}^- = \phi.$$

Using monotonicity we obtain that $T_c(u_{n,m}) \ll \phi$. Consequently, taking $c > \|\phi\|_{L^\infty(\Omega)}$, we see that $u_{n,m} \ll \phi$ and

$$u_{n,m} + B_p^J u_{n,m} + \frac{1}{n}|u_{n,m}|^{p-2}u_{n,m}^+ - \frac{1}{m}|u_{n,m}|^{p-2}u_{n,m}^- = \phi.$$

Using that $u_{n,m}$ is increasing in n (and decreasing in m), that $u_{n,m} \ll \phi$, and the the monotone convergence, passing to the limit in n we get u_m a solution to

$$u_m + B_p^J u_m - \frac{1}{m}|u_m|^{p-2}u_m^- = \phi,$$

and $u_m \ll \phi$.

Now, u_m is decreasing in m. Then, taking limits in m we obtain a solution u to

$$u + B_p^J u = \phi.$$

Let now $\phi \in L^p(\Omega)$. Take $\phi_n \in L^\infty(\Omega)$, $\phi_n \to \phi$ in $L^p(\Omega)$. Then, by the previous step, there exists $u_n = (I + B_p^J)^{-1}\phi_n$. Using that B_p^J is completely accretive we get $u_n \to u$ in $L^p(\Omega)$, and then also $B_p^J u_n \to B_p^J u$ in $L^{p'}(\Omega)$. Then, we conclude that $u + B_p^J u = \phi$. $\square$

If $\mathcal{B}_p^J$ denotes the closure of B_p^J in $L^1(\Omega)$, then by Theorem 2.4 we obtain that $\mathcal{B}_p^J$ is m-completely accretive in $L^1(\Omega)$, consequently, by Theorem 1.7:

THEOREM 2.5. *Let $T > 0$ and $u_0 \in L^1(\Omega)$. Then there exists a unique mild solution u of*

$$\begin{cases} u'(t) + B_p^J u(t) = 0, & t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{2.3}$$

Now, by Theorem 1.10, thanks to the complete accretivity of B_p^J and the range condition (2.2), we have:

COROLLARY 2.6. *If $u_0 \in L^p(\Omega)$, the unique mild solution of* (2.3) *is a strong solution of problem* (2.1) *and a solution in the sense of Definition* 2.1.

If $1 < p \leq 2$, since $D(B_p^J) = L^1(\Omega)$ and B_p^J is closed in $L^1(\Omega) \times L^1(\Omega)$, by Proposition 1.19:

COROLLARY 2.7. *Let $1 < p \leq 2$. If $u_0 \in L^1(\Omega)$, the unique mild solution of* (2.3) *is a strong solution of problem* (2.1) *and a solution in the sense of Definition* 2.1.

Moreover, we have the following contraction principle:

COROLLARY 2.8. *For $q \in [1, +\infty]$, if $u_{i0} \in L^q(\Omega)$, $i = 1, 2$, we have:*

$$\|(u_1(t) - u_2(t))^+\|_{L^q(\Omega)} \leq \|(u_{10} - u_{20})^+\|_{L^q(\Omega)} \quad \textit{for every } t \in [0, T].$$

2.1.2. Rescaling the kernel. Convergence to the local p–Laplacian.

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. For fixed $p > 1$ we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right),$$

where $C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p \, dz$ is a normalizing constant.

The solution u_ε of problem (2.1), with the kernel J replaced by $J_{p,\varepsilon}$, converges, as the scaling parameter ε goes to zero, to the solution of the classical p–Laplacian evolution problem with homogeneous Neumann boundary conditions:

$$\left\{ \begin{array}{ll} v_t = \Delta_p v & \text{in } \Omega \times (0,T), \\ |\nabla v|^{p-2}\nabla v \cdot \eta = 0 & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = u_0(x) & \text{in } \Omega, \end{array} \right. \tag{2.4}$$

where η is the unit outward normal on $\partial\Omega$ and $\Delta_p v = \mathrm{div}(|\nabla v|^{p-2}\nabla v)$ is the so–called p–Laplacian of v.

Some facts about the local p-Laplacian equation.

Associated to the p–Laplacian with homogeneous boundary condition, the following operator $B_p \subset L^1(\Omega) \times L^1(\Omega)$ is defined: $(v, \hat{v}) \in B_p$ if and only if $\hat{v} \in L^1(\Omega)$, $v \in W^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \xi \, dx = \int_\Omega \hat{v} \xi \, dx \quad \text{for every} \quad \xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega),$$

and it is proved that B_p is a completely accretive operator in $L^1(\Omega)$ with dense domain satisfying a range condition, which implies that its closure $\mathcal{B}_p$ in $L^1(\Omega) \times L^1(\Omega)$ is an m-completely accretive operator in $L^1(\Omega)$ with dense domain (see [**6**] or [**7**]).

In [**5**] and [**8**] it is shown that for any $u_0 \in L^1(\Omega)$, the unique mild solution $v(t) = e^{-t\mathcal{B}_p} u_0$ given by Crandall-Liggett's exponential formula is the unique entropy solution of

$$\begin{cases} v_t = \Delta_p v & \text{in } \Omega \times (0,T), \\[2ex] |\nabla v|^{p-2} \nabla v \cdot \eta = 0 & \text{on } \partial\Omega \times (0,T), \\[2ex] v(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

A formal calculation for $N = 1$. Let $u(x)$ be a smooth function and consider

$$A_\varepsilon(u)(x) = \frac{1}{\varepsilon^{p+1}} \int_{\mathbb{R}} J\left(\frac{x-y}{\varepsilon}\right) |u(y)-u(x)|^{p-2}(u(y)-u(x))\, dy.$$

Changing variables, $y = x + \varepsilon z$, we get

$$A_\varepsilon(u)(x) = \frac{1}{\varepsilon^{p}} \int_{\mathbb{R}} J(z)|u(x+\varepsilon z)-u(x)|^{p-2}(u(x+\varepsilon z)-u(x))\, dz. \tag{2.5}$$

Expanding in powers of ε we obtain

$$\begin{aligned} |u(x+\varepsilon z)-u(x)|^{p-2} &= \varepsilon^{p-2}\left|u'(x)z + \tfrac{u''(x)}{2}\varepsilon z^2 + O(\varepsilon^2)\right|^{p-2} \\ &= \varepsilon^{p-2}|u'(x)|^{p-2}|z|^{p-2} + \varepsilon^{p-1}(p-2)|u'(x)z|^{p-4}u'(x)z\frac{u''(x)}{2}z^2 + O(\varepsilon^p), \end{aligned}$$

and

$$u(x+\varepsilon z)-u(x) = \varepsilon u'(x)z + \frac{u''(x)}{2}\varepsilon^2 z^2 + O(\varepsilon^3).$$

Hence, (2.5) becomes

$$\begin{aligned} A_\varepsilon(u)(x) &= \frac{1}{\varepsilon}\int_{\mathbb{R}} J(z)|z|^{p-2}z\, dz|u'(x)|^{p-2}u'(x) \\ &+\frac{1}{2}\int_{\mathbb{R}} J(z)|z|^{p}\, dz\,\big((p-2)|u'(x)|^{p-2}u''(x) + |u'(x)|^{p-2}u''(x)\big) + O(\varepsilon), \end{aligned}$$

that is,

$$A_\varepsilon(u)(x) = \frac{1}{\varepsilon}\int_{\mathbb{R}} J(z)|z|^{p-2}z\,dz|u'(x)|^{p-2}u'(x)$$

$$+\frac{1}{2}\int_{\mathbb{R}} J(z)|z|^{p}\,dz(|u'(x)|^{p-2}u'(x))' + O(\varepsilon).$$

Using that J is radially symmetric, the first integral vanishes and therefore

$$\lim_{\varepsilon\to 0} A_\varepsilon(u)(x) = C(|u'(x)|^{p-2}u'(x))',$$

where

$$C = \frac{1}{2}\int_{\mathbb{R}} J(z)|z|^p\,dz.$$

The objective is to make this formal calculation rigorous.

Tools:

- a precompactness result (a variant of [**23**, Theorem 4]).
- Nonlinear Semigroup Theory.

For a function g defined in a set D, we denote by $\overline{g}$ to its extension by 0 outside D. $BV(D)$ is the space of bounded variation functions.

THEOREM 2.9 (The precompactness result). *Let $1 \leq q < +\infty$ and $D \subset \mathbb{R}^N$ open. Let $\rho : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, non identically zero. Let $\{f_n\}_n \subset L^q(D)$ such that*

$$\int_D \int_D n^N |f_n(y) - f_n(x)|^q \rho(n(y-x))\, dx\, dy \leq \frac{M}{n^q}. \tag{2.6}$$

1. *If $\{f_n\}$ is weakly convergent in $L^q(D)$ to f, then:*

(i) *For $q > 1$, $f \in W^{1,q}(D)$, and moreover*

$$(\rho(z))^{1/q} \chi_D\left(x + \frac{1}{n}z\right) \frac{\overline{f}_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \rightharpoonup (\rho(z))^{1/q}\, z \cdot \nabla f(x)$$

weakly in $L^q(D) \times L^q(\mathbb{R}^N)$.

(ii) *For $q = 1$, $f \in BV(D)$, and moreover*

$$\rho(z)\chi_D\left(. + \frac{1}{n}z\right) \frac{\overline{f}_n\left(. + \frac{1}{n}z\right) - f_n(.)}{1/n} \rightharpoonup \rho(z) z \cdot Df$$

in the sense of measures.

2. *Suppose D is a smooth bounded domain in $\mathbb{R}^N$ and $\rho(x) \geq \rho(y)$ if $|x| \leq |y|$. Then there exists a subsequence $\{f_{n_k}\}$ such that*

(i) *if $q > 1$, $f_{n_k} \to f$ in $L^q(D)$ with $f \in W^{1,q}(D)$;*

(ii) *if $q = 1$, $f_{n_k} \to f$ in $L^1(D)$ with $f \in BV(D)$.*

THEOREM 2.10. *Suppose $J(x) \geq J(y)$ if $|x| \leq |y|$. For any $\phi \in L^\infty(\Omega)$,*

$$\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1} \phi \to (I + B_p)^{-1} \phi \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0. \tag{2.7}$$

PROOF. For $\varepsilon > 0$, let $u_\varepsilon = \left(I + B_p^{J_{p,\varepsilon}}\right)^{-1} \phi$. Then

(2.8)

$$\int_\Omega u_\varepsilon \xi - \frac{C_{J,p}}{\varepsilon^{p+N}} \int_\Omega \int_\Omega J\left(\frac{x-y}{\varepsilon}\right) |u_\varepsilon(y) - u_\varepsilon(x)|^{p-2}(u_\varepsilon(y) - u_\varepsilon(x))\, dy\, \xi(x)\, dx$$
$$= \int_\Omega \phi\xi \qquad \text{for every } \xi \in L^\infty(\Omega).$$

Aim: find a sequence $\varepsilon_n \to 0$ such that $u_{\varepsilon_n} \to v$ in $L^p(\Omega)$, $v \in W^{1,p}(\Omega)$ and $v = (I + B_p)^{-1} \phi$, that is,

$$\int_\Omega v\xi + \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \xi = \int_\Omega \phi\xi \quad \text{for every } \xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega).$$

We have that $u_\varepsilon \ll \phi$; therefore, by Proposition 1.16, there exists a sequence $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n} \rightharpoonup v \quad \text{weakly in } L^p(\Omega) \text{ and in } L^2(\Omega), \quad \text{and } v \ll \phi.$$

Consequently, $\|u_{\varepsilon_n}\|_{L^\infty(\Omega)},\ \|v\|_{L^\infty(\Omega)} \leq \|\phi\|_{L^\infty(\Omega)}$.

Changing variables, we can rewrite (2.8) as

$$\int_\Omega \phi(x)\xi(x)\,dx - \int_\Omega u_\varepsilon(x)\xi(x)\,dx$$

$$= \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi_\Omega(x+\varepsilon z)\left|\frac{\overline{u}_\varepsilon(x+\varepsilon z)-u_\varepsilon(x)}{\varepsilon}\right|^{p-2} \tag{2.9}$$

$$\times\frac{\overline{u}_\varepsilon(x+\varepsilon z)-u_\varepsilon(x)}{\varepsilon}\frac{\overline{\xi}(x+\varepsilon z)-\xi(x)}{\varepsilon}\,dx\,dz.$$

Taking $\varepsilon=\varepsilon_n$ and $\xi=u_{\varepsilon_n}$ in (2.9), we get

$$\int_\Omega\int_\Omega \frac{1}{2}\frac{C_{J,p}}{\varepsilon_n^N} J\left(\frac{x-y}{\varepsilon_n}\right)\left|\frac{u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p\,dx\,dy$$

$$= \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi_\Omega(x+\varepsilon_n z)\left|\frac{\overline{u}_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p\,dx\,dz \le M.$$

That is,

$$\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)\chi_\Omega(x+\varepsilon_n z)\left|\frac{\overline{u}_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dz \le M,$$

$$u_{\varepsilon_n} \rightharpoonup v \quad \text{weakly in } L^p(\Omega).$$

Therefore, by Theorem 2.9, $v\in W^{1,p}(\Omega)$ and

(2.10)

$$\left(\frac{C_{J,p}}{2}J(z)\right)^{1/p}\chi_\Omega(x+\varepsilon_n z)\frac{\overline{u}_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\rightharpoonup\left(\frac{C_{J,p}}{2}J(z)\right)^{1/p} z\cdot\nabla v(x)$$

weakly in $L^p(\Omega)\times L^p(\mathbb{R}^N)$. Moreover, there exists $\chi\in L^{p'}(\Omega)\times L^{p'}(\mathbb{R}^N)$ such that

$$(J(z))^{1/p'}\chi_\Omega(x+\varepsilon_n z)\left|\frac{\overline{u}_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^{p-2}\frac{\overline{u}_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}$$

$$\rightharpoonup (J(z))^{1/p'}\chi(x,z)$$

weakly in $L^{p'}(\Omega)\times L^{p'}(\mathbb{R}^N)$.

Therefore, passing to the limit in (2.9) for $\varepsilon=\varepsilon_n$, we get

$$\int_\Omega v\xi+\int_{\mathbb{R}^N}\int_\Omega\frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla\xi(x)\,dx\,dz=\int_\Omega\phi\xi \tag{2.11}$$

for every smooth ξ and by approximation for every $\xi\in W^{1,p}(\Omega)$.

We now show that

$$\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi(x,z)z\cdot\nabla\xi(x)\,dx\,dz = \int_\Omega |\nabla v|^{p-2}\,\nabla v\cdot\nabla\xi. \tag{2.12}$$

Taking $\xi = u_{\varepsilon_n}$ in (2.9):

$$\int_\Omega u_\varepsilon^2(x)dx + \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi_\Omega(x+\varepsilon z)\left|\frac{\overline{u}_\varepsilon(x+\varepsilon z)-u_\varepsilon(x)}{\varepsilon}\right|^p dxdz = \int_\Omega \phi(x)u_{\varepsilon_n}dx.$$

Taking now limits, using that $\displaystyle\int_\Omega v^2 \le \liminf_n \int_\Omega u_{\varepsilon_n}^2$ and (2.11) for $\xi = v$ we get:

$$\begin{aligned}&\limsup_n \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi_\Omega(x+\varepsilon_n z)\left|\frac{\overline{u}_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dz\\ &\le \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi(x,z)\,z\cdot\nabla v(x)\,dx\,dz.\end{aligned} \tag{2.13}$$

By the monotonicity Lemma 2.3, for every ρ smooth,

$$
\begin{aligned}
&-\frac{C_{J,p}}{\varepsilon_n^{p+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right)|\rho(y)-\rho(x)|^{p-2}(\rho(y)-\rho(x))\,dy\,(u_{\varepsilon_n}(x)-\rho(x))\,dx\\
&\le -\frac{C_{J,p}}{\varepsilon_n^{p+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right)|u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x)|^{p-2}\\
&\qquad\qquad\times(u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x))\,dy\,(u_{\varepsilon_n}(x)-\rho(x))\,dx.
\end{aligned}
$$

Using the same change of variable that we used above and taking limits, on account of (2.10) and (2.13), we obtain, for every smooth ρ,

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)|z\cdot\nabla\rho(x)|^{p-2}z\cdot\nabla\rho(x)\,z\cdot(\nabla v(x)-\nabla\rho(x))\,dx\,dz\\
&\le \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot(\nabla v(x)-\nabla\rho(x))\,dx\,dz,
\end{aligned}
$$

and then for every $\rho\in W^{1,p}(\Omega)$.

Taking $\rho = v \pm \lambda\xi$, $\lambda > 0$ and $\xi \in W^{1,p}(\Omega)$, and letting $\lambda \to 0$, we get

$$\int_{\mathbb{R}^N}\int_{\Omega} \frac{C_{J,p}}{2} J(z)\chi(x,z) z \cdot \nabla\xi(x)\, dx\, dz$$

$$= \int_{\mathbb{R}^N} \frac{C_{J,p}}{2} J(z) \int_{\Omega} |z \cdot \nabla v(x)|^{p-2} \left(z \cdot \nabla v(x)\right) \left(z \cdot \nabla \xi(x)\right)\, dx\, dz.$$

Consequently,

$$\int_{\mathbb{R}^N}\int_{\Omega} \frac{C_{J,p}}{2} J(z)\chi(x,z) z \cdot \nabla\xi(x)\, dx\, dz = \int_{\Omega} \mathbf{a}(\nabla v) \cdot \nabla \xi \quad \text{for every } \xi \in W^{1,p}(\Omega),$$

where

$$\mathbf{a}_j(\xi) = C_{J,p} \int_{\mathbb{R}^N} \frac{1}{2} J(z)\, |z \cdot \xi|^{p-2}\; z \cdot \xi\, z_j\, dz.$$

Then, since

$$\mathbf{a}(\xi) = |\xi|^{p-2}\xi, \tag{2.14}$$

we obtain that (2.12) is true and

$$v = (I + B_p)^{-1}\, \phi.$$

And the proof finishes using Theorem 2.9, 2. $\square$

The proof of (2.14) is an exercise: Use that $\mathbf{a}$ is positively homogeneous of degree $p-1$, that J is a radial function, and an adequate change of variables.

From the above theorem, and Theorem 1.12 we obtain:

THEOREM 2.11. *Assume that* $J(x) \geq J(y)$ *if* $|x| \leq |y|$. *Let* $T > 0$ *and* $u_0 \in L^q(\Omega)$, $p \leq q < +\infty$. *Let* u_ε *be the unique solution of* (2.1) *with* J *replaced by* $J_{p,\varepsilon}$ *and* v *the unique solution of* (2.4). *Then*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon(\cdot,t) - v(\cdot,t)\|_{L^q(\Omega)} = 0. \tag{2.15}$$

Moreover, if $1 < p \leq 2$, (2.15) *holds for any* $u_0 \in L^q(\Omega)$, $1 \leq q < +\infty$.

PROOF. Since B_p^J is completely accretive and satisfies the range condition (2.2), to get (2.15) it is enough to see that

$$\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1} \phi \to (I + B_p)^{-1} \phi \quad \text{in } L^q(\Omega) \text{ as } \varepsilon \to 0$$

for any $\phi \in L^\infty(\Omega)$ (see Theorem 1.12). Taking into account that

$$\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1} \phi \ll \phi,$$

the above convergence follows by (2.7). □

PROOF OF THE PRECOMPACTNESS THEOREM 2.9. From (2.6),

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_D \rho(z)\chi_D\left(x+\frac{1}{n}z\right)\left|\frac{\overline{f}_n\left(x+\frac{1}{n}z\right)-f_n(x)}{1/n}\right|^q dx\,dz \\
&= \int_D\int_D n^N\rho(n(x-y))\left|\frac{f_n(y)-f_n(x)}{1/n}\right|^q dx\,dy \le M.
\end{aligned}
\tag{2.16}
$$

On the other hand, if $\varphi\in\mathcal{D}(D)$ and $\psi\in\mathcal{D}(\mathbb{R}^N)$, for n large enough,

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(\rho(z))^{1/q}\int_D \chi_D\left(x+\frac{1}{n}z\right)\frac{\overline{f}_n\left(x+\frac{1}{n}z\right)-f_n(x)}{1/n}\varphi(x)\,dx\,\psi(z)\,dz \\
&= -\int_{\mathbb{R}^N}(\rho(z))^{1/q}\int_D f_n(x)\frac{\varphi(x)-\overline{\varphi}\left(x-\frac{1}{n}z\right)}{1/n}\,dx\,\psi(z)\,dz.
\end{aligned}
\tag{2.17}
$$

Let us start with the case 1(i): suppose $f_n\to f$ weakly in $L^q(D)$. By (2.16), up to a subsequence,

$$
(\rho(z))^{1/q}\chi_D\left(x+\frac{1}{n}z\right)\frac{\overline{f}_n\left(x+\frac{1}{n}z\right)-f_n(x)}{1/n}\rightharpoonup(\rho(z))^{1/q}g(x,z)\ \text{weak–}L^q(D)\times L^q(\mathbb{R}^N).
$$

Therefore, passing to the limit in (2.17), we get

$$\int_{\mathbb{R}^N} (\rho(z))^{1/q} \int_D g(x,z)\varphi(x)\,dx\,\psi(z)\,dz$$

$$= -\int_{\mathbb{R}^N} (\rho(z))^{1/q} \int_D f(x)\,z\cdot\nabla\varphi(x)\,dx\,\psi(z)\,dz.$$

Consequently,

$$\int_D g(x,z)\varphi(x)\,dx = -\int_D f(x)\,z\cdot\nabla\varphi(x)\,dx, \qquad \forall\, z \in \mathrm{int}(\mathrm{supp}(\rho)).$$

This implies $f \in W^{1,q}(D)$ and

$$(\rho(z))^{1/q}\, g(x,z) = (\rho(z))^{1/q}\, z\cdot\nabla f(x) \quad \text{in } D\times\mathbb{R}^N.$$

Let us now prove 1(ii). By (2.16), there exists a bounded Radon measure $\mu \in \mathcal{M}(D\times\mathbb{R}^N)$ such that, up to a subsequence,

$$\rho(z)\chi_D\left(x+\frac{1}{n}z\right)\frac{\overline{f}_n\left(x+\frac{1}{n}z\right)-f_n(x)}{1/n} \rightharpoonup \mu(x,z) \quad \text{weakly in } \mathcal{M}(D\times\mathbb{R}^N).$$

Hence, passing to the limit in (2.17), we get

$$\int_{D\times\mathbb{R}^N} \varphi(x)\psi(z)d\mu(x,z) = -\int_{D\times\mathbb{R}^N} \rho(z)f(x)\, z\cdot\nabla\varphi(x)\psi(z)\,dx\,dz.$$

Now, applying the *disintegration theorem* to the measure μ, we get that:

$$f \in BV(D)$$

and

$$\mu(x,z) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(x)\rho(z)z_i\mathcal{L}^N(z).$$

The proof of 2 follows the same steps of the proof of [**23**, Theorem 4].

$\square$

2.2. The Neumann problem for the nonlocal total variation flow for non–degenerate kernels

Motivated by problems in image processing, the Neumann problem for the total variation flow is studied in [**3**] (see also [**4**]):

$$\begin{cases} v_t = \operatorname{div}\left(\dfrac{Dv}{|Dv|}\right) & \text{in } \Omega \times (0,T), \\ \dfrac{Dv}{|Dv|} \cdot \eta = 0 & \text{on } \partial\Omega \times (0,T), \\ v(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases} \tag{2.18}$$

The operator $\operatorname{div}\left(\dfrac{Dv}{|Dv|}\right)$ is also known as 1-Laplacian: $\Delta_1 v$.

The aim of this section is to study the nonlocal version of problem (2.18), which can be written formally as

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y) \frac{u(y,t) - u(x,t)}{|u(y,t) - u(x,t)|} \, dy, \quad x \in \Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases} \tag{2.19}$$

Here, again, $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with compact support, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x) dx = 1$.

2.2.1. Existence and uniqueness.

DEFINITION 2.12. A *solution* of (2.19) in $[0,T]$ is a function
$$u \in W^{1,1}(0,T;L^1(\Omega))$$
which satisfies $u(x,0) = u_0(x)$ a.e. $x \in \Omega$ and
$$u_t(x,t) = \int_\Omega J(x-y)g(x,y,t)\,dy \quad \text{a.e. in } \Omega \times (0,T),$$
for some $g \in L^\infty(\Omega \times \Omega \times (0,T))$ with $\|g\|_\infty \leq 1$ such that $g(x,y,t) = -g(y,x,t)$ and
$$J(x-y)g(x,y,t) \in J(x-y)\mathrm{sgn}(u(y,t)-u(x,t)),$$
where sgn is the multivalued sign function.

To prove the existence and uniqueness of this kind of solutions, the idea is to take the limit as $p \searrow 1$ to the solutions of (2.1) studied previously, and use again Nonlinear Semigroup Theory.

So, we begin by introducing an operator in $L^1(\Omega)$ associated to our problem.

DEFINITION 2.13. We define the operator B_1^J in $L^1(\Omega)\times L^1(\Omega)$ by $\hat{u}\in B_1^J u$ if and only if $u,\hat{u}\in L^1(\Omega)$, and there exists $g\in L^\infty(\Omega\times\Omega)$, $g(x,y)=-g(y,x)$ for almost all $(x,y)\in\Omega\times\Omega$, $\|g\|_\infty\le 1$, such that

$$J(x-y)g(x,y)\in J(x-y)\operatorname{sgn}(u(y)-u(x)) \quad \text{a.e. } (x,y)\in\Omega\times\Omega \tag{2.20}$$

and

$$\hat{u}(x)=-\int_\Omega J(x-y)g(x,y)\,dy \quad \text{a.e. } x\in\Omega.$$

It is not difficult to see that:

- (2.20) is equivalent to

$$-\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,u(x)\,dx=\frac{1}{2}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|\,dy\,dx;$$

- $L^1(\Omega)=D(B_1^J)$;
- B_1^J is closed in $L^1(\Omega)\times L^1(\Omega)$;
- and B_1^J is positively homogeneous of degree zero.

THEOREM 2.14. *The operator B_1^J is completely accretive and satisfies the range condition*

$$L^\infty(\Omega) \subset \mathrm{R}(I + B_1^J).$$

PROOF. Let $\hat{u}_i \in B_1^J u_i$, $i = 1, 2$. Then there exists $g_i \in L^\infty(\Omega \times \Omega)$, $\|g_i\|_\infty \leq 1$, $g_i(x,y) = -g_i(y,x)$, $J(x-y)g_i(x,y) \in J(x-y)\mathrm{sgn}(u_i(y) - u_i(x))$ for almost all $(x,y) \in \Omega \times \Omega$, such that

$$\hat{u}_i(x) = -\int_\Omega J(x-y)g_i(x,y)\,dy \quad \text{a.e. } x \in \Omega, \quad i = 1, 2.$$

Let $q \in P_0$. We have

$$\int_\Omega (\hat{u}_1(x) - \hat{u}_2(x))q(u_1(x) - u_2(x))\,dx$$

$$= \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(g_1(x,y) - g_2(x,y))\left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x))\right)\,dx\,dy.$$

Therefore,

$$\int_\Omega (\hat{u}_1(x) - \hat{u}_2(x)) q(u_1(x) - u_2(x))\, dx$$

$$= \frac{1}{2} \int \int_{\{(x,y): u_1(y) \neq u_1(x), u_2(y) = u_2(x)\}} J(x-y)(g_1(x,y) - g_2(x,y))$$

$$\times \left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x))\right)\, dx\, dy$$

$$+ \frac{1}{2} \int \int_{\{(x,y): u_1(y) = u_1(x), u_2(y) \neq u_2(x)\}} J(x-y)(g_1(x,y) - g_2(x,y))$$

$$\times \left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x))\right)\, dx\, dy$$

$$+ \frac{1}{2} \int \int_{\{(x,y): u_1(y) \neq u_1(x), u_2(y) \neq u_2(x)\}} J(x-y)(g_1(x,y) - g_2(x,y))$$

$$\times \left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x))\right)\, dx\, dy,$$

$$\geq 0.$$

Hence, B_1^J is a completely accretive operator.

Let us prove not that B_1^J satisfies the range condition.

We will see that for any $\phi \in L^\infty(\Omega)$,

$$\lim_{p\to 1+} (I + B_p^J)^{-1}\phi = (I + B_1^J)^{-1}\phi \quad \text{weakly in } L^1(\Omega).$$

Write $u_p := \left(I + B_p^J\right)^{-1}\phi$ for $1 < p < +\infty$. Then

$$u_p(x) - \int_\Omega J\left(x-y\right)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))\,dy = \phi(x) \quad \text{a.e. } x \in \Omega.$$

Thus, for every $\xi \in L^\infty(\Omega)$, we can write

$$\begin{aligned} &\int_\Omega u_p\xi - \int_\Omega\int_\Omega J\left(x-y\right)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))\,dy\,\xi(x)\,dx \\ &= \int_\Omega \phi\xi. \end{aligned} \tag{2.21}$$

We have $u_p \ll \phi$. Hence, by Proposition 1.16, there exists a sequence $p_n \to 1$ such that

$$u_{p_n} \rightharpoonup u \quad \text{weakly in } L^1(\Omega), \quad u \ll \phi.$$

Consequently, we also have $\|u_{p_n}\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)} \le \|\phi\|_{L^\infty(\Omega)}$.

Now, since

$$\left| |u_{p_n}(y) - u_{p_n}(x)|^{p_n-2} \left(u_{p_n}(y) - u_{p_n}(x)\right) \right| \leq (2\|\phi\|_\infty)^{p_n-1},$$

there exists $g(x,y)$ such that

$$|u_{p_n}(y) - u_{p_n}(x)|^{p_n-2} \left(u_{p_n}(y) - u_{p_n}(x)\right) \rightharpoonup g(x,y)$$

weakly in $L^1(\Omega\times\Omega)$, $g(x,y) = -g(y,x)$ for almost all $(x,y) \in \Omega\times\Omega$, and $\|g\|_\infty \leq 1$.

Passing to the limit in (2.21) for $p = p_n$, we get

$$\int_\Omega u\xi - \int_\Omega \int_\Omega J(x-y)g(x,y)\,dy\,\xi(x)\,dx = \int_\Omega \phi\xi \tag{2.22}$$

for every $\xi \in L^\infty(\Omega)$, and consequently

$$u(x) - \int_\Omega J(x-y)g(x,y)\,dy = \phi(x) \qquad \text{a.e. in } \Omega.$$

Then, to finish the proof we have to show that

$$-\int_\Omega \int_\Omega J(x-y)g(x,y)\,dy\,u(x)\,dx = \frac{1}{2}\int_\Omega \int_\Omega J(x-y)|u(y)-u(x)|\,dy\,dx. \tag{2.23}$$

In fact, by (2.21) with $p = p_n$, $\xi = u_{p_n}$, and (2.22) with $\xi = u$,

$$\begin{aligned} &\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\left|u_{p_n}(y)-u_{p_n}(x)\right|^{p_n}\,dy\,dx \\ &= \int_\Omega \phi u_{p_n} - \int_\Omega u_{p_n}u_{p_n} = \int_\Omega \phi u - \int_\Omega uu - \int_\Omega \phi(u-u_{p_n}) \\ &\quad + \int_\Omega 2u(u-u_{p_n}) - \int_\Omega (u-u_{p_n})(u-u_{p_n}) \\ &\le -\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,u(x)\,dx - \int_\Omega \phi(u-u_{p_n}) + \int_\Omega 2u(u-u_{p_n}), \end{aligned}$$

and so,

$$\begin{aligned} \limsup_{n\to+\infty}\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\left|u_{p_n}(y)-u_{p_n}(x)\right|^{p_n}\,dy\,dx \\ \le -\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,u(x)\,dx. \end{aligned}$$

By the monotonicity Lemma 2.3,

$$-\int_\Omega\int_\Omega J(x-y)|\rho(y)-\rho(x)|^{p_n-2}(\rho(y)-\rho(x))\,dy\,(u_{p_n}(x)-\rho(x))\,dx$$

$$\leq -\int_\Omega\int_\Omega J(x-y)|u_{p_n}(y)-u_{p_n}(x)|^{p_n-2}(u_{p_n}(y)-u_{p_n}(x))\,dy\,(u_{p_n}(x)-\rho(x))dx.$$

Therefore, taking limits in n,

$$-\int_\Omega\int_\Omega J(x-y)\,\mathrm{sgn}_0(\rho(y)-\rho(x))\,dy\,(u(x)-\rho(x))\,dx$$

$$\leq -\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,(u(x)-\rho(x))\,dx.$$

Taking $\rho = u \pm \lambda u$, $\lambda > 0$, and letting $\lambda \to 0$, we get (2.23), and the proof is finished.

$\square$

THEOREM 2.15. *Let* $u_0 \in L^1(\Omega)$. *Then there exists a unique solution of* (2.19).

Moreover, if u_i *is a solution in* $[0,T]$ *of* (2.19) *with initial data* $u_{i0} \in L^1(\Omega)$, $i = 1, 2$, *then*

$$\int_\Omega (u_1(t) - u_2(t))^+ \leq \int_\Omega (u_{10} - u_{20})^+ \quad \text{for every } t \in [0,T].$$

PROOF. As a consequence of the above results, by Theorem 1.7, we have that the abstract Cauchy problem

$$\begin{cases} u'(t) + B_1^J u(t) \ni 0, & t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{2.24}$$

has a unique mild solution u for every initial datum $u_0 \in L^1(\Omega)$ and $T > 0$. Moreover, due to the complete accretivity of the operator $B_{1,\psi}^J$, the mild solution of (2.24) is a strong solution and a solution in the sense of Definition 2.12.

$\square$

2.2.2. Rescaling the kernel. Convergence to the total variation flow.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. We will see that the solutions of problem (2.19):

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)\frac{u(y,t)-u(x,t)}{|u(y,t)-u(x,t)|}\,dy, & x\in\Omega,\ t>0, \\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}$$

with the kernel J rescaled in a suitable way, converge, as the scaling parameter goes to zero, to the solutions of the Neumann problem for the total variation flow (2.18):

$$\begin{cases} v_t = \operatorname{div}\left(\dfrac{Dv}{|Dv|}\right) & \text{in } \Omega\times(0,T), \\ \dfrac{Dv}{|Dv|}\cdot\eta = 0 & \text{on } \partial\Omega\times(0,T), \\ v(\cdot,0)=u_0 & \text{in } \Omega. \end{cases}$$

Solution to (2.18) were also obtained in [**3**] using the techniques of completely accretive operators and the Crandall-Liggett semigroup generation theorem. To this end, the following operator in $L^1(\Omega)$ was defined:

The 1*–Laplacian operator with Dirichlet boundary conditions.*

$(v, \hat{v}) \in B_1$ if and only if $v, \hat{v} \in L^1(\Omega)$, $T_k(v) \in BV(\Omega)$ for all $k > 0$ and there exists $\zeta \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|\zeta\|_\infty \leq 1$ such that

$$\hat{v} = -\mathrm{div}(\zeta) \quad \text{in } \mathcal{D}'(\Omega)$$

and

$$\int_\Omega (\xi - T_k(v))\hat{v}\, dx \leq \int_\Omega \zeta \cdot \nabla \xi \, dx - |DT_k(v)|(\Omega),$$

$\forall\, \xi \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, $\forall\, k > 0$.

And it was proved:

THEOREM 2.16. *The operator B_1 is m-completely accretive in $L^1(\Omega)$ with dense domain. For any $u_0 \in L^1(\Omega)$ the semigroup solution $v(t) = e^{-tB_1}u_0$ is a strong solution of problem* (2.18)*.*

Now we return to the analysis of the nonlocal problem and set

$$J_{1,\varepsilon}(x) := \frac{C_{J,1}}{\varepsilon^{1+N}} J\left(\frac{x}{\varepsilon}\right),$$

with ${C_{J,1}}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|\,dz$ being a normalizing constant.

Let u_ε be the solution of problem (2.19) with J replaced by $J_{1,\varepsilon}$ and the same initial condition u_0. The main result now states that these functions u_ε converge strongly to the solution of the local problem (2.18).

Theorem 2.17. *Suppose $J(x) \geq J(y)$ if $|x| \leq |y|$. Let $T > 0$ and $u_0 \in L^1(\Omega)$. Let u_ε be the unique solution in $[0,T]$ of (2.19) with J replaced by $J_{1,\varepsilon}$ and v the unique solution of (2.18). Then*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_\varepsilon(\cdot,t) - v(\cdot,t)\|_{L^1(\Omega)} = 0.$$

Arguing as in the proof of Theorem 2.11, since the solutions of the above theorem coincide with the semigroup solutions, by Theorem 1.12, to prove Theorem 2.17 it is enough to obtain the following result:

Theorem 2.18. *Suppose $J(x) \geq J(y)$ if $|x| \leq |y|$. Then, for any $\phi \in L^\infty(\Omega)$,*

$$\left(I + B_1^{J_{1,\varepsilon}}\right)^{-1}\phi \to (I + B_1)^{-1}\phi \quad \textit{in } L^1(\Omega) \textit{ as } \varepsilon \to 0.$$

PROOF. Given $\varepsilon > 0$, we set $u_\varepsilon := \left(I + B_1^{J_{1,\varepsilon}}\right)^{-1} \phi$. Then there exists $g_\epsilon \in L^\infty(\Omega \times \Omega)$, $g_\varepsilon(x,y) = -g_\varepsilon(y,x)$ for almost all $x, y \in \Omega$, $\|g_\varepsilon\|_\infty \leq 1$, such that

$$J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y) \in J\left(\frac{x-y}{\varepsilon}\right) \operatorname{sgn}(u_\varepsilon(y) - u_\varepsilon(x)) \quad \text{for a.e. } x, y \in \Omega$$

and

$$-\frac{C_{J,1}}{\varepsilon^{1+N}} \int_\Omega J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y) dy = \phi(x) - u_\varepsilon(x) \qquad \text{for a.e. } x \in \Omega. \tag{2.25}$$

Moreover $u_\varepsilon \ll \phi$. Hence, by Proposition 1.16, there exists a sequence $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n} \rightharpoonup v \quad \text{weakly in } L^1(\Omega), \quad u \ll \phi.$$

Consequently, $\|u_{\varepsilon_n}\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)} \leq \|\phi\|_{L^\infty(\Omega)}$.

Observe that

$$-\int_\Omega \int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right) g_{\varepsilon_n}(x,y) dy\, u_{\varepsilon_n}(x)\, dx = \frac{1}{2} \int_\Omega \int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right) |u_{\varepsilon_n}(y) - u_{\varepsilon_n}(x)|\, dy\, dx. \tag{2.26}$$

Changing variables and having in mind (2.26), we get

$$\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,1}}{2} J(z)\chi_\Omega(x+\varepsilon_n z)\left|\frac{\overline{u}_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right| dx\,dz$$

$$= \int_\Omega\int_\Omega \frac{1}{2}\frac{C_{J,1}}{{\varepsilon_n}^N} J\left(\frac{x-y}{\varepsilon_n}\right)\left|\frac{u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right| dx\,dy$$

$$= \int_\Omega (\phi(x)-u_{\varepsilon_n}(x))u_{\varepsilon_n}(x)\,dx \le M, \qquad \forall\, n\in\mathbb{N}.$$

Therefore, by Theorem 2.9, $v\in BV(\Omega)$,

$$\frac{C_{J,1}}{2} J(z)\chi_\Omega(x+\varepsilon_n z)\frac{\overline{u}_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n} \rightharpoonup \frac{C_{J,1}}{2} J(z) z\cdot Dv$$

weakly as measures and

$$u_{\varepsilon_n}\to v \qquad \text{strongly in } L^1(\Omega).$$

Moreover, we also can assume that

$$J(z)\chi_\Omega(x+\varepsilon_n z)\overline{g}_{\varepsilon_n}(x, x+\varepsilon_n z) \rightharpoonup \Lambda(x,z)$$

weakly* in $L^\infty(\Omega)\times L^\infty(\mathbb{R}^N)$, with $\Lambda(x,z)\le J(z)$ almost everywhere in $\Omega\times\mathbb{R}^N$.

From (2.25) we have

$$\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\int_\Omega J(z)\chi_\Omega(x+\varepsilon_n z)\overline{g}_{\varepsilon_n}(x,x+\varepsilon_n z)\,dz\frac{\overline{\xi}(x+\varepsilon_n z)-\xi(x)}{\varepsilon_n}\,dx$$

$$=\int_\Omega(\phi(x)-u_{\varepsilon_n}(x))\xi(x)\,dx, \qquad \forall\,\xi\in L^\infty(\Omega).$$

And passing to the limit we get

$$\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\int_\Omega\Lambda(x,z)z\cdot\nabla\xi(x)\,dx\,dz=\int_\Omega(\phi(x)-v(x))\xi(x)\,dx \tag{2.27}$$

for all smooth ξ and, by approximation, for all $\xi\in L^\infty(\Omega)\cap W^{1,1}(\Omega)$.

We denote by $\zeta=(\zeta_1,\ldots,\zeta_N)$ the vector field defined by

$$\zeta_i(x):=\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,z)z_i\,dz, \qquad i=1,\ldots,N.$$

Then $\zeta\in L^\infty(\Omega,\mathbb{R}^N)$, and from (2.27),

$$-\mathrm{div}(\zeta)=\phi-v \quad \text{ in } \mathcal{D}'(\Omega).$$

Let us show that

$$\|\zeta\|_\infty\le 1. \tag{2.28}$$

Given $\xi \in \mathbb{R}^N \setminus \{0\}$, consider R_ξ the rotation such that $R_\xi^t(\xi) = \mathbf{e}_1|\xi|$. Then, if we make the change of variables $z = R_\xi(y)$, we obtain

$$\zeta(x) \cdot \xi = \frac{C_{J,1}}{2} \int_{\mathbb{R}^N} \Lambda(x,z) z \cdot \xi \, dz = \frac{C_{J,1}}{2} \int_{\mathbb{R}^N} \Lambda(x, R_\xi(y)) R_\xi(y) \cdot \xi \, dy$$

$$= \frac{C_{J,1}}{2} \int_{\mathbb{R}^N} \Lambda(x, R_\xi(y)) y_1 |\xi| \, dy.$$

On the other hand, since J is a radial function and $\Lambda(x,z) \leq J(z)$ almost everywhere,

$$C_{J,1}^{-1} = \frac{1}{2} \int_{\mathbb{R}^N} J(z)|z_1| \, dz$$

and

$$|\zeta(x) \cdot \xi| \leq \frac{C_{J,1}}{2} \int_{\mathbb{R}^N} J(y)|y_1| \, dy |\xi| = |\xi| \quad \text{a.e. } \ x \in \Omega.$$

Therefore, (2.28) holds.

Since $v \in L^\infty(\Omega)$, to finish the proof we only need to show that

$$\int_\Omega (\xi - v)(\phi - v) \, dx \leq \int_\Omega \zeta \cdot \nabla \xi \, dx - |Dv|(\Omega) \quad \forall \, \xi \in W^{1,1}(\Omega) \cap L^\infty(\Omega).$$

For w smooth we have that

$$\int_\Omega (\phi(x) - u_{\varepsilon_n}(x))(w(x) - u_{\varepsilon_n}(x))\,dx$$

$$= \frac{C_{J,1}}{2} \int_{\mathbb{R}^N} \int_\Omega J(z)\chi_\Omega(x+\varepsilon_n z)\overline{g}_{\varepsilon_n}(x, x+\varepsilon_n z)\,dz \frac{\overline{w}(x+\varepsilon_n z) - w(x)}{\varepsilon_n}\,dx$$

$$- \frac{C_{J,1}}{2} \int_{\mathbb{R}^N} \int_\Omega J(z)\chi_\Omega(x+\varepsilon_n z)\left|\frac{\overline{u}_{\varepsilon_n}(x+\varepsilon_n z) - u_{\varepsilon_n}(x)}{\varepsilon_n}\right|\,dx.$$

Then, taking limits as $n \to \infty$, we get

$$\int_\Omega (w - v)(\phi - v)\,dx$$

$$\leq \frac{C_{J,1}}{2} \int_\Omega \int_{\mathbb{R}^N} \Lambda(x,z) z \cdot \nabla w(x)\,dx\,dz - \frac{C_{J,1}}{2} \int_\Omega \int_{\mathbb{R}^N} |J(z) z \cdot Dv|$$

$$= \int_\Omega \zeta \cdot \nabla w\,dx - \frac{C_{J,1}}{2} \int_\Omega \int_{\mathbb{R}^N} |J(z) z \cdot Dv| = \int_\Omega \zeta \cdot \nabla w\,dx - \int_\Omega |Dv|,$$

for all smooth w and, by approximation, for all $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. $\square$

2.3. The Dirichlet problem for fractional 1–Lapacian evolution

Our aim here is to study the evolution equation associated to a nonlinear version of the fractional Laplacian, the fractional 1–Laplacian with Dirichlet boundary conditions, that formally we write as

$$
\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \frac{u(y,t)-u(x,t)}{|u(y,t)-u(x,t)|}\, dy, & x \in \Omega,\ t>0. \\ u(x,t) = 0, & x \in \mathbb{R}^N \setminus \Omega,\ t>0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.29}
$$

where $0 < s < 1$ and Ω is a bounded smooth domain in $\mathbb{R}^N$.

Changing the ambient space. We have to change the underlying space, now L^p is not adequate for these kind of nonlocal problems but they will be fractional Sobolev spaces. Let the (s,p)–Gagliardo seminorm of a measurable function u in Ω be

$$
[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(y)-u(x)|^p}{|x-y|^{N+sp}}\, dxdy \right)^{\frac{1}{p}}.
$$

We consider the fractional Sobolev space

$$
W^{s,p}(\Omega) = \{ u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} < +\infty \},
$$

which is a Banach space respect to the norm $\|u\|_{W^{s,p}(\Omega)} := [u]_{W^{s,p}(\Omega)} + \|u\|_{L^p(\Omega)}$.

We also consider by $\widetilde{W}_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the norm

$$u \mapsto [u]_{W^{s,p}(\mathbb{R}^N)} + \|u\|_{L^p(\Omega)}.$$

Functions in the space $\widetilde{W}_0^{s,p}(\Omega)$ can be defined in the whole space $W^{s,p}(\mathbb{R}^N)$ by extending then by zero outside Ω, as we will consider.

See [**32**] and [**24**] for a good overview of fractional Sobolev spaces.

For $1 < p < \infty$, we can define the fractional p–Laplacian $\Delta_p^s u$ trough the Euler-Lagrange associated to the minimization of $[u]^p_{W^{s,p}(\mathbb{R}^N)}$:

$$\Delta_p^s u(x) := \text{P.V.} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} |u(y)-u(x)|^{p-2}(u(y)-u(x))\, dy, \quad x \in \mathbb{R}^N.$$

Proposition 2.19 ([**48**]). *For any $f \in L^2(\Omega)$, there exists a unique $u \in \widetilde{W}_0^{s,p}(\Omega) \cap L^2(\Omega)$ solving the Dirichlet problem*

$$\begin{cases} u(x) - \Delta_p^s u(x) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

in the following sense:

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} |u(y)-u(x)|^{p-2}(u(y)-u(x))(\varphi(y)-\varphi(x))\, dydx = \int_\Omega (f-u)\varphi,$$

for all $\varphi \in \widetilde{W}_0^{s,p}(\Omega) \cap L^2(\Omega)$.

We now define formally the *fractional 1–Laplacian operator of order s* of a function $u \in W^{s,1}(\mathbb{R}^N)$:

$$\Delta_1^s u(x) = \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \frac{u(y)-u(x)}{|u(y)-u(x)|}\, dy, \quad x \in \mathbb{R}^N.$$

Solutions to the Dirichlet problem associated with this operator Δ_1^s will be in a larger space than $\widetilde{W}_0^{s,1}(\Omega)$, they live in the space

$$\mathcal{W}_0^{s,1}(\Omega) := \{u \in L^1(\Omega) : [u]_{W^{s,1}(\mathbb{R}^N)} < \infty \text{ and } u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

Definition 2.20. Given $v \in L^2(\Omega)$, we say that $u \in \mathcal{W}_0^{s,1}(\Omega)$ is a *weak solution* to the Dirichlet problem

$$\begin{cases} -\Delta_1^s u(x) = v(x) & \text{in } \Omega \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.30}$$

if there exists $\eta \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, $\eta(x,y) = -\eta(y,x)$ for almost all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $\|\eta\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} \leq 1$, such that

$$\eta(x,y) \in \operatorname{sign}(u(y)-u(x)) \quad \text{a.e. } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N, \quad \text{and}$$

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \eta(x,y)(\varphi(y)-\varphi(x))\, dy dx = \int_\Omega v(x)\varphi(x)\, dx$$

for all $\varphi \in \mathcal{W}_0^{s,1}(\Omega) \cap L^2(\Omega)$.

Definition 2.21. Given $u_0 \in L^2(\Omega)$, we say that u is a *solution* of the Dirichlet problem (2.29) in $[0,T]$, if $u \in W^{1,1}(0,T;L^2(\Omega))$, $u(0,\cdot) = u_0$, and, for almost all $t \in (0,T)$,

$$\begin{cases} u_t(t,\cdot) = \Delta_1^s u(t,\cdot) & \text{in } \Omega, \\ u(t,\cdot) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

in the sense of Definition 2.20; that is, if there exists $\eta(t,\cdot,\cdot) \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ such that

$$\eta(t,x,y) = -\eta(t,y,x) \quad \text{for almost all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

$$\|\eta(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \leq 1,$$

$$\eta(t,x,y) \in \operatorname{sign}(u(t,y) - u(t,x)) \qquad \text{for a.e. } (t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N,$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta(t,x,y)(\varphi(y)-\varphi(x))\,dydx = -\int_\Omega u_t(t,x)\varphi(x)\,dx$$

for all $\varphi \in \mathcal{W}_0^{s,1}(\Omega) \cap L^2(\Omega)$.

2.3.1. Existence and uniqueness. To study the Dirichlet problem (2.29) we consider the energy functional:

$$\mathcal{D}_1^s(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,|u(y)-u(x)|\,dxdy & \text{if } u \in \mathcal{W}_0^{s,1}(\Omega)\cap L^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus \mathcal{W}_0^{s,1}(\Omega). \end{cases}$$

$\mathcal{D}_1^s$ is convex and lower semi-continuous in $L^2(\Omega)$. Then, the subdifferential $\partial\mathcal{D}_1^s$ is a maximal monotone operator in $L^2(\Omega)$. We characterize the subdifferential $\partial\mathcal{D}_1^s$ in the following way.

Definition 2.22. We define in $L^2(\Omega)\times L^2(\Omega)$ the operator $D_{1,s}$ as:

$$(u,v)\in D_{1,s} \iff u,v\in L^2(\Omega) \text{ and } u \text{ is a weak solution to problem (2.30).}$$

Theorem 2.23. *The operator $D_{1,s}$ is m–completely accretive in $L^2(\Omega)$ with dense domain. Moreover,*

$$D_{1,s} = \partial\mathcal{D}_1^s.$$

Proof. The proof of the complete accretiveness is similar to the one in the previous section. So, let us see that the operator $D_{1,s}$ satisfies the range condition

$$L^2(\Omega)\subset R(I+D_{1,s}). \tag{2.31}$$

In what follows, C will denote a constant independent of p that may change from one line to another.

Set $p^\star := \frac{Np}{N-sp}$ for the fractional critical exponent for $1 \le p < \frac{N}{s}$.

For $1 < p < \frac{N}{s}$, take $s_p := \frac{N}{(p^\star_s)'}$. We have $0 < s_p < 1$ for all $1 < p < (N^\star)' = \frac{N}{N+s-1} \le \frac{N}{s}$.

Then, given $f \in L^2(\Omega)$, for $1 < p < (N^\star)'$, applying Proposition 2.19, there exists $u_p \in \widetilde{W}_0^{s_p,p}(\Omega) \cap L^2(\Omega)$ such that (since $N + s_p p = (N+s)p$)

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{(N+s)p}} |u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))(\varphi(y)-\varphi(x))\, dydx = \int_\Omega (f-u_p)\varphi, \tag{2.32}$$

for all $\varphi \in \widetilde{W}_0^{s_p,p}(\Omega) \cap L^2(\Omega)$. Moreover, $u_p \ll f$ and, hence,

$$\|u_p\|_{L^q(\Omega)} \le \|f\|_{L^q(\Omega)} \qquad \forall 1 < p < (N^\star)', \quad \text{for any } 1 \le q \le 2. \tag{2.33}$$

By (2.33), there exists a sequence $p_n \downarrow 1$, such that

$$u_{p_n} \rightharpoonup u \quad \text{weakly in } L^2(\Omega), \text{ and } \|u\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}.$$

On the other hand, taking $\varphi = u_p$ in (2.32) we have

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{(N+s)p}} |u_p(y)-u_p(x)|^p\, dydx = \int_\Omega (f-u_p)u_p \le C \quad \forall 1 < p < (N^\star)'. \tag{2.34}$$

Now, since

$$\int_\Omega \int_\Omega \frac{1}{|x-y|^{N+s}} |u_p(y) - u_p(x)| \, dydx$$

$$\leq \left(\int_\Omega \int_\Omega \frac{1}{|x-y|^{(N+s)p}} |u_p(y) - u_p(x)|^p \, dydx \right)^{1/p} |\Omega \times \Omega|^{1/p'},$$

from (2.34) we get

$$\|u_p\|_{W^{s,1}(\Omega)} \leq C \qquad \forall 1 < p < (N^\star)'.$$

Hence, by the compact embedding Theorem 2.7 in [**24**], we have that for a subsequence of $\{p_n\}$, denoted equal,

$$u_{p_n} \to u \quad \text{strongly in } L^1(\Omega) \quad \text{and} \quad u \in \mathcal{W}_0^{s,1}(\Omega).$$

For $k > 0$ we set

$$C_{p,k} := \left\{ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N : \left| \frac{u_p(y) - u_p(x)}{|x-y|^{N+s}} \right| > k \right\}.$$

Then, by (2.34),

$$|C_{p,k}| \leq \frac{C}{k^p}. \tag{2.35}$$

On the other hand,

$$\left|\left|\frac{u_p(y)-u_p(x)}{|x-y|^{N+s}}\right|^{p-2}\frac{u_p(y)-u_p(x)}{|x-y|^{N+s}}\chi_{\mathbb{R}^N\times\mathbb{R}^N\setminus C_{p,k}}(x,y)\right|\leq k^{p-1}\qquad \forall (x,y)\in\mathbb{R}^N\times\mathbb{R}^N.$$

Therefore, for any $k\in\mathbb{N}$ there exists a subsequence of $\{p_n\}_n$, denoted by $\{p_{n_j^k}\}_j$, such that

$$\left|\frac{u_{p_{n_j^k}}(y)-u_{p_{n_j^k}}(x)}{|x-y|^{N+s}}\right|^{p_{n_j^k}-2}\frac{u_{p_{n_j^k}}(y)-u_{p_{n_j^k}}(x)}{|x-y|^{N+s}}\chi_{\mathbb{R}^N\times\mathbb{R}^N\setminus C_{p_{n_j^k},k}}(x,y)\overset{j\to\infty}{\rightharpoonup}\eta_k(x,y),$$

weakly* in $L^\infty(\mathbb{R}^N\times\mathbb{R}^N)$, with η_k antisymmetric such that $\|\eta_k\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}\leq 1$.

Now there exist a subsequence of $\{\eta_k\}_k$, $\{\eta_{k_j}\}_j$ such that,

$$\eta_{k_j}\overset{j\to\infty}{\rightharpoonup}\eta\quad \text{weakly* in } L^\infty(\mathbb{R}^N\times\mathbb{R}^N),$$

with η antisymmetric and

$$\|\eta\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}\leq 1.$$

Let us finally pass to the limit in (2.32).

Let us first take $\varphi \in \mathcal{D}(\Omega)$. For a fixed $1 < q_0 < \frac{N}{N+s-1}$, we can extend φ as 0 outside Ω, and then $\varphi \in W^{r_0,q_0}(\mathbb{R}^N)$ with $r_0 = \frac{(N+s)q_0 - N}{q_0} < 1$. Fix $k \in \mathbb{N}$. From (2.32):

(2.36)

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}-2} \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \chi_{\mathbb{R}^N\times\mathbb{R}^N\setminus C_{p_{n_j^k},k}}(x,y) \frac{\varphi(y)-\varphi(x)}{|x-y|^{N+s}}\, dydx$$

$$- \int_\Omega (f - u_{p_{n_j^k}})\varphi$$

$$= -\frac{1}{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}-2} \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \chi_{C_{p_{n_j^k},k}}(x,y) \frac{\varphi(y)-\varphi(x)}{|x-y|^{N+s}}\, dydx,$$

Now, for $p_{n_j^k} < q_0$, using Hölder's inequality, (2.34) and (2.35),

$$\left| \int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}-2} \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \chi_{C_{p_{n_j^k},k}}(x,y) \frac{\varphi(y)-\varphi(x)}{|x-y|^{N+s}}\, dydx \right|$$

$$= \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}} dydx \right)^{(p_{n_j^k}-1)/p_{n_j^k}} \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\varphi(y)-\varphi(x)|^{q_0}}{|x-y|^{N+r_0q_0}}\, dydx \right)^{1/q_0} |C_{p_{n_j^k},k}|^{\frac{q_0 - p_{n_j^k}}{p_{n_j^k} q_0}}$$

$$\leq \frac{C_\varphi}{k^{1-p_{n_j^k}/q_0}}.$$

Therefore, taking limits as $j \to \infty$ in (2.36), we get

$$\left| \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \eta_k(x,y)(\varphi(y)-\varphi(x))\, dydx - \int_\Omega (f-u)\varphi \right| \leq \frac{C_\varphi}{k}.$$

In particular,

$$\left| \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \eta_{k_j}(x,y)(\varphi(y)-\varphi(x))\, dydx - \int_\Omega (f-u)\varphi \right| \leq \frac{C_\varphi}{k_j}.$$

Therefore, taking now the limit as $j \to \infty$, we obtain that

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \eta(x,y)(\varphi(y)-\varphi(x))\, dydx - \int_\Omega (f-u)\varphi = 0. \tag{2.37}$$

Suppose now that $\varphi \in \mathcal{W}_0^{s,1}(\Omega) \cap L^2(\Omega)$. As in [**24**, Lemma 2.3], there exists $\varphi_n \in \mathcal{D}(\Omega)$ such that

$$\varphi_n \to \varphi \quad \text{in } L^2(\Omega) \quad \text{as } n \to +\infty,$$

and

$$[\varphi_n]_{W^{s,1}(\mathbb{R}^N)} \to [\varphi]_{W^{s,1}(\mathbb{R}^N)} \quad \text{as } n \to +\infty.$$

By Fatou's Lemma and (2.37), we have

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{|x-y|^{N+s}}\,|\varphi(y)-\varphi(x)| - \frac{1}{|x-y|^{N+s}}\eta(x,y)(\varphi(y)-\varphi(x))\right)dydx$$

$$\leq \liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{|x-y|^{N+s}}\,|\varphi_n(y)-\varphi_n(x)| - \frac{1}{|x-y|^{N+s}}\,\eta(x,y)(\varphi_n(y)-\varphi_n(x))\right)dydx$$

$$= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}|\varphi(y)-\varphi(x)|dydx - \int_\Omega (f-u)\varphi,$$

which implies

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}\eta(x,y)(\varphi(y)-\varphi(x))\,dydx \geq \int_\Omega (f-u)\varphi$$

for all $\varphi \in \mathcal{W}_0^{s,1}(\Omega)\cap L^2(\Omega)$. But we obtain an equality, since the above inequality is also true for $-\varphi$.

To finish the proof of (2.31), we only need to show that

$$\eta(x,y) \in \operatorname{sign}(u(y)-u(x)) \quad \text{a.e. } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N. \tag{2.38}$$

By (2.34) for p_n, and taking $\varphi = u$ in (2.37), we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{(N+s)p_n}} \left|u_{p_n}(y)-u_{p_n}(x)\right|^{p_n} dydx = \int_\Omega (f(x)-u_{p_n}(x))u_{p_n}(x)dx \\
&\quad = \int_\Omega (f(x)-u(x))u(x)dx - \int_\Omega f(x)(u(x)-u_{p_n}(x))dx \\
&\qquad +2\int_\Omega u(x)((u(x)-u_{p_n}(x))dx - \int_\Omega (u(x)-u_{p_n}(x))^2 dx \\
&\quad \leq \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \eta(x,y)(u(y)-u(x))\, dydx - \int_\Omega f(x)(u(x)-u_{p_n}(x))dx \\
&\qquad +2\int_\Omega u(x)((u(x)-u_{p_n}(x))dx.
\end{aligned}
$$

Then, taking limit as $n \to \infty$, we get

$$
\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{(N+s)p_n}} \left|u_{p_n}(y)-u_{p_n}(x)\right|^{p_n} dydx \\
&\quad \leq \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \eta(x,y)(u(y)-u(x))\, dydx.
\end{aligned}
$$

On the other hand, given $\epsilon > 0$ we can find $A \supset \Omega$ with $|A| < +\infty$ such that

$$\int_{\mathbb{R}^N \setminus A} \frac{1}{|x-y|^{N+s}} dy \leq \frac{\epsilon}{\|f\|_{L^1(\Omega)}} \qquad \forall\, x \in \Omega.$$

Then,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \left|u_{p_n}(y) - u_{p_n}(x)\right| dydx \\
&= \int_{\Omega}\int_{\mathbb{R}^N \setminus A} \frac{1}{|x-y|^{N+s}} \left|u_{p_n}(y) - u_{p_n}(x)\right| dydx \\
&\qquad\qquad\qquad + \frac{1}{2}\int_A\int_A \frac{1}{|x-y|^{N+s}} \left|u_{p_n}(y) - u_{p_n}(x)\right| dydx \\
&= \int_{\Omega} |u_p(x)| \left(\int_{\mathbb{R}^N \setminus A} \frac{1}{|x-y|^{N+s}} dy \right) dx \\
&\qquad\qquad\qquad + \frac{1}{2}\int_A\int_A \frac{1}{|x-y|^{N+s}} \left|u_{p_n}(y) - u_{p_n}(x)\right| dydx \\
&\leq \epsilon + \frac{1}{2}\int_A\int_A \frac{1}{|x-y|^{N+s}} \left|u_{p_n}(y) - u_{p_n}(x)\right| dydx.
\end{aligned}$$

By the lower semi-continuity in $L^1(\mathbb{R}^N)$ of $[\,\cdot\,]_{W^{s,1}(\Omega)}$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}\,|u(y)-u(x)|\,dydx\\
&\qquad\le \liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}\,|u_{p_n}(y)-u_{p_n}(x)|\,dydx\\
&\qquad\le \epsilon+\liminf_{n\to\infty}\frac{1}{2}\int_{A}\int_{A}\frac{1}{|x-y|^{N+s}}\,|u_{p_n}(y)-u_{p_n}(x)|\,dydx\\
&\qquad\le \epsilon+\liminf_{n\to\infty}\frac{1}{2}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{(N+s)p_n}}\,|u_{p_n}(y)-u_{p_n}(x)|^{p_n}\,dydx\right)^{1/p_n}|A\times A|^{1/p_n{}'}\\
&\qquad\le \epsilon+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}\,\eta(x,y)(u(y)-u(x))\,dydx.
\end{aligned}$$

Therefore,

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}\,|u(y)-u(x)|\,dydx$$

$$\le \epsilon+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}\,\eta(x,y)(u(y)-u(x))\,dydx,$$

from where it follows (2.38), since ϵ was arbitrary.

Finally, let us see that $D_{1,s} = \partial \mathcal{D}_1^s$.

Given $(u,v) \in D_{1,s}$, there exists $\eta \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, $\eta(x,y) = -\eta(y,x)$ for almost all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $\|\eta\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} \leq 1$, such that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta(x,y)(\varphi(y)-\varphi(x))\,dydx$$

$$= \int_\Omega v(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in \mathcal{W}_0^{s,1}(\Omega)\cap L^2(\Omega),$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}|u(y)-u(x)|\,dydx = \int_\Omega v(x)u(x)\,dx.$$

Then, given $w \in \mathcal{W}_0^{s,1}(\Omega)\cap L^2(\Omega)$, we have

$$\int_\Omega v(x)(w(x)-u(x))\,dx = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta(x,y)\left(w(y)-w(x)\right)\,dydx - \mathcal{D}_1^s(u)$$

$$\leq \mathcal{D}_1^s(w) - \mathcal{D}_1^s(u).$$

Therefore, $(u,v) \in \partial\mathcal{D}_1^s$, and consequently $D_{1,s} \subset \partial\mathcal{D}_1^s$. Then, since $D_{1,s}$ is m–accretive in $L^2(\Omega)$, we have $\partial\mathcal{D}_1^s = D_{1,s}$.

$\square$

THEOREM 2.24. *For every $u_0 \in L^2(\Omega)$ there exists a unique solution of the Dirichlet problem* (2.29) *in* $(0, T)$ *for any* $T > 0$. *Moreover, if* $u_{i,0} \in L^2(\Omega)$ *and* u_i *are solutions of the Dirichlet problem* (2.29) *in* $(0, T)$ *with initial data* $u_{i,0}$, $i = 1, 2$, *respectively, then*

$$\int_\Omega (u_1(t) - u_2(t))^+ \leq \int_\Omega (u_{1,0} - u_{2,0})^+ \quad \textit{for every } t \in (0, T).$$

2.3.2. Rescaling.

We now study the limit as $s \to 1$ in the nonlocal fractional $1-$Laplacian evolution problem

$$\begin{cases} u_t(t, x) = L_{1,s} \Delta_1^s u(t, x) & \text{in } \ (0, T) \times \Omega, \\ u(t, x) = 0 & \text{in } \ (0, T) \times (\mathbb{R}^N \setminus \Omega), \\ u(0, x) = u_0(x) & \text{in } \ \Omega, \end{cases} \tag{2.39}$$

where the scale factor is $L_{1,s} = \frac{2}{K_{1,N}}(1 - s)$, $K_{1,N} = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} |e_1 \cdot \sigma| d\mathcal{H}^{N-1}(\sigma)$.

THEOREM 2.25. *Given $s_n \to 1^-$, let u_n be the solution of* (2.39) *for $s = s_n$. Then, if u is the solution of the Dirichlet* 1*–Laplacian problem*

$$\begin{cases} u_t(t,x) = \Delta_1 u(t,x), & \textit{in} \ \ (0,T)\times\Omega, \\ u(t,x) = 0, & \textit{on} \ \ (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x), & \textit{in} \ \ \Omega, \end{cases}$$

we have $\quad \lim_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t) - u(t)\|_{L^2(\Omega)} = 0.$

We have full convergence as $s \to 1$ (without the need of considering subsequences) since the solution to the limit problem is unique.

Results in this direction have been obtained in [**41**] in the stationary case (see also [**22**] and [**24**]).

PROOF. Consider the energy functionals

$$\Phi_{s_n}(u) = \begin{cases} \dfrac{1-s_n}{K_{1,N}} \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)-u(x)|}{|x-y|^{N+s_n}}\,dxdy & \text{if } u \in \mathcal{W}_0^{s_n,1}(\Omega)\cap L^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus \mathcal{W}_0^{s_n,1}(\Omega), \end{cases}$$

and

$$\Phi(u) := \begin{cases} |Du|(\Omega) + \displaystyle\int_{\partial\Omega} |u| & \text{if } u \in BV(\Omega)\cap L^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus BV(\Omega). \end{cases}$$

Then, u_n is the strong solution of the abstract Cauchy problem

$$\begin{cases} u_n'(t) + \partial\Phi_{s_n}(u_n(t)) \ni 0, & \text{a.e. } \ t \in (0,T), \\ u_n(0) = u_0, \end{cases}$$

and also (see [**4**]) u is the strong solution of the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Phi(u(t)) \ni 0, & \text{a.e. } \ t \in (0,T), \\ u(0) = u_0. \end{cases}$$

Let us now check the Mosco convergence of the functionals Φ_{s_n} to Φ, that is,

$$\forall\, u \in \mathrm{Dom}(\Phi) \ \ \exists u_n \in \mathrm{Dom}(\Phi_{s_n}) : u_n \to u \ \text{ and } \ \Phi(u) \geq \limsup_{n\to\infty} \Phi_{s_n}(u_n);$$

and

$$\text{if } u_n \rightharpoonup u \quad \text{then } \ \Phi(u) \leq \liminf_{n\to\infty} \Phi_{s_n}(u_n). \tag{2.40}$$

Set $\tilde\Omega := \Omega + B(0,1)$. Observe that

$$\Phi_{s_n}(u) = \Psi_{s_n}(u) + \frac{2(1-s_n)}{K_{1,N}} \int_\Omega \left(\int_{\mathbb{R}^N\setminus\tilde\Omega} \frac{1}{|x-y|^{N+s_n}} dy \right) |u(x)| dx, \tag{2.41}$$

where

$$\Psi_{s_n}(u) = \frac{1-s_n}{K_{1,N}} \int_{\tilde\Omega}\int_{\tilde\Omega} \frac{|u(y)-u(x)|}{|x-y|^{N+s_n}}\, dxdy.$$

Observe also that

$$\lim_{n\to\infty} \frac{2(1-s_n)}{K_{1,N}} \int_\Omega \left(\int_{\mathbb{R}^N \setminus \tilde{\Omega}} \frac{1}{|x-y|^{N+s_n}} dy \right) |u(x)| dx = 0. \tag{2.42}$$

Given $u \in \mathrm{Dom}(\Phi^{\Omega,1}) = BV(\Omega) \cap L^2(\Omega)$, we consider $u_n = u\chi_\Omega$; we have $u_n \in \mathcal{W}_0^{s_n,1}(\Omega) \cap L^2(\Omega)$. Now, in [**31**] (see also [**23**]) J. Dávila proves that

$$\lim_{n\to\infty} \Psi_{s_n}(u_n) = |Du|(\tilde{\Omega}).$$

But $|Du|(\tilde{\Omega}) = |Du|(\Omega) + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}$, hence

$$\lim_{n\to\infty} \Phi_{s_n}(u_n) = \Phi(u).$$

To prove (2.40) we can suppose that $\{\Phi_{s_n}(u_n) \ : \ n \in \mathbb{N}\}$ is bounded. Therefore, $\{\Psi_{s_n}(u_n) \ : \ n \in \mathbb{N}\}$ is also bounded and consequently, from [**23**] and [**51**],

$$u_n \to u \quad \text{strongly in } L^1(\tilde{\Omega})$$

and

$$|Du|(\tilde{\Omega}) \leq \liminf_n \Psi_{s_n}^{\tilde{\Omega},1}(u_n).$$

Now, since $|Du|(\tilde{\Omega}) = |Du|(\Omega) + \int_{\partial\Omega} |u|$, from (2.41) and (2.42), we get (2.40). $\square$

2.4. Poincaré type inequalities

For several classical partial differential equations the solutions belong to appropriate Sobolev spaces. Hence, Poincaré type inequalities play a key role in their analysis. When considering nonlocal problems with non–degenerate kernels, we look for solutions in L^p spaces; however, we can prove nonlocal analogs of Poincaré type inequalities that also play a role for these problems.

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$.

Proposition 2.26. *Let $q \geq 1$ and Ω a bounded domain in $\mathbb{R}^N$.*

1. Let $\psi \in L^q(\Omega_J \setminus \overline{\Omega})$. There exists $\lambda(J, \Omega, q) > 0$ such that

$$\lambda \int_\Omega |u(x)|^q \, dx \leq \int_\Omega \int_{\Omega_J} J(x-y)|u_\psi(y) - u(x)|^q \, dy \, dx + \int_{\Omega_J \setminus \overline{\Omega}} |\psi(y)|^q \, dy$$

for all $u \in L^q(\Omega)$.

2. There exists $\beta(J, \Omega, q) > 0$ such that

$$\beta \int_\Omega \left| u - \frac{1}{|\Omega|} \int_\Omega u \right|^q \leq \frac{1}{2} \int_\Omega \int_\Omega J(x-y)|u(y) - u(x)|^q \, dy \, dx,$$

for every $u \in L^q(\Omega)$.

PROOF.

Proof of the first inequality. Take $r, \alpha > 0$ such that $J(x) \geq \alpha$ in $B(0,r)$. Let

$$B_0 = \{x \in \Omega_J \setminus \overline{\Omega} \ : \ d(x, \Omega) \leq r/2\},$$

$$B_1 = \{x \in \Omega \ : \ d(x, B_0) \leq r/2\},$$

and

$$B_j = \left\{ x \in \Omega \setminus \bigcup_{k=1}^{j-1} B_k \ : \ d(x, B_{j-1}) \leq r/2 \right\}, \quad j = 2, 3, \ldots.$$

Observe that we can cover Ω by a finite number of nonnull sets $\{B_j\}_{j=1}^{l_r}$.

Now, for $j = 1, \ldots, l_r$,

$$\int_\Omega \int_{\Omega_J} J(x-y)|u_\psi(y) - u(x)|^q \, dy \, dx \geq \int_{B_j} \int_{B_{j-1}} J(x-y)|u_\psi(y) - u(x)|^q \, dy \, dx,$$

and

$$\int_{B_j}\int_{B_{j-1}} J(x-y)|u_\psi(y)-u(x)|^q\,dy\,dx$$

$$\geq \frac{1}{2^q}\int_{B_j}\int_{B_{j-1}} J(x-y)|u(x)|^q\,dy\,dx - \int_{B_j}\int_{B_{j-1}} J(x-y)|u_\psi(y)|^q\,dy\,dx$$

$$= \frac{1}{2^q}\int_{B_j}\left(\int_{B_{j-1}} J(x-y)\,dy\right)|u(x)|^q\,dx - \int_{B_{j-1}}\left(\int_{B_j} J(x-y)\,dx\right)|u_\psi(y)|^q\,dy$$

$$\geq \alpha_j\int_{B_j}|u(x)|^q\,dx - \beta\int_{B_{j-1}}|u_\psi(y)|^q\,dy,$$

where

$$\alpha_j = \frac{1}{2^q}\min_{x\in\overline{B_j}}\int_{B_{j-1}} J(x-y)\,dy > 0$$

(since $J(x)\geq\alpha$ in $B(0,r)$) and

$$\beta = \int_{\mathbb{R}^N} J(x)\,dx.$$

Hence

$$\int_{\Omega}\int_{\Omega_J} J(x-y)|u_\psi(y)-u(x)|^q\,dy\,dx \geq \alpha_j \int_{B_j} |u(x)|^q\,dx - \beta \int_{B_{j-1}} |u_\psi(y)|^q\,dy.$$

Therefore, since $u_\psi(y)=\psi(y)$ if $y\in B_0$, $u_\psi(y)=u(y)$ if $y\in B_j$, $j=1,\ldots,l_r$, $B_j\cap B_i=\emptyset$, for all $i\neq j$ and $|\Omega\setminus\bigcup_{j=1}^{j_r}B_j|=0$, it is easy to see, by cancelation, that there exists $\lambda=\lambda(J,\Omega,q)>0$ such that

$$\lambda\int_{\Omega}|u|^q \leq \int_{\Omega}\int_{\Omega_J} J(x-y)|u_\psi(y)-u(x)|^q\,dy\,dx + \int_{B_0}|\psi|^q.$$

Proof of the second inequality. It is enough to prove that there exists a constant c such that

(2.43)

$$\|u\|_q \leq c\left(\left(\int_{\Omega}\int_{\Omega} J(x-y)|u(y)-u(x)|^q dydx\right)^{1/q} + \left|\int_{\Omega} u\right|\right) \ \forall u\in L^q(\Omega).$$

Let $r>0$ be such that $J(x)\geq\alpha>0$ in $B(0,r)$. There exists $\{x_i\}_{i=1}^m\subset\Omega$ such that $\Omega\subset\bigcup_{i=1}^m B(x_i,r/2)$. Take $0<\delta<r/2$ such that $B(x_i,\delta)\subset\Omega$ for all $i=1,\ldots,m$. Then, for any $\hat{x}_i\in B(x_i,\delta)$, $i=1,\ldots,m$,

$$\Omega=\bigcup_{i=1}^m (B(\hat{x}_i,r)\cap\Omega). \tag{2.44}$$

Let us argue by contradiction. Suppose that (2.43) is false. Then there exists $u_n \in L^q(\Omega)$, with $\|u_n\|_{L^q(\Omega)} = 1$, satisfying

$$1 \geq n \left(\left(\int_\Omega \int_\Omega J(x-y)|u_n(y) - u_n(x)|^q dydx \right)^{1/q} + \left| \int_\Omega u_n \right| \right) \quad \forall\, n \in \mathbb{N}.$$

Consequently,

$$\lim_n \int_\Omega \int_\Omega J(x-y)|u_n(y) - u_n(x)|^q \, dy \, dx = 0 \tag{2.45}$$

and

$$\lim_n \int_\Omega u_n = 0.$$

Let

$$F_n(x,y) = J(x-y)^{1/q}|u_n(y) - u_n(x)|$$

and

$$f_n(x) = \int_\Omega J(x-y)|u_n(y) - u_n(x)|^q \, dy.$$

From (2.45), it follows that

$$f_n \to 0 \quad \text{in } L^1(\Omega).$$

Passing to a subsequence if necessary, we can assume that

$$f_n(x) \to 0 \quad \forall\, x \in \Omega \setminus B_1, \quad B_1 \text{ null.} \tag{2.46}$$

On the other hand, by (2.45), we also have that

$$F_n \to 0 \quad \text{in } L^q(\Omega \times \Omega).$$

So we can assume that, up to a subsequence,

$$F_n(x,y) \to 0 \quad \forall\, (x,y) \in \Omega \times \Omega \setminus C, \quad C \text{ null.} \tag{2.47}$$

Take $B_2 \subset \Omega$ the null set satisfying

$$\text{for all } x \in \Omega \setminus B_2, \text{ the section } C_x \text{ of } C \text{ is null.} \tag{2.48}$$

Let $\hat{x}_1 \in B(x_1, \delta) \setminus (B_1 \cup B_2)$; then there exists a subsequence such that, in the same notation,

$$u_n(\hat{x}_1) \to \lambda_1 \in [-\infty, +\infty].$$

Consider now $\hat{x}_2 \in B(x_2, \delta) \setminus (B_1 \cup B_2)$; then, up to a subsequence, we can assume that

$$u_n(\hat{x}_2) \to \lambda_2 \in [-\infty, +\infty].$$

So, successively, for $\hat{x}_m \in B(x_m, \delta) \setminus (B_1 \cup B_2)$, there exists a subsequence, again denoted the same way, such that

$$u_n(\hat{x}_m) \to \lambda_m \in [-\infty, +\infty].$$

By (2.47) and (2.48),

$$u_n(y) \to \lambda_i \quad \forall\, y \in (B(\hat{x}_i, r) \cap \Omega) \setminus C_{\hat{x}_i}.$$

Now, by (2.44),

$$\Omega = (B(\hat{x}_1, r) \cap \Omega) \cup \left(\bigcup_{i=2}^{m} (B(\hat{x}_i, r) \cap \Omega) \right).$$

Hence, since Ω is a bounded domain, there exists $i_2 \in \{2, \ldots, m\}$ such that

$$(B(\hat{x}_1, r) \cap \Omega) \cap (B(\hat{x}_{i_2}, r) \cap \Omega) \neq \emptyset.$$

Therefore, $\lambda_1 = \lambda_{i_2}$. Let us call $i_1 := 1$. Again, since

$$\Omega = ((B(\hat{x}_{i_1}, r) \cap \Omega) \cup ((B(\hat{x}_{i_1}, r) \cap \Omega)) \cup \left(\bigcup_{i \in \{1,\ldots,m\} \setminus \{i_1, i_2\}} (B(\hat{x}_i, r) \cap \Omega) \right),$$

there exists $i_3 \in \{1, \ldots, m\} \setminus \{i_1, i_2\}$ such that

$$((B(\hat{x}_{i_1}, r) \cap \Omega) \cup (B(\hat{x}_{i_1}, r) \cap \Omega)) \cap (B(\hat{x}_{i_3}, r) \cap \Omega) \neq \emptyset.$$

Consequently, $\lambda_{i_1} = \lambda_{i_2} = \lambda_{i_3}$. And using the same argument we get

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m = \lambda.$$

If $|\lambda| = +\infty$, we have shown that

$$|u_n(y)|^q \to +\infty \quad \text{for almost every } y \in \Omega,$$

which contradicts $\|u_n\|_{L^q(\Omega)} = 1$ for all $n \in \mathbb{N}$. Hence λ is finite.

On the other hand, by (2.46), $f_n(\hat{x}_i) \to 0$, $i = 1, \ldots, m$. Hence,

$$F_n(\hat{x}_1, \cdot) \to 0 \quad \text{in } L^q(\Omega).$$

Since $u_n(\hat{x}_1) \to \lambda$, from the above we conclude that

$$u_n \to \lambda \quad \text{in } L^q(B(\hat{x}_i, r) \cap \Omega).$$

Using again a compactness argument we get

$$u_n \to \lambda \quad \text{in } L^q(\Omega).$$

By (2.4), $\lambda = 0$, so

$$u_n \to 0 \quad \text{in } L^q(\Omega),$$

which contradicts $\|u_n\|_{L^q(\Omega)} = 1$. $\square$

THEME 3

Some applications

3.1. A nonlocal version of the Aronsson-Evans-Wu model for sandpiles

The continuous models for the dynamics of a sandpile introduced by G. Aronsson, L. C. Evans and Y. Wu in [**15**] (see also ([**53**]) is a model in the form of a variational inequality based on the requirement that the slope of sandpile is at most one.

However, a *"more realistic model"* would require the slope constraint only on a larger scale. This is *grosso modo* the case for the nonlocal model presented here.

This nonlocal model is the counterpart of the local Aronsson-Evans-Wu model obtained as the limit as $p \to \infty$ of Cauchy problems for the p–Laplacian evolution.

By reescaling, the local sandpile model can be recovered from the nonlocal one.

3.1.1. The Aronsson-Evans-Wu model for sandpiles.

Assume that u_0 (the initial state of the sandpile) is a Lipschitz function with compact support such that

$$\|\nabla u_0\|_\infty \leq 1,$$

and f (the source of sand) is a smooth nonnegative function with compact support in $\mathbb{R}^N \times (0,T)$. Set $v_\infty(x,t)$ to describe the amount of the sand at the point x at time t, the main assumption being that the sandpile is stable when the slope is less than or equal to one and unstable if not. This model states that v_∞ satisfies

$$\begin{cases} f(\cdot,t) - (v_\infty)_t(.,t) \in \partial F_\infty(v_\infty(\cdot,t)) & \text{a.e. } t \in (0,T), \\ v_\infty(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{3.1}$$

where

$$F_\infty(v) = \begin{cases} 0 & \text{if } v \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N),\ |\nabla v| \leq 1, \\ +\infty & \text{otherwise} \end{cases}$$

that is,

$$\begin{cases} f(\cdot,t) - (v_\infty)_t(\cdot,t) \in \partial \mathbb{I}_{K_0}(v_\infty(\cdot,t)) & \text{a.e. } t \in (0,T), \\ v_\infty(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$K_0 := \left\{ u \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) \ : \ |\nabla u| \leq 1 \right\}.$$

Problem (3.1) is obtained by taking limits, as $p \to \infty$, to

$$\begin{cases} (v_p)_t - \Delta_p v_p = f & \text{in } \mathbb{R}^N \times (0,T), \\ v_p(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{3.2}$$

where f is adding material to an evolving system, where mass particles are continually rearranged by the p–Laplacian diffusion.

Let us define for $1 < p < \infty$ the functional

$$F_p(v) = \begin{cases} \dfrac{1}{p} \displaystyle\int_{\mathbb{R}^N} |\nabla v(y)|^p \, dy & \text{if } \; v \in L^2(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \\ +\infty & \text{if } \; v \in L^2(\mathbb{R}^N) \setminus W^{1,p}(\mathbb{R}^N). \end{cases}$$

The PDE problem (3.2) is the abstract Cauchy problem associated to ∂F_p:

$$\begin{cases} f(\cdot,t) - (v_p)_t(\cdot,t) \in \partial F_p(v_p(\cdot,t)) & \text{a.e. } \; t \in (0,T), \\ v_p(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

They prove the existence of a sequence $p_i \to +\infty$ and a limit function v_∞ such that, for each $T > 0$,

$$\begin{aligned} & v_{p_i} \to v_\infty && \text{in } L^2(\mathbb{R}^N \times (0,T)) \text{ and a.e.,} \\ & \nabla v_{p_i} \rightharpoonup \nabla v_\infty, \; (v_{p_i})_t \rightharpoonup (v_\infty)_t && \text{weakly in } \; L^2(\mathbb{R}^N \times (0,T)), \end{aligned}$$

and v_∞ satisfies (3.1).

3.1.2. Limit as $p \to \infty$ in a nonlocal p–Laplacian Cauchy problem.

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with support $\overline{B}(0,1)$, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)\,dx = 1$.

In [**11**] we study the existence and uniqueness of solutions of the following nonlocal p–Laplacian Cauchy problem

(3.3)

$$\begin{cases} (u_p)_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)|u_p(y,t)-u_p(x,t)|^{p-2}(u_p(y,t)-u_p(x,t))dy + f(x,t), \\ \hfill x \in \mathbb{R}^N,\ t > 0, \\ u_p(0) = u_0 \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N). \end{cases}$$

If we set

$$G_p^J(u) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)|u(y)-u(x)|^p\,dy\,dx,$$

then problem (3.3) is the abstract Cauchy problem associated to ∂G_p^J:

$$\begin{cases} f(\cdot,t) - (u_p)_t(\cdot,t) \in \partial G_p^J(u_p(\cdot,t)) & \text{a.e. } t \in (0,T), \\ u_p(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

Formally, taking limits to

$$G_p^J(u) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)|u(y)-u(x)|^p \, dy \, dx,$$

we get the functional

$$G_\infty^1(u) = \begin{cases} 0 & \text{if } \ |u(x)-u(y)| \leq 1, \ \ \text{for } |x-y| \leq 1, \\ \\ +\infty & \text{otherwise,} \end{cases}$$

that can be seen as the indicatrice of

$$K_1 := \left\{ u \in L^2(\mathbb{R}^N) \ : \ |u(x)-u(y)| \leq 1 \text{ for } \ |x-y| \leq 1 \right\}.$$

Then the *nonlocal limit problem* should be

$$\begin{cases} f(\cdot,t) - u_t(\cdot,t) \in \partial \mathbb{I}_{K_1}(u(\cdot,t)) & \text{a.e. } t \in (0,T), \\ \\ u(x,0) = u_0(x). \end{cases} \tag{3.4}$$

LEMMA 3.1. *Given $u \in L^1(\Omega)$ such that*

$$\{(x,y) \in \Omega \times \Omega : \ |u(x) - u(y)| > 1, \ |x - y| \le 1\}$$

is a null set of $\Omega \times \Omega$, there exists $\hat{u} \in K_{d_1}(\Omega)$ such that $u = \hat{u}$ a.e. in Ω.

THEOREM 3.2. *Let $T > 0$, $f \in L^2(0,T; L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$, $u_0 \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $|u_0(x) - u_0(y)| \le 1$ for $|x - y| \le 1$, and let u_p be the unique solution of* (3.3)*, $p \ge 2$. Then,*

$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(\cdot,t) - u_\infty(\cdot,t)\|_{L^2(\mathbb{R}^N)} = 0,$$

where u_∞ is the unique solution of (3.4)*.*

PROOF. To prove the result it is enough to show that the functionals

$$G_p^J(u) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)|u(y) - u(x)|^p \, dy \, dx$$

converge to

$$G_\infty^1(u) = \begin{cases} 0 & \text{if } \ |u(x) - u(y)| \le 1 \ \text{ for } |x-y| \le 1, \\ +\infty & \text{otherwise,} \end{cases}$$

as $p \to \infty$, in the sense of Mosco.

First, let us check that

$$\mathrm{Epi}(G_\infty^1) \subset \operatorname{s-}\liminf_{p\to\infty} \mathrm{Epi}(G_p^J). \tag{3.5}$$

To this end let $(u,\lambda) \in \mathrm{Epi}(G^1_\infty)$. We can assume that $u \in K_1$ and $\lambda \geq 0$ (since $G^1_\infty(u) = 0$). Now for $R(p) > 0$ take

$$v_p = u\chi_{B(0,R(p))} \qquad \text{and} \qquad \lambda_p = G^J_p(v_p) + \lambda.$$

Then, as $\lambda \geq 0$, we have $(v_p, \lambda_p) \in \mathrm{Epi}(G^J_p)$. It is obvious that if $R(p) \to +\infty$ as $p \to +\infty$, we have

$$v_p \to u \quad \text{in} \ \ L^2(\mathbb{R}^N),$$

and, if we choose $R(p) = p^{\frac{1}{4N}}$,

$$G^J_p(v_p) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J\,(x-y)\,|v_p(y) - v_p(x)|^p \; dy\, dx \leq C \frac{R(p)^{2N}}{p} \to 0$$

as $p \to \infty$, and (3.5) holds.

Finally, let us prove that

$$\operatorname*{w-lim\,sup}_{p\to\infty} \mathrm{Epi}(G^J_p) \subset \mathrm{Epi}(G^1_\infty).$$

To this end, consider a sequence $(u_{p_j}, \lambda_{p_j}) \in \mathrm{Epi}(G^J_{p_j})$ $(p_j \to \infty)$, that is,

$$G^J_{p_j}(u_{p_j}) \leq \lambda_{p_j},$$

with $u_{p_j} \rightharpoonup u$ and $\lambda_{p_j} \to \lambda$. Therefore we obtain that $0 \leq \lambda$, since

$$0 \leq G^J_{p_j}(u_{p_j}) \leq \lambda_{p_j} \to \lambda.$$

On the other hand, we have that

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\left(x-y\right)\left|u_{p_j}(y)-u_{p_j}(x)\right|^{p_j}\,dy\,dx\right)^{1/p_j}\leq (Cp_j)^{1/p_j}.$$

Now, fix a bounded domain $\Omega\subset\mathbb{R}^N$ and $q<p_j$. Then, by the above inequality,

$$\begin{aligned}&\left(\int_{\Omega}\int_{\Omega} J\left(x-y\right)\left|u_{p_j}(y)-u_{p_j}(x)\right|^{q}\,dy\,dx\right)^{1/q}\\ &\leq \left(\int_{\Omega}\int_{\Omega} J\left(x-y\right)\,dy\,dx\right)^{(p_j-q)/p_jq}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\left(x-y\right)\left|u_{p_j}(y)-u_{p_j}(x)\right|^{p_j}\,dy\,dx\right)^{1/p_j}\\ &\leq \left(\int_{\Omega}\int_{\Omega} J\left(x-y\right)\,dy\,dx\right)^{(p_j-q)/p_jq}(Cp_j)^{1/p_j}.\end{aligned}$$

Hence, we can extract a subsequence, if necessary, and consider $p_j\to\infty$ to obtain

$$\left(\int_{\Omega}\int_{\Omega} J\left(x-y\right)\left|u(y)-u(x)\right|^{q}\,dy\,dx\right)^{1/q}\leq \left(\int_{\Omega}\int_{\Omega} J\left(x-y\right)\,dy\,dx\right)^{1/q}.$$

Now, just letting $q\to\infty$, we get

$$|u(x)-u(y)|\leq 1\quad\text{a.e. } (x,y)\in\Omega\times\Omega,\ x-y\in\operatorname{supp}(J).$$

As Ω was arbitrary, we can conclude that

$$u\in K_1.$$

$\square$

3.1.3. A nonlocal sandpile problem.

For $\varepsilon > 0$ we rescale the functional G^1_∞ as follows:

$$G^\varepsilon_\infty(u) = \begin{cases} 0 & \text{if } \ |u(x)-u(y)| \le \varepsilon, \ \text{ for } |x-y| \le \varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

In other words, $G^\varepsilon_\infty = \mathbb{I}_{K_\varepsilon}$, where

$$K_\varepsilon := \{u \in L^2(\mathbb{R}^N) \ : \ |u(x)-u(y)| \le \varepsilon, \text{ for } |x-y| \le \varepsilon\}.$$

Consider the gradient flow associated to the functional G^ε_∞ with a source term f

$$\begin{cases} f(\cdot,t) - u_t(\cdot,t) \in \partial \mathbb{I}_{K_\varepsilon}(u(\cdot,t)) & \text{a.e. } t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{3.6}$$

and the problem

$$\begin{cases} f(\cdot,t) - (v_\infty)_t(\cdot,t) \in \partial \mathbb{I}_{K_0}(v_\infty(\cdot,t)) & \text{a.e. } t \in (0,T), \\ v_\infty(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{3.7}$$

where

$$K_0 := \left\{u \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) \ : \ |\nabla u| \le 1\right\}.$$

Observe that if $u \in K_0$, then $|\nabla u| \le 1$. Hence, $|u(x)-u(y)| \le |x-y|$, and then $u \in K_\varepsilon$. That is, $K_0 \subset K_\varepsilon$. We have the following theorem.

THEOREM 3.3. *Let* $T > 0$, $f \in L^2(0,T;L^2(\mathbb{R}^N))$, $u_0 \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ *such that* $\|\nabla u_0\|_\infty \leq 1$ *and consider* $u_{\infty,\varepsilon}$ *the unique solution of* (3.6). *Then, if* v_∞ *is the unique solution of* (3.7), *we have*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_{\infty,\varepsilon}(\cdot,t) - v_\infty(\cdot,t)\|_{L^2(\mathbb{R}^N)} = 0.$$

Consequently, we are approximating the sandpile model described in Subsection 3.1.1 by a nonlocal model.

In this nonlocal approximation a configuration of sand is stable when its height u satisfies $|u(x) - u(y)| \leq \varepsilon$ if $|x - y| \leq \varepsilon$.

This is a sort of measure of how large is the size of irregularities of the sand; the sand can be completely irregular for sizes smaller than ε but it has to be arranged for sizes greater than ε.

The nonlocal version of the Aronsson-Evans-Wu model for sandpiles has been characterized by N. Igbida in [**39**], where, moreover, the connection with the stochastic process introduced in [**36**] is shown, see also [**38**].

For a model under homogeneous Dirichet boundary condtions see [**12**].

Proof of Theorem 3.3. Since $u_0 \in K_0$, we have that $u_0 \in K_\varepsilon$ for all $\varepsilon > 0$, and consequently the existence of $u_{\infty,\varepsilon}$ is guaranteed.

By Theorem 1.13, to prove the result it is enough to show that $\mathbb{I}_{K_\varepsilon}$ converges to $\mathbb{I}_{K_0}$ in the sense of Mosco. It is easy to see that

$$K_{\varepsilon_1} \subset K_{\varepsilon_2} \quad \text{if} \;\; \varepsilon_1 \leq \varepsilon_2. \tag{3.8}$$

Since $K_0 \subset K_\varepsilon$ for all $\varepsilon > 0$, we have

$$K_0 \subset \bigcap_{\varepsilon > 0} K_\varepsilon.$$

On the other hand, if

$$u \in \bigcap_{\varepsilon > 0} K_\varepsilon,$$

we have

$$|u(y) - u(x)| \leq |y - x|, \quad \text{a.e} \;\; x, y \in \mathbb{R}^N,$$

from which it follows that $u \in K_0$. Therefore,

$$K_0 = \bigcap_{\varepsilon > 0} K_\varepsilon. \tag{3.9}$$

Note that

$$\mathrm{Epi}(\mathbb{I}_{K_0}) = K_0 \times [0, \infty), \quad \mathrm{Epi}(\mathbb{I}_{K_\varepsilon}) = K_\varepsilon \times [0, \infty) \;\; \forall\, \varepsilon > 0. \tag{3.10}$$

By (3.9) and (3.10), we obtain

$$\operatorname{Epi}(\mathbb{I}_{K_0}) \subset \operatorname*{s\text{-}lim\,inf}_{\varepsilon\to 0} \operatorname{Epi}(\mathbb{I}_{K_\varepsilon}).$$

On the other hand, given $(u,\lambda) \in \operatorname{w-lim\,sup}_{\varepsilon\to 0} \operatorname{Epi}(\mathbb{I}_{K_\varepsilon})$ there exists $(u_{\varepsilon_k},\lambda_k) \in K_{\varepsilon_k} \times [0,\infty)$ such that $\varepsilon_k \to 0$ and

$$u_{\varepsilon_k} \rightharpoonup u \quad \text{in } L^2(\mathbb{R}^N), \quad \lambda_k \to \lambda \ \text{ in } \mathbb{R}.$$

By (3.8), given $\varepsilon > 0$, there exists k_0, such that $u_{\varepsilon_k} \in K_\varepsilon$ for all $k \geq k_0$. Then, since K_ε is a closed convex set, we get $u \in K_\varepsilon$, and, by (3.9), we obtain that $u \in K_0$. Consequently,

$$\operatorname*{w\text{-}lim\,sup}_{\varepsilon\to 0} \operatorname{Epi}(\mathbb{I}_{K_\varepsilon}) \subset \operatorname{Epi}(\mathbb{I}_{K_0}).$$

$\square$

Explicit solutions.

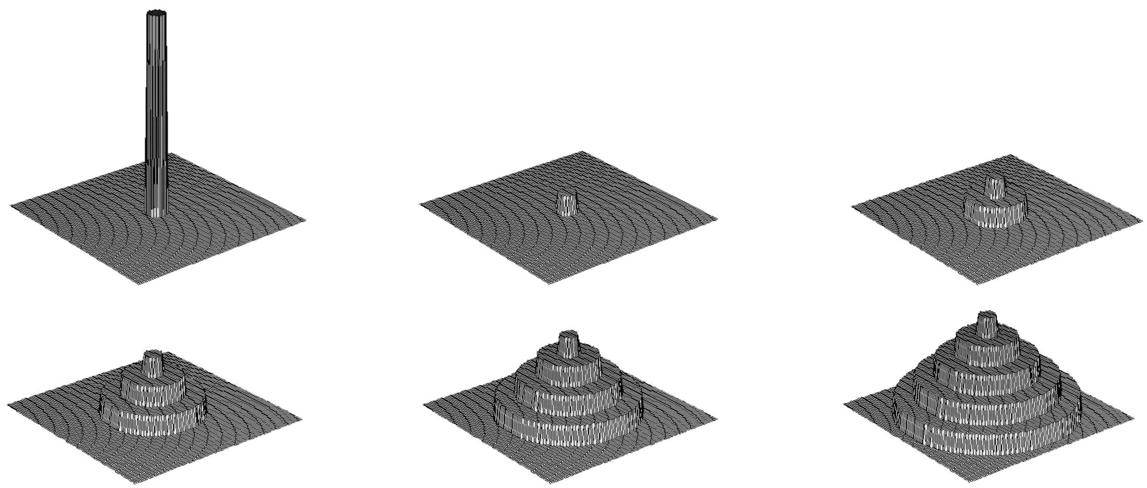

FIGURE 1. Source and five time steps.

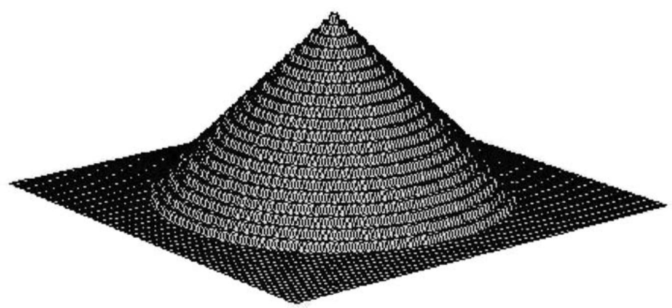

FIGURE 2. Letting $\varepsilon \to 0$.

3.2. A Monge-Kantorovich mass transport problem for a discrete distance

3.2.1. A mass transport interpretation of the sandpile model.

The Monge mass transport problem, as proposed by Monge in 1781, deals with the optimal way of moving points from one mass distribution to another so that a total work done is minimized. In the classical Monge problem the cost function used to define the work for transport one unit of mass from a point x to a point y is the Euclidean distance $d_e(x,y)$, and this problem has been intensively studied and generalized in different directions that correspond to different classes of cost functions, specially convex cost functions.

Here we deal with a cost that lacks of convexity. The purpose is to transport an amount of sand located somewhere to a hole at other place taking into account the number of steps needed to move each part of sand to its final destination, and trying to do it making as less as possible steps, that is, we will define the work using:

$$d_1(x,y) = \begin{cases} 0 & \text{if } \ x = y, \\ 1 & \text{if } \ 0 < |x-y| \le 1, \\ 2 & \text{if } \ 1 < |x-y| \le 2, \\ \vdots & \end{cases}$$

that count the number of steps.

Monge problem.

Given two measures (for simplicity, take them absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^N$) f_+, f_- in $\mathbb{R}^N$, and supposing the overall condition of mass balance

$$\int_{\mathbb{R}^N} f_+\, dx = \int_{\mathbb{R}^N} f_-\, dy,$$

the Monge problem associated to a distance d is given by

$$\text{minimize} \int d(x, s(x))\, f_+(x) dx$$

among the set of maps s that transport f_+ into f_-, that is, among s such that

$$\int_{\mathbb{R}^N} h(s(x)) f_+(x)\, dx = \int_{\mathbb{R}^N} h(y) f_-(y)\, dy$$

for each continuous function $h : \mathbb{R}^N \to \mathbb{R}$.

In general, the Monge problem is ill-posed. In 1942, L. V. Kantorovich ([**44**]) proposed to study a relaxed version of the Monge problem and, what is more relevant here, introduced a dual variational principle:

Monge–Kantorovich problem.

Let $\pi(f^+, f^-)$ the set of transport plans between f^+ and f^-, that is the set of non-negative Radon measures μ in $\Omega \times \Omega$ such that $\mathrm{proj}_x(\mu) = f^+(x)\,dx$ and $\mathrm{proj}_y(\mu) = f^-(y)\,dy$. The Monge-Kantorovich problem looks for a measure (optimal transport plan) $\mu^* \in \pi(f^+, f^-)$ which minimizes the cost functional

$$\mathcal{K}_d(\mu) := \int_{\Omega\times\Omega} d(x,y)\,d\mu(x,y),$$

in the set $\pi(f^+, f^-)$.

In general, $\inf\{\mathcal{K}_d(\mu) : \mu \in \pi(f^+, f^-)\} \leq \inf\{\mathcal{F}_d(T) : T \in \mathcal{A}(f^+, f^-)\}$.

On the other hand, if d is a lower semicontinuous cost function, we have existence of an optimal transport plan $\mu^* \in \pi(f^+, f^-)$ solving the Monge-Kantorovich problem, and we have following dual formulation of this minimization problem:

$$\min\{\mathcal{K}_d(\mu) : \mu \in \pi(f^+, f^-)\} = \max_{u\in K_d} \int_{\mathbb{R}^N} u(x)(f_+(x) - f_-(x))dx, \tag{3.11}$$

where

$$K_d(\Omega) := \left\{u \in L^2(\Omega) \ : \ |u(x) - u(y)| \leq d(x,y) \ \text{ for all } x, y \in \Omega\right\}.$$

The maximizers u^* of the right-hand side of (3.11) are called *Kantorovich potentials*.

For $\mathbb{I}_{K_d(\Omega)}$ the indicator function of $K_d(\Omega)$ we have that the Euler-Lagrange equation associated with the variational problem

$$\sup \left\{ \mathcal{P}_{f^+,f^-}(u) \ : \ u \in K_{d_1}(\Omega) \right\}$$

is the equation

$$f^+ - f^- \in \partial \mathbb{I}_{K_d(\Omega)}(u). \tag{3.12}$$

That is, the Kantorovich potentials of (3.11) are solutions of (3.12).

For the distance d_1, K_{d_1} is given by

$$K_{d_1} := \left\{ u \in L^2(\mathbb{R}^N) \ : \ |u(x) - u(y)| \leq 1 \text{ for } |x-y| \leq 1 \right\},$$

thast is just K_1 given in the nonlocal model for sandpiles studied above.

So we can interprete that nonlocal model for sandpiles using these new terminology (as for the local problem). Remember that the height u of the sandpile evolves following

$$\begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial \mathbb{I}_{K_1}(u(t,\cdot)) \quad \text{a.e. } t \in (0,T), \\ u(x,0) = u_0(x), \end{cases}$$

where f is a source. Then, at each moment of time t, the height function $u(t,\cdot)$ of the sandpile is deemed also to be the potential generating the Monge-Kantorovich reallocation of $f(t,\cdot)$ to $u_t(t,\cdot)$ when the cost distance considered is d_1.

The Evans and Gangbo approach.

For the Euclidean distance d_e Evans and Gangbo ([**34**]) found a solution of

$$f^+ - f^- \in \partial \mathbb{I}_{K_{d_e}(\Omega)}(u) \tag{3.13}$$

as a limit, as $p \to \infty$, of solutions to the following local $p-$Laplacian problem with Dirichlet boundary conditions in a sufficiently large ball $B_R(0)$:

$$\begin{cases} -\Delta_p u_p = f^+ - f^- & B_R(0), \\ u_p = 0, & \partial B_R(0). \end{cases}$$

Moreover, they characterized the solutions to (3.13) by means of a PDE:

THEOREM 3.4. **(Evans-Gangbo Theorem).** *There exists* $u^* \in \mathrm{Lip}_1(\Omega, d_e)$ *such that*

$$\int_\Omega u^*(x)(f^+(x) - f^-(x))\, dx = \max\left\{ \int_\Omega u(x)(f^+(x) - f^-(x))\, dx \,:\, u \in \mathrm{Lip}_1(\Omega, d_e) \right\};$$

and there exists $0 \le a \in L^\infty(\Omega)$ *such that*

$$f^+ - f^- = -\mathrm{div}(a \nabla u^*) \quad \textit{in} \quad \mathcal{D}'(\Omega). \tag{3.14}$$

Furthermore $|\nabla u^*| = 1$ *a.e. on the set* $\{a > 0\}$.

The function a that appear in the previous result is the Lagrange multiplier corresponding to the constraint $|\nabla u^*| \le 1$; it is called the *transport density*.

Moreover, Evans and Gangbo use this PDE to find a proof of existence of an optimal transport map for the classical Monge problem, different to the first one given by Sudakov in 1979 by means of probability methods.

Our main aim is to perform such program for the discrete distance, but now the potentials cannot be characterized with standard differentiation. We give an Euler-Lagrange equation for the Kantorovich potentials obtained as a limit of nonlocal $p-$Laplacian problems. In [**40**] we show how this result allows to construct optimal transport plans .

3.2.2. A nonlocal version of the Evans-Gangbo approach to optimal mass transport.

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a non-negative continuous radial function with $\mathrm{supp}(J) = \overline{B_1(0)}$, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)\,dx = 1$.

Proposition 3.5 ([**40**]). *Let $f \in L^2(\Omega)$ and $p > 2$. Then the functional*

$$F_p(u) = \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^p\,dy\,dx - \int_\Omega f(x)u(x)\,dx$$

has a unique minimizer u_p in $S_p := \left\{u \in L^p(\Omega) \ : \ \int_\Omega u(x)\,dx = 0\right\}$.

THEOREM 3.6. *Let $f^+, f^- \in L^2(\Omega)$ be two non-negative Borel functions satisfying the mass balance condition. Let u_p be the minimizer in Proposition 3.5 for $f = f^+ - f^-$, $p > 2$. Then, there exists a subsequence $\{u_{p_n}\}_{n\in\mathbb{N}}$ having as weak limit a Kantorovich potential u for $f^\pm$ and the metric cost function d_1, that is,*

$$\int_\Omega u(x)(f^+(x) - f^-(x))\,dx = \max_{v\in K_1} \int_\Omega v(x)(f^+(x) - f^-(x))\,dx.$$

PROOF. For $q \geq 1$, we set

$$|||u|||_q := \left(\int_\Omega \int_\Omega J(x-y)|u(y) - u(x)|^q\,dxdy \right)^{\frac{1}{q}}.$$

By Hölder's inequality, for $r \geq q$:

$$|||u|||_q \leq \left(\int_\Omega \int_\Omega J(x-y)|u(y) - u(x)|^r\,dxdy \right)^{\frac{1}{r}} \left(\int_\Omega \int_\Omega J(x-y)\,dxdy \right)^{\frac{r-q}{rq}},$$

that is,

$$|||u|||_q \leq |||u|||_r \left(\int_\Omega \int_\Omega J(x-y)\,dxdy \right)^{\frac{r-q}{rq}} \quad \text{for } (r,q),\ r \geq q. \tag{3.15}$$

Using that $F_p(u_p) \leq F_p(0) = 0$ and the Poincaré's inequality (2.26).2 we get

$$|||u_p|||_p^p \leq 2p \int_\Omega f(x)u_p(x)\,dx \leq 2p\,\|f\|_2||u_p||_2 \leq \frac{2p\|f\|_2}{(2\beta_2)^{1/2}}|||u_p|||_2.$$

Then, for $2 \le q < p$, using (3.15) twice (for (p,q) and for $(q,2)$),

$$\begin{aligned} |||u_p|||_q^p &\le |||u_p|||_p^p \left(\int_\Omega \int_\Omega J(x-y)\,dxdy \right)^{\frac{p-q}{q}} \\ &\le \frac{2p\|f\|_2}{(2\beta_2)^{1/2}} |||u_p|||_2 \left(\int_\Omega \int_\Omega J(x-y)\,dxdy \right)^{\frac{p-q}{q}} \\ &\le \frac{2p\|f\|_2}{(2\beta_2)^{1/2}} |||u_p|||_q \left(\int_\Omega \int_\Omega J(x-y)\,dxdy \right)^{\frac{p-q}{q}+\frac{q-2}{2q}} . \end{aligned}$$

Consequently,

$$|||u_p|||_q \le \left(\frac{2p\|f\|_2}{(2\beta_2)^{1/2}} \right)^{\frac{1}{p-1}} \left(\int_\Omega \int_\Omega J(x-y)\,dxdy \right)^{\frac{1}{q}-\frac{1}{2(p-1)}} . \tag{3.16}$$

Then, $\{|||u_p|||_q : p > q\}$ is bounded. Hence, by Poincaré's inequality (2.26).2, we have that $\{u_p : p > q\}$ is bounded in $L^q(\Omega)$. Therefore, we can assume that $u_p \rightharpoonup u$ weakly in $L^q(\Omega)$. By a diagonal process, we have that there is a sequence $p_n \to \infty$, such that $u_{p_n} \rightharpoonup u$ weakly in $L^m(\Omega)$, as $n \to +\infty$, for all $m \in \mathbb{N}$. Thus, $u \in L^\infty(\Omega)$. Since the functional $v \mapsto |||v|||_q$ is weakly lower semi-continuous, having in mind (3.16), we have

$$|||u|||_q \le \left(\int_\Omega \int_\Omega J(x-y)\,dxdy \right)^{\frac{1}{q}} .$$

Therefore, $\lim_{q\to+\infty} |||u|||_q \le 1$, from where it follows that $|u(x)-u(y)| \le d_1(x,y)$ a.e. in $\Omega\times\Omega$. Now, thanks to Lemma 3.1 we can suppose, that $u \in K_{d_1}(\Omega)$.

Let us now see that u is a Kantorovich potential associated with the metric d_1. Fix $v \in K_{d_1}(\Omega)$. Then,

$$-\int_\Omega f u_p \le \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u_p(y)-u_p(x)|^p\,dxdy - \int_\Omega f(x)u_p(x)\,dx = F_p(u_p)$$

$$\le F_p\left(v - \frac{1}{|\Omega|}\int_\Omega v\right) = \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|v(y)-v(x)|^p dxdy - \int_\Omega f(x)v(x)\,dx$$

$$\le \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)dxdy - \int_\Omega f(x)v(x)\,dx,$$

where we have used $\int_\Omega f = 0$ for the second equality and the fact that $v \in K_{d_1}(\Omega)$ for the last inequality. Hence, taking limit as $p \to \infty$, we obtain that

$$\int_\Omega u(x)(f^+(x)-f^-(x))\,dx \ge \int_\Omega v(x)(f^+(x)-f^-(x))\,dx. \qquad \square$$

We will now characterize the Euler-Lagrange equation associated with the variational problem $\sup\left\{\mathcal{P}_{f^+,f^-}(u) \ : \ u \in K_{d_1}(\Omega)\right\}$, that is, we characterize $f^+ - f^- \in \partial\mathbb{I}_{K_{d_1}(\Omega)}(u)$.

Let $\mathcal{M}_b^a(\Omega \times \Omega)$ be the set of bounded antisymmetric Radon measures in $\Omega \times \Omega$. And define the multivalued operator B_1 in $L^2(\Omega)$ as follows: $(u, v) \in B_1$ if and only if $u \in K_1$, $v \in L^2(\Omega)$, and there exists $\sigma \in \mathcal{M}_b^a(\Omega \times \Omega)$ such that

$$\sigma = \sigma \llcorner \{(x, y) \in \Omega \times \Omega \ : \ |x - y| \leq 1\},$$

$$\int_{\Omega \times \Omega} \xi(x) \, d\sigma(x, y) = \int_\Omega \xi(x) v(x) \, dx, \qquad \forall \, \xi \in C_c(\Omega),$$

and

$$|\sigma|(\Omega \times \Omega) \leq 2 \int_\Omega v(x) u(x) \, dx.$$

Theorem 3.7. *The following characterization holds:* $\partial \mathbb{I}_{K_{d_1}(\Omega)} = B_1$.

Proof. Let us first see that $B_1 \subset \partial \mathbb{I}_{K_{d_1}(\Omega)}$. Let $(u, v) \in B_1$, to see that $(u, v) \in \partial \mathbb{I}_{K_{d_1}(\Omega)}$ we need to prove that

$$0 \leq \int_\Omega v(x)(u(x) - \xi(x)) \, dx \quad \forall \, \xi \in K_{d_1}(\Omega).$$

By approximation we can assume that $\xi \in K_{d_1}(\Omega)$ is continuous. Then,

$$\begin{aligned}\int_\Omega v(x)(u(x)-\xi(x))\,dx &\geq \frac{1}{2}|\sigma|(\Omega\times\Omega) - \int_\Omega v(x)\xi(x)\,dx \\ &= \frac{1}{2}|\sigma|(\Omega\times\Omega) - \int_{\Omega\times\Omega}\xi(x)\,d\sigma(x,y) \\ &= \frac{1}{2}|\sigma|(\Omega\times\Omega) - \frac{1}{2}\int_{\Omega\times\Omega}(\xi(x)-\xi(y))\,d\sigma(x,y) \geq 0,\end{aligned}$$

where in the last equality we have used the antisymmetry of σ. Therefore, we have $B_1 \subset \partial\mathbb{I}_{K_{d_1}(\Omega)}$. Since $\partial\mathbb{I}_{K_{d_1}(\Omega)}$ is a maximal monotone operator, to see that the operators are equal we only need to show that for every $f \in L^2(\Omega)$ there exists $u \in K_{d_1}(\Omega)$ such that

$$u + B_1(u) \ni f.$$

Given $p > N$ and $f \in L^2(\Omega)$, there exists a unique solution $u_p \in L^\infty(\Omega)$ of the nonlocal p-Laplacian problem

(3.17)

$$u_p(x) - \int_\Omega J(x-y)\,|u_p(y)-u_p(x)|^{p-2}\,(u_p(y)-u_p(x))\,dy = T_p(f)(x) \qquad \forall\, x\in\Omega,$$

where $T_k(r) := \max\{\min\{k,r\},-r\}$. We get $u \in K_{d_1}(\Omega)$ such that

$$u_p \to u \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad p \to +\infty, \tag{3.18}$$

with $u + \partial \mathbb{I}_{K_{d_1}(\Omega)}(u) \ni f$.

Observe that

$$\int_\Omega (f(x)-u(x))(w(x)-u(x))\,dx \le 0 \qquad \forall\, w \in K_1,$$

and consequently $u = P_{K_{d_1}(\Omega)}(f)$.

Multiplying (3.17) by u_p and integrating, we get

$$\int_\Omega (T_p(f)(x)-u_p(x))u_p(x)\,dx = \frac{1}{2}\int_{\Omega\times\Omega} J(x-y)\,|u_p(y)-u_p(x)|^p\,dxdy, \tag{3.19}$$

from where it follows that

$$\int_{\Omega\times\Omega} J(x-y)\,|u_p(y)-u_p(x)|^p\,dxdy + \int_\Omega |u_p(x)|^2\,dx \le \|f\|^2_{L^2(\Omega)}. \tag{3.20}$$

If we set $\sigma_p(x,y) := J(x-y)\,|u_p(y)-u_p(x)|^{p-2}\,(u_p(y)-u_p(x))$, by Hölder's inequality,

$$\int_{\Omega\times\Omega} |\sigma_p(x,y)|\,dxdy = \int_{\Omega\times\Omega} J(x-y)\,|u_p(y)-u_p(x)|^{p-1}\,dxdy$$
$$\leq \left(\int_{\Omega\times\Omega} J(x-y)\,|u_p(y)-u_p(x)|^{p}\,dxdy\right)^{\frac{p-1}{p}} \left(\int_{\Omega\times\Omega} J(x-y)\,dxdy\right)^{\frac{1}{p}}$$
$$= \left(\int_{\Omega\times\Omega} J(x-y)\,|u_p(y)-u_p(x)|^{p}\,dxdy\right)^{\frac{p-1}{p}}.$$

Now, by (3.20), we have

$$\int_{\Omega\times\Omega} |\sigma_p(x,y)|\,dxdy \leq \left(\|f\|^2_{L^2(\Omega)}\right)^{\frac{p-1}{p}}.$$

Hence, $\{\sigma_p \,:\, p\geq 2\}$ is bounded in $L^1(\Omega\times\Omega)$, and consequently we can assume that

$$\sigma_p(.,.) \rightharpoonup \sigma \quad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega\times\Omega). \tag{3.21}$$

Obviously, since each σ_p is antisymmetric, $\sigma\in\mathcal{M}^a_b(\Omega\times\Omega)$. And, moreover, since $\mathrm{supp}(J)=\overline{B_1(0)}$, we have $\sigma = \sigma \llcorner \{(x,y)\in\Omega\times\Omega \,:\, |x-y|\leq 1\}$.

On the other hand, given $\xi \in C_c(\Omega)$, by (3.17), (3.18) and (3.21), we get

$$\int_{\Omega\times\Omega} \xi(x)\, d\sigma(x,y) = \lim_{p\to+\infty} \int_{\Omega\times\Omega} \xi(x)\sigma_p(x,y)\, dx\, dy$$
$$= \lim_{p\to+\infty} \int_{\Omega\times\Omega} J(x-y)\, |u_p(y)-u_p(x)|^{p-2}\, (u_p(y)-u_p(x))\xi(x)\, dx\, dy$$
$$= \lim_{p\to+\infty} \int_{\Omega} (T_p(f)(x) - u_p(x))\xi(x)\, dx = \int_{\Omega} (f(x)-u(x))\xi(x)\, dx.$$

Then, to finish, we only need to show that $|\sigma|(\Omega\times\Omega) \le 2\int_\Omega (f(x)-u(x))u(x)\, dx$. In fact, by (3.21), we have

$$|\sigma|(\Omega\times\Omega) \le \liminf_{p\to+\infty} \int_\Omega \int_\Omega |\sigma_p(x,y)|\, dx\, dy.$$

Now, by (3.19),

$$\int_{\Omega\times\Omega} |\sigma_p(x,y)|\, dx\, dy \le \left(\int_{\Omega\times\Omega} J(x-y)\, |u_p(y)-u_p(x)|^p\, dxdy \right)^{\frac{p-1}{p}}$$
$$= \left(2\int_\Omega (T_p(f)(x)-u_p(x))u_p(x)\, dx \right)^{\frac{p-1}{p}} = 2^{\frac{p-1}{p}} \left(\int_\Omega (T_p(f)(x)-u_p(x))u_p(x)\, dx \right)^{\frac{p-1}{p}}.$$

Therefore

$$|\sigma|(\Omega\times\Omega) \le 2\int_\Omega (f(x)-u(x))u(x)\, dx. \qquad \square$$

As consequence of the above result, we have that $u^* \in K_{d_1}(\Omega)$ is a Kantorovich potential for d_1, f^+, f^-, if and only if

$$f^+ - f^- \in B_1(u^*).$$

that is, if $u^* \in K_1$ and there exists $\sigma^* \in \mathcal{M}_b^a(\Omega \times \Omega)$, such that

$$\begin{cases} [\sigma^*]^+ = [\sigma^*]^+ \llcorner \{(x,y) \in \Omega \times \Omega \ : \ u^*(x) - u^*(y) = 1, \ |x-y| \leq 1\}, \\ [\sigma^*]^- = [\sigma^*]^- \llcorner \{(x,y) \in \Omega \times \Omega \ : \ u^*(y) - u^*(x) = 1, \ |x-y| \leq 1\}, \\ \displaystyle\int_{\Omega\times\Omega} \xi(x) d\sigma^*(x,y) = \int_\Omega \xi(x)(f^+(x) - f^-(x))\, dx \quad \forall \xi \in C_c(\Omega), \end{cases}$$

and

$$\frac{1}{2}|\sigma^*|(\Omega \times \Omega) = \int_\Omega (f^+(x) - f^-(x)) u^*(x)\, dx.$$

We want to highlight that the above equations plays the role of (3.14). Moreover, the potential u_1^* and the measure σ_1^* encode all the information that is needed to construct an optimal transport plan associated with the problem (see [**40**]).

Of course all these developments can be done in the same way for the discrete distance with steps of size ε,

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } \ x = y, \\ \varepsilon & \text{if } \ 0 < |x-y| \leq \varepsilon, \\ 2\varepsilon & \text{if } \ \varepsilon < |x-y| \leq 2\varepsilon, \\ \vdots & \end{cases}$$

In [**40**] we give the connection between the Monge-Kantorovich problem with the discrete distance d_ε and the classical Monge-Kantorovich problem with the Euclidean distance, proving that, when the length of the step tends to zero, these discrete/nonlocal problems give an approximation to the classical one; in particular, we recover the PDE formulation given by Evans-Gangbo.

3.3. From the Dirichlet problem for the nonlocal p–Laplacian to ...

Let $J:\mathbb{R}^N\to\mathbb{R}$ be a nonnegative continuous radial function with compact support, $J(0)>0$ and $\int_{\mathbb{R}^N} J(x)\,dx=1$.

THEOREM 3.8. *Given $\psi\in L^\infty(\Omega_J\setminus\Omega)$and $p>1$, there exists a solution to the homogeneous nonlocal p-Laplacian Dirichlet problem:*

(3.22)
$$\begin{cases} -\displaystyle\int_{\Omega_J} J(x-y)|(u_p)_\psi(y)-u_p(x)|^{p-2}((u_p)_\psi(y)-u_p(x))dy=0, & x\in\Omega,\\ u_p=\psi, & x\in\Omega_J\setminus\overline{\Omega}. \end{cases}$$

PROOF. Let us consider the functional

$$\mathcal{F}_p(u):=\frac{1}{2p}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|u_\psi(y)-u_\psi(x)|^p dydx,\quad u\in L^p(\Omega).$$

Set

$$\theta:=\inf_{u\in L^p(\Omega)}\mathcal{F}_p(u),$$

and let $\{u_n\}$ be a minimizing sequence. Then,

$$\theta=\lim_{n\to\infty}\mathcal{F}_p(u_n)\quad\text{and}\quad K:=\sup_{n\in\mathbb{N}}\mathcal{F}_p(u_n)<+\infty\,.$$

The Poincarév inequality 2.26.1 yields

$$\lambda \int_\Omega |u_n(x)|^p \, dx \leq \int_\Omega \int_{\Omega_J} J(x-y)|(u_n)_\psi(y) - u_n(x)|^p \, dy \, dx + \int_{\Omega_J \setminus \overline{\Omega}} |\psi(y)|^p \, dy$$

$$= 2p\mathcal{F}_p(u_n) + \int_{\Omega_J \setminus \overline{\Omega}} |\psi(y)|^p \, dy \leq 2pK + \int_{\Omega_J \setminus \overline{\Omega}} |\psi(y)|^p \, dy.$$

Therefore, we obtain that

$$\int_\Omega |u_n(x)|^p \, dx \leq C \quad \forall n \in \mathbb{N}.$$

Hence, up to a subsequence, we have

$$u_n \rightharpoonup u_p \quad \text{in} \quad L^p(\Omega).$$

Furthermore, using the weak lower semi-continuity of the functional $\mathcal{F}_p$, we get

$$\mathcal{F}_p(u_p) = \inf_{u \in L^p(\Omega)} \mathcal{F}_p(u).$$

Thus, given $\lambda > 0$ and $w \in L^p(\Omega)$ (we extend it to $\Omega_J \setminus \Omega$ by zero), we have

$$0 \leq \frac{\mathcal{F}_p(u_p + \lambda w) - \mathcal{F}_p(u_p)}{\lambda},$$

or equivalently,

$$0 = \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)\left[\frac{|(u_p)_\psi(y)+\lambda w_\psi(y)-((u_p)_\psi(x)+\lambda w_\psi(x))|^p - |(u_p)_\psi(y)-(u_p)_\psi(x)|^p}{p\lambda}\right] dydx.$$

Now, since $p>1$, we pass to the limit as $\lambda \downarrow 0$ to deduce

$$0 \le \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|(u_p)_\psi(y)-(u_p)_\psi(x)|^{p-2}((u_p)_\psi(y)-(u_p)_\psi(x))((w)_\psi(y)-(w)_\psi(x))dydx.$$

Taking $\lambda<0$ and proceeding as above we obtain the reverse inequality. Consequently, we conclude that

$$\begin{aligned} 0 &= \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|(u_p)_\psi(y)-(u_p)_\psi(x)|^{p-2}((u_p)_\psi(y)-(u_p)_\psi(x))((w)_\psi(y)-(w)_\psi(x))dydx \\ &= -\int_{\Omega_J}\int_{\Omega_J} J(x-y)|(u_p)_\psi(y)-(u_p)_\psi(x)|^{p-2}((u_p)_\psi(y)-(u_p)_\psi(x))dy(w)_\psi(x)dx. \end{aligned}$$

In particular, since $w=0$ in $\Omega_J\setminus\Omega$, it follows that

$$0 = -\int_{\Omega}\int_{\Omega_J} J(x-y)|(u_p)_\psi(y)-u_p(x)|^{p-2}((u_p)_\psi(y)-u_p(x))dyw(x)dx,$$

which shows that u_p is a solution of (3.22). $\square$

Moreover, for u_p, the solution to (3.22) for $\psi\in L^\infty(\Omega_J\setminus\Omega)$, we have that

$$\|u_p\|_\infty \le \|\psi\|_\infty.$$

3.3.1. ($p \to +\infty$) A best Lipschitz extension problem for a discrete distance.

3.3.1.1. *From the Dirichlet nonlocal problem to a discrete infinity Laplace problem.* Let $\Omega_\varepsilon := \Omega + B_\varepsilon(0)$, $\psi \in L^p(\Omega_1 \setminus \overline{\Omega})$ and

$$\begin{aligned} \mathrm{B}^{J_\varepsilon}_{p,\psi}(u)(x) \ := & - \int_\Omega J_\varepsilon(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy \\ & - \int_{\Omega_\varepsilon \setminus \overline{\Omega}} J_\varepsilon(x-y)|\psi(y)-u(x)|^{p-2}(\psi(y)-u(x))\,dy, \quad x \in \Omega, \end{aligned}$$

We have:

- There exists a unique $u_p^\varepsilon \in L^p(\Omega)$ such that
$$\mathrm{B}^{J_\varepsilon}_{p,\psi}(u_p^\varepsilon) = 0.$$
- $u_p^\varepsilon \to u_\varepsilon \in L^\infty(\Omega)$ strongly in any $L^q(\Omega)$ as $p \to +\infty$.
- $(u_\varepsilon)_\psi := u_\varepsilon \chi_\Omega + f\chi_{\Omega_\varepsilon \setminus \Omega}$ is the unique solution of
$$\begin{cases} -\Delta^\varepsilon_\infty u = 0 & \text{in } \ \Omega, \\ u = \psi & \text{on } \ \Omega_\varepsilon \setminus \Omega, \end{cases}$$

where $\Delta^\varepsilon_\infty u(x) := \sup_{y \in \overline{B}_\varepsilon(x)} u(y) + \inf_{y \in \overline{B}_\varepsilon(x)} u(y) - 2u(x)$ is the discrete infinity Laplace operator. This is in fact the value function of a TUG-OF-WAR game.

TUG-OF-WAR GAME.

• There are two players moving a token inside Ω_ε. The token is placed at an initial position $x_0 \in \Omega$.

• At the kth stage of the game, player I and player II select points x_k^I and x_k^{II} respectively, both belonging to $\overline{B}(x_{k-1}, \varepsilon)$.

• The token is then moved to x_k, where x_k is chosen randomly between x_k^I or x_k^{II} with equal probability.

• After the kth stage of the game, if $x_k \in \Omega$ then the game continues to stage $k+1$.

• Otherwise, if $x_k \in \Omega_\varepsilon \setminus \Omega$, the game ends and player II pays player I the amount $\psi(x_k)$, where $\psi : \Omega_\varepsilon \setminus \Omega \to \mathbb{R}$ is the final payoff function of the game.

Dynamic Programming Principle. The value of the game is the minimum (max.) amount that player I (II) expects to win (lose):

$$u_\varepsilon(x) = \frac{1}{2} \sup_{y \in \overline{B}_\varepsilon(x)} u_\varepsilon(y) + \frac{1}{2} \inf_{y \in \overline{B}_\varepsilon(x)} u_\varepsilon(y).$$

Peres, Schramm, Sheffield and Wilson ([**50**]) prove that

$$\lim_{\varepsilon \to 0} u_\varepsilon = h,$$

where h is the absolutely minimizing Lispchitz extension (AMLE) of ψ to $\overline{\Omega}$, that is (G. Aronsson [**14**]):

- $h_{|\partial\Omega} = \psi$ and $L_d(h, \overline{\Omega}) = L_d(\psi, \partial\Omega)$ (h is a minimal Lipschitz extension),
- for every open set $D \subset\subset \Omega$,

$$L_d(h, D) \leq L_d(v, D) \qquad \forall v : \; h_{|\partial D} = v_{|\partial D},$$

where L_d stands for the Lipschitz constant respect to d.

To obtain this AMLE extension of a datum f, Aronsson proposed to take the limit as $p \to \infty$ in

$$\begin{cases} -\Delta_p u_p = 0 & \text{in } \Omega, \\ u_p = \psi & \text{on } \partial\Omega. \end{cases}$$

That is, to obtain (Bhattacharya, DiBenedetto and Manfredi [**21**]) the unique (Jensen [**42**]) viscosity solution to

$$\begin{cases} -\Delta_\infty u_\infty = 0 & \text{in } \Omega, \\ u_\infty = \psi & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_\infty u := \sum_{i,j=1}^N u_{x_i} u_{x_j} u_{x_i x_j}$ is the infinity Laplace operator.

Is $(u_\varepsilon)_\psi$ *the best Lipschitz extension with respect to some distance?*

The distance to be considered is the discrete distance

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } \ x = y, \\ \varepsilon & \text{if } \ 0 < |x-y| \le \varepsilon, \\ 2\varepsilon & \text{if } \ \varepsilon < |x-y| \le 2\varepsilon, \\ \vdots & \end{cases}$$

We see that $(u_\varepsilon)_\psi$ is the best Lipschitz extension to Ω_ε of the function ψ, defined on the strip $\Omega_\varepsilon \setminus \Omega$, w.r.t. this distance, but not in the usual sense.

Given $u : \Omega_\varepsilon \to \mathbb{R}$ and $D \subset \Omega$, we define

$$\begin{aligned} L_\varepsilon(u,D) \; &:= \sup_{\substack{x \in D,\, y \in D_\varepsilon \\ |x-y| \le \varepsilon}} \frac{|u(x)-u(y)|}{\varepsilon} \\ \text{(}D\text{ convex)} \; &= \sup_{x\in D,\, y\in D_\varepsilon,\, x\neq y} \frac{|u(x)-u(y)|}{d_\varepsilon(x,y)} \ge L_{d_\varepsilon}(u,D) \end{aligned}$$

3.3.1.2. *ε–Absolutely minimizing Lispchitz extensions.*

DEFINITION 3.9. Let ψ defined on $\Omega_\varepsilon \setminus \Omega$. A function $h : \Omega_\varepsilon \to \mathbb{R}$ is an $\mathrm{AMLE}_\varepsilon$ of ψ to Ω_ε if

(i) $h = \psi$ in $\Omega_\varepsilon \setminus \Omega$,
(ii) $\forall\ D \subset X$ and v such that $v = h$ in $\Omega_\varepsilon \setminus D$, then $L_\varepsilon(h, D) \leq L_\varepsilon(v, D)$.

For convex Ω, $h \in \mathrm{AMLE}_\varepsilon(\psi, \Omega_\varepsilon)$ iff

(i) $h \in \mathrm{MLE}_{d_\varepsilon}(\psi, \Omega_\varepsilon)$,
(ii) $\forall\ D \subset X$ and v such that $v = h$ in $\Omega_\varepsilon \setminus D$, then $L_\varepsilon(h, D) \leq L_\varepsilon(v, D)$.

THEOREM 3.10. *Let $\psi : \Omega_\varepsilon \setminus \Omega \to \mathbb{R}$ be bounded. Then, u is a solution of*

$$\begin{cases} -\Delta_\infty^\varepsilon u = 0 & \text{in } \ \Omega, \\ u = \psi & \text{on } \ \Omega_\varepsilon \setminus \Omega, \end{cases} \tag{3.23}$$

if and only if

$$u : \Omega_\varepsilon \to \mathbb{R} \quad \text{is} \quad \mathrm{AMLE}_\varepsilon(\psi, \Omega).$$

PROOF. Without loss of generality we will take $\varepsilon = 1$ along the proof. Let us first take u a solution of (3.23) and suppose that u is not $\mathrm{AMLE}_1(f, \Omega)$. Then, there exists

$D \subset \Omega$ and $v : \Omega_1 \to \mathbb{R}$, $v = u$ in $\Omega_1 \setminus D$, such that $L_1(v, D) < L_1(u, D)$. Set $\delta := L_1(u, D) - L_1(v, D) > 0$, and let $n \in \mathbb{N}$, $n > 3$, such that

$$\sup_D u - \inf_D u \leq (n-1) L_1(u, D). \tag{3.24}$$

Take $(x_0, y_0) \in D \times \Omega_1$, $|x_0 - y_0| \leq 1$, such that

$$L_1(u, D) - \frac{\delta}{n} \leq |u(x_0) - u(y_0)| \leq L_1(u, D).$$

We have that $\Delta^1_\infty u(x_0) = 0$ and $\Delta^1_\infty u(y_0) = 0$ if $y_0 \in \Omega$. Let us suppose that $u(y_0) \geq u(x_0)$ (the other case being similar), which implies

$$L_1(u, D) - \frac{\delta}{n} \leq u(y_0) - u(x_0) \leq L_1(u, D). \tag{3.25}$$

If $y_0 \notin D$, set $y_1 = y_0$. If $y_0 \in D$, since $\Delta^1_\infty u(y_0) = 0$ and $x_0 \in \overline{B}_1(y_0)$, we have

$$\sup_{y \in \overline{B}_1(y_0)} u(y) - u(y_0) = u(y_0) - \inf_{y \in \overline{B}_1(y_0)} u(y) \geq u(y_0) - u(x_0) \geq L_1(u, D) - \frac{\delta}{n}.$$

Hence, there exists $y_1 \in \overline{B}_1(y_0)$ such that

$$u(y_1) - u(y_0) \geq L_1(u, D) - \frac{2\delta}{n}.$$

Also, since $\Delta^1_\infty u(x_0) = 0$, we have

$$u(x_0) - \inf_{x\in \overline{B}_1(x_0)} u(x) = \sup_{x\in \overline{B}_1(x_0)} u(x) - u(x_0) \geq u(y_0) - u(x_0) \geq L_1(u, D) - \frac{\delta}{n},$$

and consequently, there exists $x_1 \in \overline{B}_1(x_0)$ such that

$$u(x_0) - u(x_1) \geq L_1(u, D) - \frac{2\delta}{n}.$$

Following this construction, and with the rule that in the case $x_j \notin D$ or $y_j \notin D$, then $x_i = x_j$ or $y_i = y_j$ for all $i \geq j$, we claim that there exists $m \leq n$ for which $x_m \notin D$ and $y_m \notin D$. In fact, if not, then either $\{x_i\}_{i=1,\ldots,n} \subset D$, either $\{y_i\}_{i=1,\ldots,n} \subset D$, with $\{x_i\}_{i=1,\ldots,n}$ and $\{y_i\}_{i=1,\ldots,n}$ satisfying

$$u(y_i) - u(y_{i-1}) \geq L_1(u, D) - \frac{2\delta}{n}, \quad y_i \in \overline{B}_1(y_{i-1}),\ i = 1, \ldots, n,$$

and

$$u(x_i) - u(x_{i-1}) \geq L_1(u, D) - \frac{2\delta}{n}, \quad x_i \in \overline{B}_1(x_{i-1}),\ i = 1, \ldots, n. \tag{3.26}$$

Let us suppose the first of these two possibilities, that is, $\{x_i\}_{i=1,\ldots,n} \subset D$. Then, having in mind (3.24), (3.25) and (3.26), we get

$$\begin{aligned}
&(n-1)L_1(u,D) \geq u(y_0) - u(x_n) \\
&= u(y_0) - u(x_0) + u(x_0) - u(x_1) + \cdots + u(x_{n-1}) - u(x_n) \\
&\geq L_1(u,D) - \tfrac{\delta}{n} + (n+1)(L_1(u,D) - \tfrac{2\delta}{n}),
\end{aligned}$$

from where it follows that

$$\frac{2n+3}{n}\delta \geq 3L_1(u,D) \geq 3\delta,$$

which is a contradiction since $n > 3$. Now, for $\{x_i, y_i\}_{i=1,\ldots,m}$, we have

$$v(y_m) - v(x_m) = u(y_m) - u(x_m) \geq 2m\left(L_1(u,D) - \frac{2\delta}{n}\right) + L_1(u,D) - \frac{\delta}{n},$$

$$v(y_m) - v(x_m) \leq (2m+1)L_1(v,D),$$

and therefore,

$$(2m+1)L_1(u,D) - (4m+1)\frac{\delta}{n} \leq (2m+1)L_1(v,D),$$

that is

$$\delta = L_1(u,D) - L_1(v,D) \leq \frac{4m+1}{2m+1}\frac{\delta}{n},$$

which implies $n \leq \frac{4m+1}{2m+1} \leq 2$, which is a contradiction since $n > 3$.

Let us now consider u an $\mathrm{AMLE}_1(f,\Omega)$ and suppose that u is not a solution of (3.23). Then, $\{x \in \Omega : \Delta^1_\infty u(x) \neq 0\} \neq \emptyset$. Let us suppose without loss of generality, that,

$$\left\{ x \in \Omega \ : \ \sup_{y \in \overline{B}_1(x)} u(y) - u(x) > u(x) - \inf_{y \in \overline{B}_1(x)} u(y) \right\} \neq \emptyset.$$

Then, there exists $\delta > 0$ and a nonempty set $D \subset \Omega$ such that

$$\sup_{y \in \overline{B}_1(x)} u(y) - u(x) > u(x) - \inf_{y \in \overline{B}_1(x)} u(y) + \delta \quad \text{for all } x \in D. \tag{3.27}$$

Consider the function $v : \Omega_1 \to \mathbb{R}$ defined by

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega_1 \setminus D, \\ u(x) + \frac{\delta}{2} & \text{if } x \in D. \end{cases}$$

Then, since u is an $\mathrm{AMLE}_1(f,\Omega)$, we have $L_1(u,D) \leq L_1(v,D)$. Now, there exists $x_0 \in D$ and $y_0 \in \overline{B}_1(x_0)$ such that

$$L_1(v,D) \leq \frac{\delta}{4} + |v(x_0) - v(y_0)|.$$

Therefore, if $v(x_0) \geq v(y_0)$, by (3.27),

$$L_1(v,D) \leq \frac{\delta}{4} + v(x_0) - v(y_0) \leq \frac{3\delta}{4} + u(x_0) - u(y_0)$$
$$\leq \frac{3\delta}{4} + u(x_0) - \inf_{x \in \overline{B}_1(x_0)} u(x) < -\frac{\delta}{4} + \sup_{x \in \overline{B}_1(x_0)} u(x) - u(x_0) < L_1(u,D),$$

which is a contradiction, and, if $v(x_0) < v(y_0)$,

$$L_1(v,D) \leq \frac{\delta}{4} + v(y_0) - v(x_0) = -\frac{\delta}{4} + v(y_0) - u(x_0),$$

so, if $y_0 \notin D$,

$$L_1(v,D) \leq -\frac{\delta}{4} + u(y_0) - u(x_0) < L_1(u,D),$$

also a contradiction, and if $y_0 \in D$, since also $x_0 \in \overline{B}_1(y_0)$, by (3.27),

$$L_1(v,D) \leq \frac{\delta}{4} + u(y_0) - u(x_0) \leq \frac{\delta}{4} + u(y_0) - \inf_{y \in \overline{B}_1(y_0)} u(y)$$
$$< -\frac{3\delta}{4} + \sup_{y \in \overline{B}_1(y_0)} u(y) - u(y_0) < L_1(u,D),$$

again a contradiction. Then, in any case we arrive to a contradiction and consequently u is a solution of (3.23). $\square$

3.3.2. ($p \to 1^+$) Median values and least gradient functions.

It is a well known fact that solutions to some partial differential equations are related to mean value properties. As a classical example we have that u is harmonic in a domain $\Omega \subset \mathbb{R}^N$ (that is, u verifies $\Delta u = 0$ in Ω) if and only if it verifies the mean value property

$$u(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\, dy,$$

for all $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \Omega$; if and only if

$$u(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\, dy + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

In [**37**], solutions to

$$\Delta_1^H u := |Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) = 0 \tag{3.28}$$

are characterized, in dimension 2, in terms of another asymptotic geometric property. It is proved that

$$u(x) - \operatorname{median}_{s \in \partial B_\varepsilon(x)} u(s) = -\frac{\varepsilon^2}{2} \Delta_1^H u(x) + o(\varepsilon^2);$$

here, the median of a continuous function over a measurable set A,

$$\mathrm{median}_{s\in A} u(s) = m,$$

is defined as the unique value m such that, for μ the 1–dimensional Hausdorff measure,

$$\mu(\{x \in A : u(x) \geq m\}) \geq \frac{\mu(A)}{2} \quad \text{and} \quad \mu(\{x \in A : u(x) \leq m\}) \geq \frac{\mu(A)}{2}.$$

Uniqueness of the value m holds for continuous functions.

On the other hand, in [**46**] it is proved that the Dirichlet problem for the 1–Laplacian operator

$$\begin{cases} -\mathrm{div}\Big(\dfrac{Du}{|Du|}\Big) = 0\,, & \text{in } \Omega\,, \\ u = h\,, & \text{on } \partial\Omega\,, \end{cases} \tag{3.29}$$

has a solution $u \in BV(\Omega)$ for every $h \in L^1(\partial\Omega)$. The relaxed energy functional associated to problem (3.29) is the functional $\Phi_h : L^{\frac{N}{N-1}}(\Omega) \to (-\infty, +\infty]$ defined by

$$\Phi_h(u) = \begin{cases} \displaystyle\int_\Omega |Du| + \int_{\partial\Omega} |u-h|\, d\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega), \\ +\infty & \text{if } u \in L^{\frac{N}{N-1}}(\Omega) \setminus BV(\Omega). \end{cases} \tag{3.30}$$

And these solutions are characterized as the functions of least gradient that appear in the theory of parametric minimal surfaces.

This problem is quite different from (3.28) since it involves giving a meaning to $\frac{\nabla u}{|\nabla u|}$ when the gradient vanishes. These difficulties were tackled in [**2**] (see also [**46**]) by means of a bounded vector field z which plays the role of $\frac{Du}{|Du|}$. Moreover there are extra difficulties for the Dirichlet boundary condition, which has to be considered in a weak sense.

Our aim here is to study solutions to the *nonlocal* 1-*Laplacian* with Dirichlet boundary condition ψ:

$$\begin{cases} -\displaystyle\int_{\Omega_J} J(x-y)\frac{u_\psi(y)-u(x)}{|u_\psi(y)-u(x)|}dy = 0, & x \in \Omega, \\ u(x) = \psi(x), & x \in \Omega_J \setminus \overline{\Omega}, \end{cases} \tag{3.31}$$

and to relate them with a *nonlocal median value property* and with *a kind of nonlocal least gradient functions*. Hereafter, $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain. And $J : \mathbb{R}^N \to \mathbb{R}$ is a continuous nonnegative radial function, compactly supported in $B_1(0)$ with $J(0) > 0$, verifying $\int_{\mathbb{R}^N} J(z)dz = 1$.

Let us define the following measure of a set $E \subset B_1(0)$:

$$\mu_J^0(E) := \int_E J(z)\, dz.$$

For $f : \mathbb{R}^N \to \mathbb{R}$ a measurable function (not necessarily continuous), we say that m is a median value of f with respect to μ_J^0 ($m \in \operatorname{median}_{\mu_J^0} f$) if

$$\mu_J^0(\{y \in B_1(0) : f(y) \geq m\}) \geq \frac{1}{2} \quad \text{and} \quad \mu_J^0(\{y \in B_1(0) : f(y) \leq m\}) \geq \frac{1}{2}.$$

We also define a weak solution to (3.31) as follows:

Definition 3.11. Let $\psi \in L^1(\Omega_J \setminus \overline{\Omega})$. We say that $u \in L^1(\Omega)$ is a weak solution to (3.31) if there exists $g : \Omega_J \times \Omega_J \to \mathbb{R}$ such that $g \in L^\infty(\Omega_J \times \Omega_J)$ with $\|g\|_\infty \leq 1$,

$$J(x-y)g(x,y) \in J(x-y)\operatorname{sign}(u_\psi(y) - u_\psi(x)) \quad \text{a.e } (x,y) \in \Omega_J \times \Omega_J, \tag{3.32}$$

and

$$-\int_{\Omega_J} J(x-y)g(x,y)\, dy = 0 \quad \text{a.e } x \in \Omega. \tag{3.33}$$

We have the following characterization of weak solutions of the nonlocal 1-Laplacian with Dirichlet boundary condition in terms of a nonlocal median value property.

THEOREM 3.12. *Given $\psi \in L^1(\Omega_J \setminus \overline{\Omega})$, we have that u is a weak solution to (3.31) with Dirichlet datum ψ if and only if, u verifies the following nonlocal median value property:*

$$u(x) \in \mathit{median}_{\mu_J^0} u_\psi(x - \cdot), \quad x \in \Omega, \tag{3.34}$$

that is, for $x \in \Omega$,

$$\mu_J^x(\{y \in B_1(x) : u_\psi(y) \geq u(x)\}) \geq \frac{1}{2} \quad \text{and} \quad \mu_J^x(\{y \in B_1(x) : u_\psi(y) \leq u(x)\}) \geq \frac{1}{2},$$

where $\mu_J^x(E) := \int_E J(x-y)dy$ for $E \subset B_1(x)$.

If we assume in Definition 3.11 that the function g is antisymmetric, we get a more restrictive concept of solution, which we call *variational solution*. It can be characterized as a minimizer of the functional $\mathcal{J}_\psi : L^1(\Omega) \to [0, +\infty[$ given by

$$\mathcal{J}_\psi(u) := \frac{1}{2} \int_{\Omega_J} \int_{\Omega_J} J(x-y)|u_\psi(y) - u_\psi(x)|\, dxdy. \tag{3.35}$$

This functional $\mathcal{J}_\psi$ is the nonlocal version of the energy functional Φ_h defined by (3.30)

Obviously any variational solution is a weak solution for the nonlocal 1–Laplacian, and the class of variational solutions is strictly smaller than the class of weak solutions.

EXAMPLE 3.13. *Let* $\Omega =]-2,2[\times]-2,2[$, *and choose* J *supported in* $B_1(0)$ *and* $\psi(x) = 1$ *if* $x \in]-2,2[\times(]2,3[\cup]-3,-2[)$, $\psi(x) = 1$ *if* $x \in (]2,3[\cup]-3,-2[)\times]-2,2[$ *and* $\psi(x) = 0$ *otherwise. In this case the constant function* $u(x) = 0$ *in* Ω *is a weak solution to the nonlocal* 1*–Laplacian (any constant function between* 0 *and* 1 *is also a solution, though any constant function above* 1 *or below* 0 *is not). However,* $u = 0$ *is not a variational solution by a similar argument to the above one. The function* $u(x) = 1$ *is a variational solution.*

THEOREM 3.14. *Let* $\psi \in L^1(\Omega_J \setminus \overline{\Omega})$. *Then* $u \in L^1(\Omega)$ *is a variational solution to* (3.31) *if and only if it is a minimizer of the functional* $\mathcal{J}_\psi$ *given in* (3.35).

Moreover there is a link between nonlocal and local problems:

THEOREM 3.15 ([**45**]). *Let* Ω *be a smooth bounded domain in* $\mathbb{R}^N$ *and* $\tilde{\psi} \in L^\infty(\partial\Omega)$. *Take a function* $\psi \in W^{1,1}(\Omega_J \setminus \overline{\Omega}) \cap L^\infty(\Omega_J \setminus \overline{\Omega})$ *such that* $\psi|_{\partial\Omega} = \tilde{\psi}$. *Assume also* $J(x) \geq J(y)$ *if* $|x| \leq |y|$. *Let* u_ϵ *be a variational solution to* (3.31) *for* $J_\varepsilon(x) := \frac{1}{\varepsilon^{N+1}} J\left(\frac{x}{\varepsilon}\right)$. *Then, up to a subsequence,*

$$u_\epsilon \to u \quad \text{in } L^1(\Omega),$$

being u *a solution to* (3.29) *with* $h = \tilde{\psi}$.

3.3.2.1. *Existence of variational solutions.*

THEOREM 3.16. *Given $\psi \in L^\infty(\Omega_J \setminus \Omega)$ there exists a variational solution, hence a weak solution, to problem* (3.31)*.*

PROOF. The previous result ensures that there exists a subsequence $p_n \to 1$, denoted by p, such that

$$u_p \to u \quad \text{weakly in } L^1(\Omega)$$

and

$$|(u_p)_\psi(y) - (u_p)_\psi(x)|^{p-2}((u_p)_\psi(y) - (u_p)_\psi(x)) \to g(x,y) \quad \text{weakly in } L^1(\Omega_J \times \Omega_J).$$

The function g is L^∞-bounded by 1, satisfies

$$-\int_{\Omega_J} J(x-y)g(x,y)\,dy = 0 \quad \text{a.e } x \in \Omega,$$

and, moreover, it is antisymmetric.

In order to see that

$$J(x-y)g(x,y) \in J(x-y)\text{sign}(u_\psi(y) - u_\psi(x)) \quad \text{a.e } (x,y) \in \Omega_J \times \Omega_J,$$

we need to prove that

(3.36)

$$-\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)\,dy u_\psi(x)\,dx = \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|u_\psi(y) - u_\psi(x)|\,dydx.$$

In fact, it holds that

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|(u_p)_\psi(y)-(u_p)_\psi(x)|^p\,dydx\\
&\qquad= -\int_{\Omega_J}\int_{\Omega_J} J(x-y)|(u_p)_\psi(y)-(u_p)_\psi(x)|^{p-2}((u_p)_\psi(y)-(u_p)_\psi(x))\,dy(u_p)_\psi(x)dx\\
&\qquad= -\int_{\Omega_J\setminus\Omega}\int_{\Omega_J} J(x-y)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))\,dy\psi(x)\,dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_p \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|u_p(y)-u_p(x)|^p\,dydx &= -\int_{\Omega_J\setminus\Omega}\int_{\Omega_J} J(x-y)g(x,y)\,dy\psi(x)\,dx\\
&= -\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)\,dyu_\psi(x)\,dx.
\end{aligned}
\tag{3.37}
$$

Now, for all $\rho\in L^\infty(\Omega)$ we have that

$$
\begin{aligned}
&-\int_{\Omega_J}\int_{\Omega_J} J(x-y)|\rho_\psi(y)-\rho_\psi(x)|^{p-2}(\rho_\psi(y)-\rho_\psi(x))\,dy\,((u_p)_\psi(x)-\rho_\psi(x))\,dx\\
&\le -\int_{\Omega_J}\int_{\Omega_J} J(x-y)|(u_p)_\psi(y)-(u_p)_\psi(x)|^{p-2}((u_p)_\psi(y)-(u_p)_\psi(x))dy((u_p)_\psi(x)-\rho_\psi(x))dx.
\end{aligned}
$$

Taking limits as $p \to 1$ and using (3.37) we get

$$
\begin{aligned}
&-\int_{\Omega_J}\int_{\Omega_J} J(x-y)\,\mathrm{sign}_0(\rho_\psi(y)-\rho_\psi(x))\,dy\,(u_\psi(x)-\rho_\psi(x))\,dx \\
&\qquad \leq -\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)\,dy\,(u_\psi(x)-\rho_\psi(x))\,dx.
\end{aligned}
$$

Taking now $\rho = u \pm \lambda u$, $\lambda > 0$, dividing by λ, and letting $\lambda \to 0$, we obtain (3.36), which finishes the proof. $\square$

3.3.2.2. *Characterization of weak solutions using a median value property.* We will use the following notation: given $x \in \Omega$ we decompose $B_1(x)$ as

$$B_1(x) = E_+^x \cup E_-^x \cup E_0^x$$

where

$$E_+^x := \{y \in B_1(x) : u_\psi(y) > u(x)\}$$

$$E_-^x := \{y \in B_1(x) : u_\psi(y) < u(x)\},$$

and

$$E_0^x := \{y \in B_1(x) : u_\psi(y) = u(x)\}.$$

Hence

$$1 = \mu_J^x(E_+^x) + \mu_J^x(E_-^x) + \mu_J^x(E_0^x),$$

and therefore

$$-\mu_J^x(E_0^x) \le \mu_J^x(E_-^x) - \mu_J^x(E_+^x) \le \mu_J^x(E_0^x), \tag{3.38}$$

is equivalent to

$$1 \le 2(\mu_J^x(E_+^x) + \mu_J^x(E_0^x)) \quad \text{and} \quad 1 \le 2(\mu_J^x(E_-^x) + \mu_J^x(E_0^x)).$$

That is,

(3.38) is equivalent to

$$\begin{cases} \mu_J^x(\{y \in B_1(x) : u_\psi(y) \ge u(x)\}) \ge \frac{1}{2} \\ \text{and} \quad \mu_J^x(\{y \in B_1(x) : u_\psi(y) \le u(x)\}) \ge \frac{1}{2}. \end{cases} \tag{3.39}$$

PROOF OF THEOREM 3.12. Let u be a weak solution to (3.31) with Dirichlet datum $\psi \in L^1(\Omega_J \setminus \overline{\Omega})$, and take g as in Definition 3.11. By (3.33) we have

$$-\int_{B_1(x)} J(x-y)g(x,y)\,dy = 0.$$

Thus,

$$\begin{aligned} 0 &= \int_{E_+^x} J(x-y)g(x,y)\,dy + \int_{E_-^x} J(x-y)g(x,y)\,dy + \int_{E_0^x} J(x-y)g(x,y)\,dy \\ &= \mu_J^x(E_+^x) - \mu_J^x(E_-^x) + \int_{E_0^x} J(x-y)g(x,y)\,dy. \end{aligned}$$

Since $g \in [-1,1]$ in E_0^x, it holds that

$$\mu_J^x(E_-^x) = \mu_J^x(E_+^x) + \int_{E_0^x} J(x-y)g(x,y)\,dy \le \mu_J^x(E_+^x) + \mu_J^x(E_0^x)$$

and

$$\mu_J^x(E_+^x) = \mu_J^x(E_-^x) - \int_{E_0^x} J(x-y)g(x,y)\,dy \le \mu_J^x(E_-^x) + \mu_J^x(E_0^x),$$

that is

$$-\mu_J^x(E_0^x) \le \mu_J^x(E_-^x) - \mu_J^x(E_+^x) \le \mu_J^x(E_0^x).$$

This proves, on account of (3.39), that u satisfies the nonlocal median value property (3.34).

Let us show now that the converse is also true.

Let u be satisfying the nonlocal median value property (3.34), that is (on account of (3.39) again),

$$-\mu_J^x(E_0^x) \leq \mu_J^x(E_-^x) - \mu_J^x(E_+^x) \leq \mu_J^x(E_0^x).$$

We want to find a function $g(x,y)$ verifying the conditions of Definition 3.11. For x such that $\mu_J^x(E_0^x) = 0$ let us define

$$g(x,y) := \begin{cases} 1 & \text{if } u_\psi(y) > u_\psi(x), \\ 0 & \text{if } u_\psi(y) = u_\psi(x), \\ -1 & \text{if } u_\psi(y) < u_\psi(x), \end{cases}$$

and if $\mu_J^x(E_0^x) > 0$,

$$g(x,y) := \begin{cases} 1 & \text{if } u_\psi(y) > u_\psi(x), \\ \frac{\mu_J^x(E_-^x) - \mu_J^x(E_+^x)}{\mu_J^x(E_0^x)} & \text{if } u_\psi(y) = u_\psi(x), \\ -1 & \text{if } u_\psi(y) < u_\psi(x). \end{cases}$$

This function g belongs to L^∞ and obviously $\|g\|_\infty \leq 1$. In addition, it verifies (3.32), that is,

$$J(x-y)g(x,y) \in J(x-y)\text{sign}(u_\psi(y) - u_\psi(x)) \quad \text{a.e } (x,y) \in \Omega_J \times \Omega_J.$$

Now, we have to check equation (3.33). In the case $\mu_J^x(E_0^x) = 0$,

$$\mu_J^x(E_+^x) = \mu_J^x(E_-^x) = \frac{1}{2},$$

and we conclude that

$$\begin{aligned}
&\int_{B_1(x)} J(x-y)g(x,y)\,dy \\
&\qquad = \int_{E_+^x} J(x-y)g(x,y)\,dy + \int_{E_-^x} J(x-y)g(x,y)\,dy + \int_{E_0^x} J(x-y)g(x,y)\,dy \\
&\qquad = \int_{E_+^x} J(x-y)\,dy - \int_{E_-^x} J(x-y)\,dy = \mu_J^x(E_+^x) - \mu_J^x(E_-^x) = \frac{1}{2} - \frac{1}{2} = 0.
\end{aligned}$$

In the case $\mu_J^x(E_0^x) > 0$,

$$\begin{aligned}
&\int_{B_1(x)} J(x-y)g(x,y)\,dy \\
&\qquad = \int_{E_+^x} J(x-y)g(x,y)\,dy + \int_{E_-^x} J(x-y)g(x,y)\,dy + \int_{E_0^x} J(x-y)g(x,y)\,dy \\
&\qquad = \int_{E_+^x} J(x-y)\,dy - \int_{E_-^x} J(x-y)\,dy + \frac{\mu_J^x(E_-^x) - \mu_J^x(E_+^x)}{\mu_J^x(E_0^x)} \int_{E_0^x} J(x-y)\,dy \\
&\qquad = \mu_J^x(E_+^x) - \mu_J^x(E_-^x) + (\mu_J^x(E_-^x) - \mu_J^x(E_+^x)) = 0.
\end{aligned}$$

This completes the proof. $\square$

3.3.2.3. *Characterization of variational solutions as minimizers of $\mathcal{J}_\psi$.*

PROOF OF THEOREM 3.14. Let u be a variational solution of problem (3.31). Then, there exists $g \in L^\infty(\Omega_J \times \Omega_J)$ antisymmetric, with $\|g\|_\infty \leq 1$ verifying (3.32) and (3.33).

Given $w \in L^1(\Omega)$, multiplying (3.33) by $w(x) - u(x)$, integrating, and having in mind (3.32) and the antisymmetry of g, we get

$$\begin{aligned}
0 &= -\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)dy(w_\psi(x) - u_\psi(x))dx \\
&= \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)[(w_\psi(y) - w_\psi(x)) - (u_\psi(y) - u_\psi(x))]dydx \\
&\leq \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|w_\psi(y) - w_\psi(x)|dydx - \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|u_\psi(y) - u_\psi(x)|dydx \\
&= \mathcal{J}_\psi(w) - \mathcal{J}_\psi(u).
\end{aligned}$$

Therefore, u is a minimizer of $\mathcal{J}_\psi$.

Assume now that u minimizes the functional $\mathcal{J}_\psi$. Theorem 3.16 shows the existence of a variational solution $\overline{u}$ of (3.31). Namely, there exists $g : \Omega_J \times \Omega_J \to \mathbb{R}$ such that $g \in L^\infty(\Omega_J \times \Omega_J)$, $\|g\|_\infty \leq 1$, $g(x,y) = -g(y,x)$ for (x,y) a.e in $\Omega_J \times \Omega_J$,

$$J(x-y)g(x,y) \in J(x-y)\text{sign}(\overline{u}_\psi(y) - \overline{u}_\psi(x)) \quad \text{a.e } (x,y) \in \Omega \times \Omega_J, \tag{3.40}$$

and

$$-\int_{\Omega_J} J(x-y)g(x,y)\,dy = 0 \quad \text{a.e } x \in \Omega. \tag{3.41}$$

Since u is a minimizer of $\mathcal{J}_\psi$, $\mathcal{J}_\psi(\overline{u}) - \mathcal{J}_\psi(u) = 0$.

On the other hand, arguing as in the other implication, we obtain that

$$0 = -\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)\,dy(\overline{u}_\psi(x) - u_\psi(x))dx$$
$$= \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)[(\overline{u}_\psi(y) - \overline{u}_\psi(x)) - (u_\psi(y) - u_\psi(x))]dydx$$
$$= \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|\overline{u}_\psi(y) - \overline{u}_\psi(x)|dydx - \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)(u_\psi(y) - u_\psi(x))dydx$$
$$= \mathcal{J}_\psi(\overline{u}) - \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)(u_\psi(y) - u_\psi(x))dydx.$$

Therefore,

$$\frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)(u_\psi(y)-u_\psi(x))dydx = \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|u_\psi(y)-u_\psi(x)|dydx.$$

Hence, $J(x-y)g(x,y) \in J(x-y)\text{sign}(u_\psi(y) - u_\psi(x))$ a.e $(x,y) \in \Omega_J \times \Omega_J$, which finishes the proof. $\square$

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