LARGE TIME BEHAVIOUR OF SOLUTIONS OF A SYSTEM OF PDE'S GOVERNING DIFFUSION PROCESSES IN A HETEROGENEOUS MEDIUM

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Introduction

We study the large time behaviour of solutions of the initial-boundary-value problem for a system of nonlinear partial differential equations of the form

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \Delta\phi_1(u_1) + \gamma(\phi_1(u_1) - \phi_2(u_2)) \ni 0 \quad \text{in} \ \ \Omega \times (0,\infty) \\ &\frac{\partial u_2}{\partial t} - \Delta\phi_2(u_2) - \gamma(\phi_1(u_1) - \phi_2(u_2)) \ni 0 \quad \text{in} \ \ \Omega \times (0,\infty) \\ &\qquad -\frac{\partial \phi_1(u_1)}{\partial \eta} \in \beta_1(u_1) \quad \text{on} \ \ \partial\Omega \times (0,\infty) \\ &\qquad -\frac{\partial \phi_2(u_2)}{\partial \eta} \in \beta_2(u_2) \quad \text{on} \ \ \partial\Omega \times (0,\infty) \\ &\qquad (u_1(0), u_2(0)) = (u_{01}, u_{02}) \quad \text{in} \ \ \Omega, \end{aligned} \tag{I}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\partial/\partial\eta$ denotes the Neumann boundary operator, β_i, ϕ_i and γ are maximal monotone graphs in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta_i(0), 0 \in \phi_i(0)$ and $0 \in \gamma(0)$. A particular case of system (I) is proposed by E. DiBenedetto and R. E. Showalter in [12] as a mathematical model for heat conduction in a composite material consisting of two components and under the assumption that the first component occurs in small isolated parts that are suspended in the second component, which implies the change of phase occurs in the second component. In this situation u_1 and u_2 represent the temperatures in the first and second components, respectively, $\phi_1 \equiv 0$ and $\phi_2(s) = bs + LH(s)$ where $b > 0, L > 0$ and H is the multivalued Heaviside step function. Based on the physical analysis in [12], one may still view (I) as a mathematical description of diffusion processes within a medium composed of two components which involves

Key words and phrases. m-T-accretive operator, heat conduction, nonlinear parabolic equations, asymptotic behaviour, diffusion in a heterogeneous medium..

This research has been partially supported by the Spanish DGICYT, Proyecto PB91-0331. The second author has been supported by a FPI grant from Spanish Ministerio de Educación y Ciencia.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

phase change. In this connection, γ is related to the surface area common to the two components. Thus, γ is a measure of the homogeneity of the material.

The special structure of the system (I) enables us to handle the problem via nonlinear semigroup theory. For the particular case of Dirichlet boundary condition, the study of the well-posedness of system (I), via nonlinear semigroup theory, is studied by X. Xu in [10] basing on the results on semilinear elliptic equations in L^1 due to H. Brezis and W. Strauss [10], and Ph. Bénilan, M. Crandall and P. Sacks [5]. We use here the same method, but we consider more general boundary conditions and more general phase changes. The boundary conditions in system (I) are very general. Different choices of β's give different boundary conditions. For instance, $\beta = \mathbb{R} \times \{0\}$ gives Neumann condition, $\beta = \{0\} \times \mathbb{R}$ gives Dirichlet condition and $\beta = \{0\}\times]-\infty, 0] \cup [0, +\infty[\times\{0\}$ gives the unilateral boundary conditions corresponding to variational inequalities introduced by J. L. Lions and G. Stampacchia in [14]. Also, different choices of ϕ's correspond to equations that arise in many applications. For instance, if $\phi(r) = |r|^m sign_0(r)$, we have: $m > 1$ is the porous medium equation, since it first arose in the study of gas flows in homogeneous porous media ([16], [2]); $m = 1$ is the classical equation of heat conduction, and $0 < m < 1$ is the so-called fast diffussion equation which occurs in the modelling of plasma ([7]).

The aim of this paper is to obtain stability results for the solutions of system (I). In [1] we have obtained stability results for the filtration equation using the Lyapunov method for semigroups of nonlinear contractions introduced by A. Pazy [19]. Here we will use the same method.

As it was said, our abstract framework is the theory of nonlinear semigroups. We refer the reader to [6], [11] and [3] for background material on nonlinear contraction semigroups.

The plan of the paper is as follows. The first section deals with the well-posedness of problem (I). We associate to system (I) an m-T-accretive operator. The mild-solution obtained via the Crandall-Liggett exponential formula for this operator will be the solution of system (I). In the second section we study the stability of solutions of system (I), showing that they stabilize as $t \to \infty$ by converging to a constant which is related with the boundary condition involved by β_i, the phase change γ and the diffusions ϕ_i.

1. Semigroup approach to systems of PDEs governing diffusion processes with phase change

From now on, Ω will be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. In this section we show that system (I) is well posed and is governed by an order-preserving contraction semigroup in $X := L^1(\Omega) \times L^1(\Omega)$, *i.e.*, we associate with system (I) an m-T-accretive operator in X. To do that, firstly we need the following definition given in [10].

Definition 1.1. *Let* $u \in W^{1,1}(\Omega)$, $v \in L^1(\Omega)$ *and* $w \in L^1(\partial\Omega)$. *We say that* u *is a weak solution of the Neumann problem*

$$-\Delta u = v, \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \eta} = w, \quad \text{on } \partial\Omega,$$

provided the following identity holds for all $f \in C^1(\overline{\Omega})$*:*

$$\int_\Omega \nabla u \cdot \nabla f = \int_\Omega v f + \int_{\partial\Omega} w f.$$

Now, we define the operator A associated with system (I) in X.

Definition 1.2. *A is the operator in X defined by: $((u_1, u_2), (v_1, v_2)) \in A$ if for $i = 1,\ 2$, there exist $h_i \in \phi_i(u_i)$, $w_i \in L^1(\partial\Omega)$, $z \in L^1(\Omega)$, such that h_1 is a weak-solution, in the sense of Definition 1.1, of*

$$-\Delta h_1 = v_1 - z \quad \text{in } \Omega,$$
$$\frac{\partial h_1}{\partial \eta} = w_1 \quad \text{on } \partial\Omega,$$

and h_2 is a weak-solution of

$$-\Delta h_2 = v_2 + z \quad \text{in } \Omega,$$
$$\frac{\partial h_2}{\partial \eta} = w_2 \quad \text{on } \partial\Omega,$$

with $-w_i \in \beta_i \circ \phi_i^{-1}(h_i)$, a.e. on $\partial\Omega$ and $z \in \gamma(h_1 - h_2)$, a.e. on Ω.

The definition above uses the fact that the trace of $h_i \in W^{1,1}(\Omega)$ on $\partial\Omega$ is well defined (Theorem 4.2 of [17]). Observe that we use the same notation h_i for h_i and its trace when convenient.

Many of the partial differential equations that can be studied by mean of Crandall-Liggett Theorem satisfy a *"comparison principle"*. This fact is equivalent to the order preserving property of the semigroup $(e^{-tA})_{t\geq 0}$ generated by A. The operators which generate order-preserving semigroups are the following:

Let E be a Banach lattice and let A be an operator in E. A is called *T-accretive* if

$$\|\big(x - \hat{x} + \lambda(y - \hat{y})\big)^+\| \geq \|(x - \hat{x})^+\|, \quad \text{for } \lambda \geq 0,\ y \in Ax,\ \hat{y} \in A\hat{x}.$$

It is clear that A is T-accretive if, and only if, its resolvents $J_\lambda := (I + \lambda A)^{-1}$ are T-contractions, *i.e.*,

$$\|(J_\lambda x - J_\lambda y)^+\| \leq \|(x - \hat{x})^+\|, \quad \text{for } \lambda \geq 0,\ x, y \in \mathcal{D}(J_\lambda).$$

Now, since every T-contraction is order-preserving, we have that if A is m-T-accretive then each e^{-tA} is order-preserving. In general, T-accretivity does not imply accretivity, but in some Banach spaces T-accretivity implies accretivity, this is the case for the spaces $L^p(\Omega)$ for $1 \leq p \leq \infty$.

The following notation wil be used whenever it is meaningful:

$$sign_0(s) = \begin{cases} 1 & \text{if } \ s > 0 \\ 0 & \text{if } \ s = 0 \\ -1 & \text{if } \ s < 0 \end{cases}$$

$$sign_0^+(s) = \begin{cases} 1 & \text{if } \ s > 0 \\ 0 & \text{if } \ s \leq 0 \end{cases}$$

Proposition 1.3. *If* γ *is a nondecreasing function with* $\gamma(0) = 0$, *then the operator* A *is T-accretive in X.*

Proof. Let $((u_1, u_2), (v_1, v_2)), ((\tilde{u}_1, \tilde{u}_2), (\tilde{v}_1, \tilde{v}_2)) \in A$ be. It is enough to prove that

$$\|[(u_1, u_2) - (\tilde{u}_1, \tilde{u}_2)]^+\| \leq \|[(f_1, f_2) - (\tilde{f}_1, \tilde{f}_2)]^+\|$$

where $f_i = u_i + v_i$ and $\tilde{f}_i = \tilde{u}_i + \tilde{v}_i$. By definition of A, there exist $h_i \in \phi_i(u_i)$, $w_i \in L^1(\partial\Omega)$, $-w_i \in \beta_i \circ \phi_i^{-1}(u_i)$, $z \in L^1(\Omega)$, $z \in \gamma(h_1 - h_2)$ and $\tilde{h}_i \in \phi_i(\tilde{u}_i)$, $\tilde{w}_i \in L^1(\partial\Omega)$, $-\tilde{w}_i \in \beta_i \circ \phi_i^{-1}(\tilde{u}_i)$, $\tilde{z} \in L^1(\Omega)$, $\tilde{z} \in \gamma(\tilde{h}_1 - \tilde{h}_2)$ such that, in weak sense

$$u_1 - \Delta h_1 + z = f_1, \quad \frac{\partial h_1}{\partial \eta} = w_1,$$

$$\tilde{u}_1 - \Delta \tilde{h}_1 + \tilde{z} = \tilde{f}_1, \quad \frac{\partial \tilde{h}_1}{\partial \eta} = \tilde{w}_1,$$

and consequently

$$u_1 - \tilde{u}_1 - \Delta(h_1 - \tilde{h}_1) + z - \tilde{z} = f_1 - \tilde{f}_1,$$

$$\frac{\partial (h_1 - \tilde{h}_1)}{\partial \eta} = w_1 - \tilde{w}_1.$$

Let $\sigma_1 \in L^\infty(\Omega)$ given by

$$\sigma_1(x) = sign_0^+(u_1(x) - \tilde{u}_1(x) + h_1(x) - \tilde{h}_1(x))$$

and $\tau_1(x) = sign_0^+(h_1(x) - \tilde{h}_1(x)), x \in \partial\Omega$. It follows from [5, Lemma D] that

$$\int_\Omega (u_1(x) - \tilde{u}_1(x))\sigma_1(x) + \int_\Omega (z(x) - \tilde{z}(x))\sigma_1(x) \leq$$

$$\leq \int_\Omega (f_1(x) - \tilde{f}_1(x))\sigma_1(x) + \int_{\partial\Omega} (w_1(x) - \tilde{w}_1(x))\tau_1(x).$$

Hence,

$$\int_\Omega (u_1(x) - \tilde{u}_1(x))^+ + \int_\Omega (z(x) - \tilde{z}(x))\sigma_1(x) \leq \int_\Omega (f_1(x) - \tilde{f}_1(x))^+.$$

Similarly

$$\int_\Omega (u_2(x) - \tilde{u}_2(x))^+ - \int_\Omega (z(x) - \tilde{z}(x))\sigma_2(x) \leq \int_\Omega (f_2(x) - \tilde{f}_2(x))^+.$$

Since $(z(x) - \tilde{z}(x))(\sigma_1(x) - \sigma_2(x)) \geq 0$ almost everywhere in Ω, adding the above expressions we conclude the proof.

Our next step is to proof the range condition for the closure $\overline{A}$ of the operator A.

Let ϕ and β be maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$ and $0 \in \phi(0)$. In order to study the filtration equation from the point of view of nonlinear semigroup theory, Ph. Bénilan [4] and Ph. Bénilan, M. G. Crandall and P. Sacks [5], define the following operator in $L^1(\Omega)$:

$A_{\phi,\beta} = \{(u,v) \in L^1(\Omega)\times L^1(\Omega) :$ there exists $h \in \phi(u)$ and there exists $w \in L^1(\partial\Omega) :$ h is a

weak solution of $-\Delta h = v$ in Ω, $\dfrac{\partial h}{\partial \eta} = w$ on $\partial\Omega$; and $-w(x) \in \beta\circ\phi^{-1}(h(x))$ a.e. on $\partial\Omega\}$.

Remark 1.4. We have the following relation between the resolvents of the operators A and $A_{\phi,\beta}$: $(u_1, u_2) = (I+\lambda A)^{-1}(v_1, v_2)$ if and only if $u_1 = (I+\lambda A_{\phi_1\beta_1})^{-1}(v_1-\lambda z)$ and $u_2 = (I+\lambda A_{\phi_2\beta_2})^{-1}(v_2+\lambda z)$. In fact, since $((u_1, u_2), \frac{1}{\lambda}((v_1, v_2) - (u_1, u_2))) \in A$, there exist $w_i \in \phi_i(u_i)$, $i = 1, 2$, $z \in \gamma(w_1 - w_2)$, $-\eta_i \in \beta_i \circ \phi_i^{-1}(w_i)$, such that (w_1, w_2) is weak solution of

$$-\Delta w_1 + z = \frac{v_1 - u_1}{\lambda}, \quad \frac{\partial w_1}{\partial \eta} = \eta_1,$$

$$-\Delta w_2 - z = \frac{v_2 - u_2}{\lambda}, \quad \frac{\partial w_2}{\partial \eta} = \eta_2,$$

from which it folows that $u_1 = (I+\lambda A_{\phi_1\beta_1})^{-1}(v_1-\lambda z)$ and $u_2 = (I+\lambda A_{\phi_2\beta_2})^{-1}(v_2+\lambda z)$.

Lemma 1.5. *Under the above general conditons, suppose that ϕ_i, ϕ_i^{-1}, β_i^{-1} and γ are Lipschitz continuous functions. Then, for any $f_1, f_2 \in L^2(\Omega)$, there exist $v_1, v_2 \in H^1(\Omega)$ weak solutions of*

$$\begin{aligned} -\Delta v_1 + \phi_1^{-1}(v_1) + \gamma(v_1 - v_2) = f_1 \quad &\text{in} \ \ \Omega, \\ -\frac{\partial v_1}{\partial \eta} = \beta_1 \circ \phi_1^{-1}(v_1) \quad &\text{on} \ \ \partial\Omega, \\ -\Delta v_2 + \phi_1^{-1}(v_2) - \gamma(v_1 - v_2) = f_2 \quad &\text{in} \ \ \Omega, \\ -\frac{\partial v_2}{\partial \eta} = \beta_2 \circ \phi_2^{-1}(v_2) \quad &\text{on} \ \ \partial\Omega. \end{aligned}$$

Proof. Let $V = H^1(\Omega) \times H^1(\Omega)$ and define $B : V \to V^*$ (being V^* the topological dual of V) by

$$\langle B(u_1,u_2),(v_1,v_2)\rangle = \int_\Omega \nabla u_1\cdot\nabla v_1 + \int_{\partial\Omega} \beta_1\circ\phi_1^{-1}(u_1)v_1 + \int_\Omega \phi_1^{-1}(u_1)v_1 + \int_\Omega \gamma(u_1-u_2)v_1 +$$

$$+\int_\Omega \nabla u_2 \cdot \nabla v_2 + \int_{\partial\Omega} \beta_2 \circ \phi_2^{-1}(u_2)v_2 + \int_\Omega \phi_2^{-1}(u_2)v_2 - \int_\Omega \gamma(u_1 - u_2)v_2,$$

for all (u_1,u_2), $(v_1,v_2) \in V$, where $(.,.)$ denotes the duality pairing between V and V^*.

By our assumptions, B is well-defined. An easy computation shows that

$$\langle B(u_1,u_2)-B(v_1,v_2),(u_1,u_2)-(v_1,v_2)\rangle \geq \Gamma(\|(u_1,u_2)-(v_1,v_2)\|_{H^1})\|(u_1,u_2)-(v_1,v_2)\|_{H^1}$$

where $\Gamma(s) = \min\{1, K_1^{-1}, K_2^{-1}\}s$, with K_i the Lipschitz constant of ϕ_i. Consequently, B is strongly monotone. On the other hand, since ϕ_i^{-1}, β_i and γ are Lispchitz continuous, it is easy to see that the mapping $\Upsilon(s) := \langle B(u_1, u_2) + s(v_1, v_2), (w_1, w_2)\rangle$ is continuous for fixed $(u_1, u_2), (v_1, v_2), (w_1, w_2) \in V$. Hence B is hemicontinuous. Then, by [19, Theorem 6. 10] (see also [13, Theorem 2.2.1]), given $(f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$, there exists $(v_1, v_2) \in H^1(\Omega) \times H^1(\Omega)$ such that $B(v_1, v_2) = (f_1, f_2)$, and from here we conclude the proof.

Suppose $u_i = \phi_i^{-1}(v_i)$, $i = 1,\ 2$, where v_i are given by the previous lemma. Then, we have that $(u_1,\ u_2) = (I + A)^{-1}(f_1,\ f_2)$, i.e., in the weak sense:

$$-\Delta\phi_1(u_1) + u_1 + \gamma(\phi_1(u_1) - \phi_2(u_2)) = f_1 \text{ in } \Omega, \tag{1}$$

$$-\Delta\phi_2(u_2) + u_2 - \gamma(\phi_1(u_1) - \phi_2(u_2)) = f_2 \text{ in } \Omega, \tag{2}$$

$$-\frac{\partial\phi_1(u_1)}{\partial\eta} = \beta_1(u_1) \quad \text{on} \quad \partial\Omega, \tag{3}$$

$$-\frac{\partial\phi_2(u_2)}{\partial\eta} = \beta_2(u_2) \quad \text{on} \quad \partial\Omega. \tag{4}$$

Now, we will proceed to derive estimates for u_1, u_2, that are independent of ϕ_i, β_i and γ. This enables us to relax the assumptions on ϕ_i, β_i and γ later on.

Lemma 1.6. *Under the assumptions of Lemma 1.5, if (u_1, u_2) is given by (1)-(4), then*

$$\|(u_1, u_2)\|_X \leq \|(f_1, f_2)\|_X.$$

Proof. Let $\sigma(x) = sign_0(u_1(x))$ in Ω and $\tau(x) = sign_0(\phi_1(u_1(x)))$ on $\partial\Omega$. Then by [5, Lemma D] applied to (1) and (3) we obtain

$$\int_\Omega |u_1| + \int_\Omega \gamma(\phi_1(u_1) - \phi_2(u_2)) sign_0(u_1) \leq \int_\Omega |f_1| + \int_{\partial\Omega} \frac{\partial\phi_1(u_1)}{\partial\eta} sign_0(\phi_1(u_1)) \leq \int_\Omega |f_1|.$$

Similarly,

$$\int_\Omega |u_2| - \int_\Omega \gamma(\phi_2(u_1) - \phi_2(u_2)) sign_0(u_2) \leq \int_\Omega |f_2|.$$

Hence, adding both expressions,

$$\|(u_1, u_2)\|_X + \int_\Omega G(x) \leq \|(f_1, f_2)\|_X,$$

with $G(x) = \gamma(\phi_2(u_1) - \phi_2(u_2))(sign_0(u_1) - sign_0(u_2)) \geq 0$ a.e. on Ω, and the proof is complete.

Lemma 1.7. *Let the situation of Lemma 1.6 hold. Then, for any $M > 0$,*

$$\int_\Omega [(|u_1| - M)^+ + (|u_2| - M)^+] \leq \int_\Omega [(|f_1| - M)^+ + (|f_2| - M)^+].$$

In particular, $\|(u_1, u_2)\|_\infty \leq \|(f_1, f_2)\|_\infty$, where $\|(v_1, v_2)\|_\infty = \sup(\|v_1\|_\infty, \|v_2\|_\infty)$.

Proof. Let $\epsilon > 0$ be and define

$$\eta_\epsilon(s) := \begin{cases} 1 & \text{if } s > \epsilon \\ \frac{1}{\epsilon}s & \text{if } |s| \le \epsilon \\ -1 & \text{if } s < -\epsilon. \end{cases}$$

By [15, Corollary A.5] $\eta^+_\epsilon(u_1 - M) \in H^1(\Omega)$. Then by (1) and (3), we have

$$\int_\Omega \nabla\phi_1(u_1)\cdot\nabla\eta^+_\epsilon(u_1 - M) + \int_{\partial\Omega}(-\frac{\partial\phi_1(u_1)}{\partial\eta})\eta^+_\epsilon(u_1 - M) +$$

$$+\int_\Omega u_1\eta^+_\epsilon(u_1 - M) + \int_\Omega \gamma(\phi_1(u_1) - \phi_2(u_2))\eta^+_\epsilon(u_1 - M) =$$

$$= \int_\Omega f_1\eta^+_\epsilon(u_1 - M).$$

Now, by [15, Corollary A.5],

$$\nabla\phi_1(u_1)\cdot\nabla\eta^+_\epsilon(u_1 - M) = \phi'(u_1)\nabla u_1\cdot(\eta^+_\epsilon)'(u_1 - M)\nabla u_1 \ge 0.$$

Moreover, since $-\frac{\partial\phi_1(u_1)}{\partial\eta} \in \beta_1(u_1)$, also $-\frac{\partial\phi_1(u_1)}{\partial\eta}\eta^+_\epsilon(u_1 - M) \ge 0$. Consequently, we have

$$\int_\Omega (u_1 - M)\eta^+_\epsilon(u_1 - M) + \int_\Omega \gamma(\phi_1(u_1) - \phi_2(u_2))\eta^+_\epsilon(u_1 - M) \le$$

$$\le \int_\Omega (f_1 - M)\eta^+_\epsilon(u_1 - M) \le \int_\Omega (f_1 - M)^+.$$

Similarly,

$$\int_\Omega (u_2 - M)\eta^+_\epsilon(u_2 - M) - \int_\Omega \gamma(\phi_1(u_1) - \phi_2(u_2))\eta^+_\epsilon(u_1 - M) \le \int_\Omega (f_2 - M)^+.$$

Adding both expressions, using again that

$$\gamma(\phi_1(u_1) - \phi_2(u_2))(sign^+_0(u_1 - M) - sign^+_0(u_2 - M)) \ge 0 \quad \text{a.e. on } \Omega,$$

and taking $\epsilon \to \infty$, we obtain

$$\int_\Omega (u_1 - M)^+ + \int_\Omega (u_2 - M)^+ \le \int_\Omega (f_1 - M)^+ + \int_\Omega (f_2 - M)^+.$$

A similar argument shows that

$$\int_\Omega (u_1 + M)^- + \int_\Omega (u_2 + M)^- \le \int_\Omega (f_1 + M)^- + \int_\Omega (f_2 + M)^-,$$

and the proof is complete.

Lemma 1.8. *Let the situation of Lemma 1.7 hold. Let* Ω' *be an open subset of* Ω *such that* $d(\Omega', \partial\Omega) = d > 0$. *Then*

$$\int_{\Omega'} (|u_1(x+y) - u_1(x)| + |u_2(x+y) - u_2(x)|)\psi(x) \leq$$

$$\leq \int_{\Omega'} (|\phi_1(u_1(x+y)) - \phi_1(u_1(x))| + |\phi_2(u_2(x+y)) - \phi_2(u_2(x))|)\Delta\psi(x) +$$

$$+ \int_{\Omega'} (|f_1(x+y) - f_1(x)| + |f_2(x+y) - f_2(x)|)\psi(x),$$

for all $\psi \in C_0^\infty(\Omega')$, $\psi \geq 0$, *and for all* $y \in \mathbb{R}^N$, $|y| < d$.

For the proof of this result we refer to [21, Lemma 4.4], taking in account the change due to the function γ.

Proposition 1.9. *Suppose* ϕ_i *and* β_i *are maximal monotone graphs in* $\mathbb{R}\times\mathbb{R}$, *with* $0 \in \phi_i(0)$, $0 \in \beta_i(0)$ *and* $D(\phi_i) = \mathbb{R}$. *Suppose* γ *is a nondecreasing function with* $D(\gamma) = \mathbb{R}$. *Then, the following holds:*

(i) $\quad L^\infty(\Omega) \times L^\infty(\Omega) \subset R(I + \lambda A)$ and $\|(I+\lambda A)^{-1}(f_1, f_2)\|_\infty \leq \|(f_1, f_2)\|_\infty$.

(ii)
$$\overline{D(A)} = L^1(\Omega) \times L^1(\Omega).$$

Proof. (i) It is enough to prove (i) for $\lambda = 1$. Let $(f_1, f_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$. Take $\epsilon > 0$ and, for $i = 1,\ 2$, let $\phi_{i\epsilon}$, $\beta_{i\epsilon}$ and γ_ϵ be the Yosida approximations of ϕ_i, β_i and γ, respectively; that is

$$\phi_{i\epsilon} = (I - (I + \epsilon\phi_i)^{-1}/\epsilon, \quad \beta_{i\epsilon} = (I - (I + \epsilon\beta_i)^{-1}/\epsilon, \quad \gamma_\epsilon = (I - (I + \epsilon\gamma)^{-1}/\epsilon.$$

For $i = 1,\ 2$, define $\tilde\phi_{i\epsilon} = \phi_{i\epsilon} + \epsilon s$, then $\tilde\phi_{i\epsilon}$ and $\tilde\phi_{i\epsilon}^{-1}$ are Lipschitz continuous. By Lemma 1.5, there exists $(u_{1\epsilon},\ u_{2\epsilon})$, a weak solution of

$$-\Delta\tilde\phi_{1\epsilon}(u_{1\epsilon}) + u_{1\epsilon} + \gamma_\epsilon(\tilde\phi_{1\epsilon}(u_{1\epsilon}) - \tilde\phi_{2\epsilon}(u_{2\epsilon})) = f_1 \quad \text{in } \Omega,$$

$$-\Delta\tilde\phi_{2\epsilon}(u_{2\epsilon}) + u_{2\epsilon} - \gamma_\epsilon(\tilde\phi_{1\epsilon}(u_{1\epsilon}) - \tilde\phi_{2\epsilon}(u_{2\epsilon})) = f_2 \quad \text{in } \Omega,$$

$$-\frac{\partial\tilde\phi_{1\epsilon}(u_{1\epsilon})}{\partial\eta} = \beta_{1\epsilon}(u_{1\epsilon}) \quad \text{on } \partial\Omega,$$

$$-\frac{\partial\tilde\phi_{2\epsilon}(u_{2\epsilon})}{\partial\eta} = \beta_{2\epsilon}(u_{2\epsilon}) \quad \text{on } \partial\Omega.$$

Now, by Lemma 1.7

$$\|(u_{1\epsilon},\ u_{2\epsilon})\|_\infty \leq 2\mu(\Omega)^{1/2}\|(f_1, f_2)\|_\infty. \tag{5}$$

Moreover, since $\tilde{\phi}_{i\epsilon}(s) \to \phi_i^\circ(s)$ ($\phi_i^\circ(s)$ denoting the element of $\phi_i(s)$ of minimun norm) we have that

$$\|\tilde{\phi}_{1\epsilon}(u_{1\epsilon})\|_\infty + \|\tilde{\phi}_{2\epsilon}(u_{2\epsilon})\|_\infty \leq C, \tag{6}$$

with C a constant only depending on Ω and $\phi_i^\circ(\pm\|(f_1, f_2)\|_\infty)$, for ϵ small enough.

Also, since $\gamma_\epsilon(s) \to \gamma(s)$, we have that

$$\|\gamma_\epsilon(\tilde{\phi}_{1\epsilon}(u_{1\epsilon}) - \tilde{\phi}_{2\epsilon}(u_{2\epsilon}))\|_\infty \leq K, \tag{7}$$

with K a constant only depending on Ω and $\gamma(\pm(\phi_i^\circ(\pm\|(f_1, f_2)\|_\infty))$, for ϵ small enough.

Now, by Remark 1.4, $u_{1\epsilon} = (I + A_{\phi_{1\epsilon}\beta_{1\epsilon}})^{-1}(f_1 - z_\epsilon)$ and $u_{2\epsilon} = (I + A_{\phi_{2\epsilon}\beta_{2\epsilon}})^{-1}(f_2 + z_\epsilon)$. Then, by [8, Theorem 12], $\phi_i(u_{i\epsilon}) \in H^2(\Omega)$ and there exist constants K_1 and K_2 such that

$$\|\tilde{\phi}_{1\epsilon}(u_{1\epsilon})\|_{H^2} + \|\tilde{\phi}_{2\epsilon}(u_{2\epsilon})\|_{H^2} \leq$$

$$\leq K_1\{\|f_1 + \tilde{\phi}_{1\epsilon}(u_{1\epsilon}) + u_{1\epsilon} - \gamma_\epsilon(\tilde{\phi}_{1\epsilon}(u_{1\epsilon}) - \tilde{\phi}_{2\epsilon}(u_{2\epsilon}))\|_2 +$$

$$+ \|f_2 + \tilde{\phi}_{2\epsilon}(u_{2\epsilon}) + u_{2\epsilon} + \gamma_\epsilon(\tilde{\phi}_{1\epsilon}(u_{1\epsilon}) - \tilde{\phi}_{2\epsilon}(u_{2\epsilon}))\|_2\} + K_2,$$

Hence, by (5),(6) and (7), $\{(\tilde{\phi}_{1\epsilon}(u_{1\epsilon}),\ \tilde{\phi}_{2\epsilon}(u_{2\epsilon}))\}_{\epsilon>0}$ is bounded in $H^2(\Omega) \times H^2(\Omega)$.

On the other hand, proceeding as in the proof of [21, Theorem 2.4], we have that $\{(u_{1\epsilon},\ u_{2\epsilon})\}_{\epsilon>0}$ is precompact in X. Moreover, by (7), $\{\gamma_\epsilon(\tilde{\phi}_{1\epsilon}(u_{1\epsilon}) - \tilde{\phi}_{2\epsilon}(u_{2\epsilon}))\}_{\epsilon>0}$ is weakly sequentially compact in $L^1(\Omega)$. Then, there exists $\epsilon_n \to 0$, such that

$$(u_{1\epsilon_n},\ u_{2\epsilon_n}) \to (u_1,\ u_2) \quad \text{strongly in } X,$$

$$\gamma_{\epsilon_n}(\tilde{\phi}_{1\epsilon_n}(u_{1\epsilon_n}) - \tilde{\phi}_{2\epsilon_n}(u_{2\epsilon_n})) \to z \quad \text{weakly in } L^1(\Omega),$$

$$(\tilde{\phi}_{1\epsilon_n}(u_{1\epsilon_n}),\ \tilde{\phi}_{2\epsilon_n}(u_{2\epsilon_n})) \to (w_1,\ w_2) \quad \text{strongly in } X,$$

$$(\tilde{\phi}_{1\epsilon_n}(u_{1\epsilon_n}),\ \tilde{\phi}_{2\epsilon_n}(u_{2\epsilon_n})) \to (w_1,\ w_2) \quad \text{strongly in } L^2(\partial\Omega) \times L^2(\partial\Omega),$$

$$\left(\frac{\partial\tilde{\phi}_{1\epsilon_n}(u_{1\epsilon_n})}{\partial\eta},\ \frac{\partial\tilde{\phi}_{2\epsilon_n}(u_{2\epsilon_n})}{\partial\eta}\right) \to \left(\frac{\partial w_1}{\partial\eta},\ \frac{\partial w_2}{\partial\eta}\right) \quad \text{strongly in } L^2(\partial\Omega) \times L^2(\partial\Omega).$$

Consequently, by [5, Lemma G], $w_1 \in \phi_1(u_1)$, $w_2 \in \phi_2(u_2)$, $z \in \gamma(w_1 - w_2)$, $-\frac{\partial w_1}{\partial\eta} \in \beta_1 \circ \phi_1^{-1}(w_1)$, $-\frac{\partial w_2}{\partial\eta} \in \beta_2 \circ \phi_2^{-1}(w_2)$. Therefore, in the weak sense

$$-\Delta w_1 + u_1 + z = f_1 \quad \text{in } \Omega,$$

$$-\Delta w_2 + u_2 - z = f_2 \quad \text{in } \Omega,$$

$$-\frac{\partial w_1}{\partial\eta} \in \beta_1(u_1) \quad \text{on } \partial\Omega,$$

$$-\frac{\partial w_2}{\partial\eta} \in \beta_2(u_2) \quad \text{on } \partial\Omega.$$

From where it follows that $L^\infty(\Omega)\times L^\infty(\Omega) \subset R(I+A)$. Also, since $\|(u_{1\epsilon},\ u_{2\epsilon})\|_\infty \leq \|(f_1,\ f_2)\|_\infty$, we have that $\|(u_1,\ u_2)\|_\infty \leq \|(f_1,\ f_2)\|_\infty$, and the proof of (i) is complete.

(ii) Let $(v_1,\ v_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$. Then by (i), $(I+\lambda A)^{-1}(v_1,\ v_2) = (u_1,\ u_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$. Now, $u_1 = (I+\lambda A_{\phi_1\beta_1})^{-1}(v_1 - \lambda z)$ and $u_2 = (I+\lambda A_{\phi_2\beta_2})^{-1}(v_2 + \lambda z)$, with $z- \in \gamma(w_1 - w_2)$, $w_i \in \phi_i(u_i)$, $i = 1,\ 2$. Hence, $(I+\lambda A_{\phi_1\beta_1})^{-1}(v_1) \to v_1$ (see [5, Teorema B']) and $v_1 - \lambda z \to v_1$. Therefore $(I+\lambda A_{\phi_1\beta_1})^{-1}(v_1 - \lambda z) \to v_1$, and similarly $(I+\lambda A_{\phi_2\beta_2})^{-1}(v_2 + \lambda z) \to v_2$. From here, since $(I+\lambda A)^{-1}(v_1,\ v_2) \in D(A)$, it follows that $(v_1,\ v_2) \in \overline{D(A)}$.

Now we can stablish the main result of this section.

Theorem 1.10. *Suppose ϕ_i and β_i are maximal monotone graphs in $\mathbb{R}\times\mathbb{R}$, with $0 \in \phi_i(0)$, $0 \in \beta_i(0)$ and $D(\phi_i) = \mathbb{R}$. Suppose γ is a nondecreasing function with $D(\gamma) = \mathbb{R}$. Then the operator $\overline{A}$ is m-T-accretive in X and*

$$\overline{D(\overline{A})} = L^1(\Omega) \times L^1(\Omega).$$

Proof. Since the closure of a T-accretive operator is T-accretive, by the above proposition $\overline{A}$ is T-accretive in X. On the other hand, by the above proposition we have

$$X = \overline{L^\infty(\Omega) \times L^\infty(\Omega)} \subset \overline{R(I+\lambda A)} = R(I+\lambda\overline{A}).$$

Consequently, $\overline{A}$ is m-T-accretive in X.

As consequence of the Crandall-Liggett Theorem and the above theorem we have that for every initial data $\mathbf{u}_0 = (u_{01}, u_{02}) \in X$ the system (I) has a mild-solution given by

$$\mathbf{u}(x,t) = S(t)\mathbf{u}_0,$$

being $\big(S(t)\big)_{t\geq 0}$ the order-preserving contraction semigroup generated by $\overline{A}$.

2. The Stabilization Results

In this section we stablish that the mild-solutions of system (I) stabilize as $t \to \infty$ by converging to a constant function. We use the Lyapunov method for semigroups of nonlinear contractions introduced by A. Pazy [19].

In all this section we assume that we are under the assumptions of Theorem 1.10, $\big(S(t)\big)_{t\geq 0}$ is the order-preserving contraction semigroup generated by $\overline{A}$ and J_λ is the resolvent of the operator $\overline{A}$.

In order to prove the stabilization theorem we need the orbits to be relatively compact. This result is obtained using the decomposition of Remark 1.4 and the precompactness result obtained in [1] for the Filtration Equation.

Theorem 2.1. *Suppose $\phi_i : \mathbb{R} \to \mathbb{R}$ are increasing continuous functions for $i = 1, 2$. Then,*

(i) If $B \times B' \subset L^\infty(\Omega)\times L^\infty(\Omega)$ is bounded, $J_\lambda(B \times B')$ is a relatively compact subset of X.

(ii) For every $\mathbf{u}_0 = (u_{01}, u_{02}) \in X$ the orbit $\gamma\big(\mathbf{u}_0\big) = \{S(t)\mathbf{u}_0 \ :\ t \geq 0\}$ is a relatively compact subset of X.

Proof. (i) Let $B \times B' \subset L^\infty(\Omega) \times L^\infty(\Omega)$ be bounded. Take $(v_{1n},\ v_{2n}) \in B \times B'$, $n = 1,\ 2,\ \ldots,$ and $(u_{1n},\ u_{2n}) = J_\lambda(v_{1n},\ v_{2n})$. By Remark 1.4, $u_{1n} = (I + \lambda A_{\phi_1\beta_1})^{-1}(v_{1n} - \lambda z_n)$ and $u_{2n} = (I + \lambda A_{\phi_2\beta_2})^{-1}(v_{2n} + \lambda z_n)$, with $z_n \in \gamma(\phi_1(u_{1n}) - \phi_2(u_{2n}))$. Now, $\{v_{in} \pm \lambda z_n\}$ are bounded sequences in $L^\infty(\Omega)$. Hence, by [1, Theorem 2.6], there exists a subsequence of $\{(u_{1n},\ u_{2n})\}$ which converges in X.

(ii) First consider $\mathbf{u}_0 \in \mathcal{D}(\overline{A}) \cap \big(L^\infty(\Omega) \times L^\infty(\Omega)\big)$. Then, since

$$\|S(t)\mathbf{u}_0\|_\infty \leq \|\mathbf{u}_0\|_\infty \quad \text{for all} \ \ t \geq 0,$$

as consequence of (i), we have that $J_\lambda(\gamma(\mathbf{u}_0))$ is a relatively compact subset of X for all $\lambda > 0$. Moreover,

$$\|S(t)\mathbf{u}_0 - J_\lambda S(t)\mathbf{u}_0\|_1 \leq \lambda \inf\{\|\mathbf{v}\|_1 \ : \ \mathbf{v} \in \overline{A}\mathbf{u}_0\}.$$

Hence, $\gamma(\mathbf{u}_0)$ is relatively compact in X.

On the other hand, it is easy to see that $\mathcal{D}(\overline{A}) \cap \big(L^\infty(\Omega) \times L^\infty(\Omega)\big)$ is dense in X. Thus, given $\mathbf{u}_0 \in X$ and $\epsilon > 0$, there exists $\mathbf{v}_0 \in \mathcal{D}(\overline{A}) \cap \big(L^\infty(\Omega) \times L^\infty(\Omega)\big)$ such that $\|\mathbf{u}_0 - \mathbf{v}_0\|_1 < \epsilon$. So we have,

$$\sup_{t\geq 0} \inf_{s \geq 0} \|S(t)\mathbf{u}_0 - S(s)\mathbf{v}_0\|_1 \leq \sup_{t \geq 0} \|S(t)\mathbf{u}_0 - S(t)\mathbf{v}_0\|_1 \leq \|\mathbf{u}_0 - \mathbf{v}_0\|_X < \epsilon.$$

From where it follows that $\gamma(\mathbf{u}_0)$ is relatively compact in X.

Now we come to the main result.

Theorem 2.2. *Let β_i be maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta_i(0)$ and $\phi_i : \mathbb{R} \to \mathbb{R}$ increasing continuous functions with $\phi_i(0) = 0$. Suppose also that $\gamma : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function. Let $(u_{01}, u_{02}) \in X$. Then, if $\mathbf{u}(x,t)$ is the mild-solution of system (I), there exists constants K_i, $K_i \in \beta_i^{-1}\{0\}$ with $\gamma(\phi_1(K_1) - \phi_2(K_2)) = 0$, such that*

$$\|\mathbf{u}(.,t) - (K_1,\ K_2)\|_X \to 0 \ \text{ as } t \to \infty.$$

Proof. Let $\mathcal{V} : X \to [0, +\infty]$ defined by

$$\mathcal{V}(u_1, u_2) = \begin{cases} \int_\Omega j_1(u_1) + \int_\Omega j_2(u_2), & \text{if } j_i(u_i) \in L^1(\Omega),\ i = 1,2 \\ \\ +\infty, & \text{if } j_i(u_i) \notin L^1(\Omega),\ i = 1 \ \text{ or } \ 2 \end{cases}$$

being $\partial j_i = \phi_i$. Since ϕ_i is increasing, it is easy to see that j_i is continuous and convex. Hence $\mathcal{V}$ is lower semicontinuous (see [9, p. 160]).

Let $\mathcal{W} : X \to [0, +\infty]$ defined by

$$\frac{1}{2}\int_\Omega \big(\nabla\phi_1(u_1)\big)^2 + \int_{\partial\Omega} \rho_1 \circ \phi_1(u_1) + \frac{1}{2}\int_\Omega \big(\nabla\phi_2(u_2)\big)^2 + \int_{\partial\Omega} \rho_2 \circ \phi_2(u_2) +$$

$$+\int_\Omega \gamma(\phi_1(u_1)-\phi_2(u_2))(\phi_1(u_1)-\phi_2(u_2)),$$

when the integrals are finite, and $+\infty$, when they are not, being $\partial\rho_i = \beta_i \circ \phi_i^{-1}$.

Let $\mathbf{u} = (u_1,\ u_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$ be. By Proposition 1.9, $(v_1,\ v_2) = J_\lambda \mathbf{u} \in L^\infty(\Omega) \times L^\infty(\Omega)$. Moreover, by Remark 1.4, $v_1 = (I + \lambda A_{\phi_1\beta_1})^{-1}(u_1 - \lambda z)$ and $v_2 = (I + \lambda A_{\phi_2\beta_2})^{-1}(u_2 + \lambda z)$, with $z = \gamma(\phi_1(u_1) - \phi_2(u_2)) \in L^\infty(\Omega)$.

Since $u_i \in L^\infty(\Omega)$, $(u_1, u_2) \in \mathcal{D}(\mathcal{V})$, and by [8, Theorem 12] it follows that $J_\lambda \mathbf{u} \in \mathcal{D}(\mathcal{V}) \cap \mathcal{D}(\mathcal{W})$. In the next step we prove that

$$\mathcal{V}(J_\lambda \mathbf{u}) + \lambda \mathcal{W}(J_\lambda \mathbf{u}) - \mathcal{V}(\mathbf{u}) \le 0. \tag{8}$$

Since ϕ_i is continuous and increasing, it is easy to see that

$$\frac{1}{\lambda}\big(\mathcal{V}(J_\lambda \mathbf{u}) - \mathcal{V}(\mathbf{u})\big) \le \int_\Omega \frac{1}{\lambda}((I+\lambda A_{\phi_1\beta_1})^{-1}(u_1 - \lambda z) - u_1)\ \phi_1((I+\lambda A_{\phi_1\beta_1})^{-1}(u_1 - \lambda z)) +$$

$$+\int_\Omega \frac{1}{\lambda}((I+\lambda A_{\phi_2\beta_2})^{-1}(u_2 + \lambda z) - u_2)\ \phi_2((I+\lambda A_{\phi_2\beta_2})^{-1}(u_1 + \lambda z)) =$$

$$= \int_\Omega \frac{1}{\lambda}((I+\lambda A_{\phi_1\beta_1})^{-1}(u_1 - \lambda z) - (u_1 - \lambda z))\ \phi_1((I+\lambda A_{\phi_1\beta_1})^{-1}(u_1 - \lambda z)) +$$

$$+\int_\Omega \frac{1}{\lambda}((I+\lambda A_{\phi_2\beta_2})^{-1}(u_2 + \lambda z) - (u_2 + \lambda z))\ \phi_2((I+\lambda A_{\phi_2\beta_2})^{-1}(u_1 + \lambda z)) -$$

$$-\int_\Omega z(\phi_1((I+\lambda A_{\phi_1\beta_1})^{-1}(u_1 - \lambda z)) - \phi_2((I+\lambda A_{\phi_2\beta_2})^{-1}(u_1 + \lambda z))).$$

Now, arguing as in [1, Theorem 3.3], we obtain that

$$\frac{1}{\lambda}\big(\mathcal{V}(J_\lambda \mathbf{u}) - \mathcal{V}(\mathbf{u})\big) \le -\mathcal{W}(J_\lambda \mathbf{u}),$$

and (8) follows.

Replacing $\mathbf{u}$ by $J_\lambda^{k-1}\mathbf{u}$ in (8) we find

$$\mathcal{V}(J_\lambda^k \mathbf{u}) + \lambda \mathcal{W}(J_\lambda^k \mathbf{u}) - \mathcal{V}\big(J_\lambda^{k-1}\mathbf{u}\big) \le 0.$$

Summing these inequalities from $k=1$ to $k=n$ and choosing $\lambda = t/n$, it yields

$$\mathcal{V}(J_{\frac{t}{n}}^n \mathbf{u}) + \sum_{k=1}^n \frac{t}{n}\mathcal{W}(J_{\frac{t}{n}}^k \mathbf{u}) - \mathcal{V}(\mathbf{u}) \le 0. \tag{9}$$

Next we define $F_n(\tau) = \mathcal{W}(J_{\frac{t}{n}}^k \mathbf{u})$ for $(k-1)t/n < \tau \le kt/n$. Then

$$\sum_{k=1}^n \frac{t}{n}\mathcal{W}(J_{\frac{t}{n}}^k \mathbf{u}) = \int_0^t F_n(\tau)\ d\tau.$$

On the other hand, by the Crandall-Liggett Theorem

$$\lim_{n\to\infty} J_{\frac{t}{n}}^k \mathbf{u} = S(\tau)\mathbf{u} \quad \text{in } X.$$

Now, arguing as in Proposition 1.9 we have that $\{\phi_i((J^k_{\frac{t}{n}}\mathbf{u})_i)\}_{n=1}^{\infty}$ are bounded sequences in $H^2(\Omega)$. Hence there exists subsequences, which we denote again as before, such that

$$\phi_i((J^k_{\frac{t}{n}}\mathbf{u})_i) \to \phi_i((S(\tau)\mathbf{u})_i, \quad \text{in } L^2(\Omega) \text{ for } i=1,2,$$

$$\phi_i((J^k_{\frac{t}{n}}\mathbf{u})_i) \to \phi_i((S(\tau)\mathbf{u})_i) \quad \text{in } L^2(\partial\Omega) \text{ for } i=1,2.$$

Consider the functionals

$$\mathcal{U}_0(v) = \begin{cases} \frac{1}{2}\int_\Omega \big(\nabla(v)\big)^2, & \text{if } \big(\nabla(v)\big)^2 \in L^1(\Omega) \\ \\ +\infty, & \text{if } \big(\nabla(v)\big)^2 \notin L^1(\Omega) \end{cases}$$

$$\mathcal{U}_i(v) = \begin{cases} \int_{\partial\Omega} \rho_i(v), & \text{if } \rho_i(v) \in L^1(\partial\Omega) \\ \\ +\infty, & \text{if } \rho_i(v) \notin L^1(\partial\Omega), \end{cases}$$

and

$$\mathcal{U}_3(u_1,u_2) = \begin{cases} \int_\Omega \gamma(\phi_1(u_1)-\phi_2(u_2)), & \text{if } \gamma(\phi_1(u_1)-\phi_2(u_2)) \in L^1(\partial\Omega) \\ \\ +\infty, & \text{if } \gamma(\phi_1(u_1)-\phi_2(u_2)) \notin L^1(\partial\Omega), \end{cases}$$

in $L^2(\Omega)$, $L^1(\partial\Omega)$ and $L^1(\Omega)\times L^1(\Omega)$, respectively. It is easy to see that these functional are lower semicontinuous; whence

$$\mathcal{W}\big(S(\tau)\mathbf{u}\big) \le \liminf_{n\to\infty} \mathcal{W}\big((J^k_{\frac{t}{n}}\mathbf{u})\big) = \liminf_{n\to\infty} F_n(\tau).$$

Now, by Fatou's Lemma

$$\int_0^t \mathcal{W}\big(S(\tau)\mathbf{u}\big)\, d\tau \le \int_0^t \liminf_{n\to\infty} F_n(\tau)\, d\tau \le \liminf_{n\to\infty} \int_0^t F_n(\tau)\, d\tau,$$

that is,

$$\int_0^t \mathcal{W}\big(S(\tau)\mathbf{u}\big)\, d\tau \le \liminf_{n\to\infty} \sum_{k=1}^n \frac{t}{n} \mathcal{W}(J^k_{\frac{t}{n}}\mathbf{u}). \tag{10}$$

Hence, taking limits as $n\to\infty$ in (9) we obtain that

$$\mathcal{V}\big(S(t)\mathbf{u}\big) + \int_0^t \mathcal{W}\big(S(\tau)\mathbf{u}\big)\, d\tau - \mathcal{V}(\mathbf{u}) \le 0,$$

from where it follows that

$$\int_0^\infty \mathcal{W}\big(S(\tau)\mathbf{u}\big)\, d\tau \le \mathcal{V}(\mathbf{u}).$$

Thus, there exists a sequence $t_n \to \infty$, such that $\mathcal{W}\big(S(t_n)\mathbf{u}\big) \to 0$ when $n \to \infty$. Now, by Theorem 2.1 there exists a subsequence $\{t_{n_k}\}$ such that

$$\lim_{k\to\infty} S(t_{n_k})\mathbf{u} = (v_1,\ v_2).$$

Since $\mathcal{W}\big(S(t_{n_k})\mathbf{u}\big) \to 0$, we have that $\{\phi_i((S(t_{n_k})\mathbf{u})_i)\}_{k=1}^{\infty}$ are bounded sequences in $H^2(\Omega)$. Hence, reasoning as before, it follows that

$$\mathcal{W}(v_1,\ v_2) \leq \liminf_{n\to\infty} \mathcal{W}\big(S(t_{n_k})\mathbf{u}\big) = 0,$$

and consequently, $\mathcal{W}(v_1,\ v_2) = 0$.

Therefore, each v_i is a constant such that $v_i \in \beta_i^{-1}(0)$ and $\gamma(\phi_1(v_1) - \phi_2(v_2)) = 0$. From where it is easy to see that $(v_1,\ v_2)$ is an equilibrium.

Now, since $L^\infty(\Omega) \times L^\infty(\Omega)$ is dense in $\overline{D(A)} = X$ and each $S(t)$ is a T-contraction, from the above we obtain easily the conclusion in the general case $\mathbf{u} \in X$.

Remark 2.3. For some particular boundary conditions we can be more precise about the constants K_1, K_2 of the above theorem. In fact: when $\beta_i = \mathbb{R} \times \{0\}$, for $i = 1, 2$, then $K_1 = K_2 = 0$, i.e., with Dirichlet boundary conditions the solutions of system (I) stabilizes to $(0,\ 0)$. On the other hand, if we have Neumann boundary conditions in both diffussion proceses, that is, $\beta_i = \{0\} \times \mathbb{R}$, for $i = 1, 2$, we have the following result.

Theorem 2.4. *Suppose we are under the assumptions of Theorem 2.2, and β_i correspond to Neumann boundary conditions, i.e., $\beta_i = \mathbb{R} \times \{0\}$. Then, for every initial data $(u_0, v_0) \in X$, if $\mathbf{u}(x,t)$ is the mild-solution of system (I) we have*

$$\lim_{t\to\infty} \|\mathbf{u}(.,t) - (K_1, K_2)\|_X = 0,$$

where K_1 and K_2 are constants satisfying

$$K_1 + K_2 = \frac{1}{\mu(\Omega)}\{\int_\Omega u_0 + \int_\Omega v_0\}.$$

Proof. By a density argument, we can assume that $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Omega)$. For $\lambda > 0$, define (u_i, v_i) by $(u_0, v_0) = (u_0, v_0)$, $(u_{i+1}, v_{i+1}) = J_\lambda(u_i, v_i)$, $i = 1, 2, \ldots$. Letting $\mathbf{u}_\lambda(t) = (u_i, v_i)$ for $i\lambda \leq t < (i+1)\lambda$, we have by the Crandall-Liggett Theorem that

$$\mathbf{u}(.,t) = \lim_{\lambda\downarrow 0} \mathbf{u}_\lambda(t) \quad \text{in} \quad X,$$

and the limit is uniform for t in compact subsets of $[0, \infty[$. Now, by Remark 1.4,

$$u_{i+1} = (I + \lambda A_{\phi_1,\beta_1})^{-1}(u_i - \lambda z_i)$$

$$v_{i+1} = (I + \lambda A_{\phi_2,\beta_2})^{-1}(v_i + \lambda z_i),$$

with $z_i = \gamma\big(\phi_1(u_i) - \phi_2(v_i)\big)$. From here, taking $f \equiv 1$ in the definition of A_{ϕ_i,β_i}, it follows that

$$\int_\Omega u_{i+1} = \int_\Omega u_i - \lambda \int_\Omega z_i$$

and

$$\int_\Omega v_{i+1} = \int_\Omega v_i + \lambda \int_\Omega z_i.$$

Hence,

$$\int_\Omega u_{i+1} + \int_\Omega v_{i+1} = \int_\Omega u_i + \int_\Omega v_i.$$

Consequently,

$$\int_\Omega \big(\mathbf{u}(.,t)\big)_1 + \int_\Omega \big(\mathbf{u}(.,t)\big)_2 = \lim_{\lambda\downarrow 0} \int_\Omega \big(\mathbf{u}_\lambda(t)\big)_1 + \int_\Omega \big(\mathbf{u}_\lambda(t)\big)_2 = \int_\Omega u_0 + \int_\Omega v_0.$$

Applying Theorem 2.2, we conclude the proof.

Acknowledgement. We want to express our thanks to Prof. F. Andreu for many stimulating talks.

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