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STABILIZATION OF SOLUTIONS OF THE FILTRATION EQUATION IN $\mathbb{R}^N$

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ABSTRACT. We prove that mild-solutions of the degenerate parabolic equation $u_t - \Delta\varphi(u) = 0$ defined in $\mathbb{R}^N \times (0,\infty)$, stabilize as $t \to \infty$ by converging to an equilibrium in an appropriated weighted L^1-space. Here φ is a nondecreasing continuous function from $\mathbb{R}$ into $\mathbb{R}$, or more generally, a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $N \geq 3$. It is also proved that the mild-solution $u(t)$ has compact support for every t if the initial datum has compact support. Previous stabilization results for this equation required more conditions on φ. Our approach uses methods developed by Ph. Bénilan and M.G. Crandall in non-linear semigroup theory.

Introduction.

We study the long-time behaviour of solutions of the degenerate parabolic equation

$$(I) \qquad \begin{cases} u_t - \Delta\varphi(u) = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

where φ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \varphi(0)$.

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Equation (I), usually called the filtration equation, is very general. Different choices of φ's arise in applications. One of the more important cases is $\varphi(r) = |r|^m sign(r)$, i.e., the equation

$$(II) \qquad \begin{cases} u_t - \Delta(|u|^m signu) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

There is an extensive literature dealing with equation (II); see *e.g.*, the expository papers of D. G. Aronson [1], L. Peletier [21] and J. L. Vazquez [23]. The case $0 < m < 1$ corresponds to a "fast diffusion process"; equations of this type appear in plasma problems ([10]); $m = 1$ is the classical equation of heat conduction and for $m > 1$ the equation is called the porous medium equation since it first arose in the study of gas flows in homogeneous porous media ([20]).

For exponents $m > (N-2)^+/N$, there exists a family of special solutions of equation (II) which plays the role of the fundamental solution of the heat equation. Such special solutions were described by G. I. Barenblatt [2] with the name of *source-type solutions*, and can be characterized as the unique weak solutions of the corresponding equation with initial data a Dirac mass,

$$u_0(x) = M\delta(x), \quad M > 0.$$

Various authors (see, S. Kamin [18], A. Friedmann and S. Kamin [16] and S. Kamin and J. L. Vazquez [19]) have used the class of source-type solutions to describe the large time behaviour of equation (II) when $m > (N-2)^+/N$. When $0 < m \leq (N-2)^+/N$ it is not possible to use these source-type solutions of Barenblatt since in this case no solution of equation (II) exists for $u_0 = \delta$ (see [9]). Now, Ph. Bénilan and M. G. Crandall showed in [6, Proposition 10] that when $0 < m < (N-2)^+/N$ $(N > 2)$ there is extinction in finite time for initial data in $L^p(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ with $p = N(1-m)/2$. Consequently, the asymptotic behaviour of the solutions of equation (II) is known when the exponent m is not the critical exponent $m_{crit} = (N-2)^+/N$.

The uniform bounds given by Ph. Bénilan and J. Berger in [4] provide information on asymptotic behaviour of solutions of problem (I) (see also [3]).

1. Preliminaries.

In this section we will define some of our notation. A weight in $\mathbb{R}^N$ is a positive continuous function $p : \mathbb{R}^N \to \mathbb{R}$ such that $\lim_{|x|\to\infty} p(x) = 0$. We denote by $\mathbb{P}$ the set of all these weights. Important weights include are the following: The weight ρ_α given by

$$\rho_\alpha(x) := (1 + |x|^2)^{-\alpha}, \qquad \alpha > 0.$$

Given a weight $p \in \mathbb{P}$, $L^1(p)$ denotes the weighted L^1-space determined by the norm

$$\|u\|_p = \int_{\mathbb{R}^N} p(x)|u(x)|dx.$$

We will use some terminology and notations from classical topology dynamics. Let $(T(t))_{t\geq 0}$ be a continuous semigroup on a metric space X. The *orbit* or *trajectory* of $u \in X$ (respect to $(T(t))_{t\geq 0}$) is the set

$$\gamma(u) = \{T(t)u : t \geq 0\},$$

and the ω-*limit set* of u is

$$\omega(u) = \{v \in X : v = \lim_{n\to\infty} T(t_n)u \text{ for some sequence } t_n \to \infty\}.$$

This set may be empty. Now, it is well-known that if $\gamma(u)$ is relatively compact, then $\omega(u)$ is a nonempty, compact and connected subset of X. Furthermore, $\omega(u)$ is positive invariant under $T(t)$, *i.e.*, $T(t)\omega(u) \subset \omega(u)$ for any $t \geq 0$. An *equilibrium* or *stationary point* $u \in X$ is a point such that $\gamma(u) = \omega(u) = \{u\}$, or equivalently, $T(t)u = u$ for all $t \geq 0$.

Our abstract framework is the theory of non-linear semigroups. We refer the reader to [8], [11] and [13] for background material on non-linear contraction semigroups.

In order to discuss $u_t - \Delta\varphi(u) = 0$ within the nonlinear semigroup theory, Ph. Bénilan and M. G. Crandall ([6]) associate an m-T-accretive operator A_φ in $L^1(\mathbb{R}^N)$ with the formal expression $A_\varphi = -\Delta\varphi(u)$, (i.e. $(I+\lambda A_\varphi)^{-1}$ is an order preserving contraction in $L^1(\mathbb{R}^N)$ and $R(1+\lambda A_\varphi) = L^1(\mathbb{R}^N)$ for all $\lambda > 0$). This is done via the results of [5]. More concretely, they obtain the following result.

Theorem 1.1. *Let φ be a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \varphi(0)$, and $0 \in \mathrm{int}D(\varphi)$ if $N = 1$ or 2. Then the operator A_φ on $L^1(\mathbb{R}^N)$ defined by*

$$A_\varphi u = \{-\Delta w \ : \ w \in L^1_{loc}(\mathbb{R}^N), -\Delta w \in L^1(\mathbb{R}^N) \text{ and } w(x) \in \varphi(u(x)) \text{ a.e. } \}$$

for $u \in L^1(\mathbb{R}^N)$ is m-T-accretive on $L^1(\mathbb{R}^N)$.

Suppose $N \geq 3$ and $0 < \alpha \leq (N-2)/2$. Then the operator B_φ on $L^1(\rho_\alpha)$ defined by

$$B_\varphi u = \{-\Delta w \ : \ w \in L^1(\rho_{\alpha+1}), -\Delta w \in L^1(\rho_\alpha) \text{ and } w(x) \in \varphi(u(x))\text{a.e.}\}$$

for $u \in L^1(\rho_\alpha)$ is m-T-accretive on $L^1(\rho_\alpha)$.

As a consequence of the Crandall-Liggett Theorem and the above theorem we have that for every initial data $u_0 \in L^1(\mathbb{R}^N)(u_0 \in L^1(\rho_\alpha))$ the problem (I) has a mild-solution given by $u(x,t) = \big(S(t)u_0\big)(x)$. $\big(S(t)\big)_{t\geq 0}$ being the order-preserving contraction semigroup generated by $A_\varphi(\ B_\varphi)$, i.e.,

$$S(t)u_0 = \lim_{n\to\infty} J^n_{t/n} u_0,$$

with $J_\lambda = (I+\lambda A_\varphi)^{-1}(J_\lambda = (I+\lambda B_\varphi)^{-1})$ the resolvent of A_φ (B_φ).

2. The asymptotic behaviour.

We need the orbits to be relatively compact to ensure that the ω-limit set is nonempty.

Lemma 2.1. *Let φ a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \varphi(0)$, and $0 \in \mathrm{int}D(\varphi)$ if $N = 1$ or 2. If $u_0 \in L^1(\mathbb{R}^N)$ and $p \in \mathbb{P}$, then the orbit $\gamma(u_0) = \{\ S(t)u_0 : t \geq 0\}$ is a relatively compact subset of $L^1(p)$.*

For $N \geq 3$, $0 < \alpha \leq (N-2)/2$ and $u_0 \in L^1(\rho_\alpha)$, the orbit $\gamma(u_0) = \{S(t)u_0 : t \geq 0\}$ is a relatively compact subset of $L^1(\rho_\alpha)$.

Proof. By Riesz Theorem [12, Theorem IV.8.21], we must prove that $\gamma(u_0)$ is a bounded subset of $L^1(p)$ satisfying:

$$(1) \qquad \lim_{y\to 0} \|\tau_y S(t)u_0 - S(t)u_0\|_{\rho_\alpha} = 0 \text{ uniformly for } t \geq 0$$

where $\tau_y u(x) := u(x+y)$ and

$$\lim_{R\to\infty} \int_{\{|x|\geq R\}} p(x)|S(t)u_0(x)|\,dx = 0 \quad \text{uniformly for } t \geq 0. \tag{2}$$

In fact: since $S(t)0 = 0$ and $S(t)$ is an $L^1(\mathbb{R}^N)$-contraction, we have

$$\|S(t)u_0\|_p \leq \|p\|_\infty \|S(t)u_0 - S(t)0\|_1 \leq \|p\|_\infty \|u_0\|_1, \text{ for } t \geq 0.$$

Hence, $\gamma(u_0)$ is bounded in $L^1(p)$. Moreover,

$$\|\tau_y S(t)u_0 - S(t)u_0\|_p \leq \|p\|_\infty \|S(t)(\tau_y u_0) - S(t)u_0\|_1 \leq \|p\|_\infty \|\tau_y u_0 - u_0\|_1,$$

from where it follows (1).

On the other hand,

$$\int_{\{|x|\geq R\}} p(x)|S(t)u_0|\,dx \leq \sup_{\{|x|\geq R\}} p(x) \int_{\mathbb{R}^N} |S(t)u_0|\,dx \leq \sup_{\{|x|\geq R\}} p(x)\|u_0\|_1,$$

so (2) follows. Thus $\gamma(u_0)$ is a precompact subset of $L^1(p)$.

Since the semigroup generated by B_φ is an extension of the one generated by A_φ, each $S(t)$ is a contraction in $L^1(\rho_\alpha)$. From the density of $L^1(\mathbb{R}^N)$ in $L^1(\rho_\alpha)$, it follows that $\gamma(u_0)$ is a precompact subset of $L^1(\rho_\alpha)$ for any $u_0 \in L^1(\rho_\alpha)$.

To obtain our stabilization results it is convenient to introduce the concept of *concentration relation* [22]: for locally integrable, radially-symmetric and nonnegative functions f, g defined in $\mathbb{R}^N$ we say that f is *more concentrated* than g, and represent this relation as $f \succ g$ or $g \prec f$, if for every $r > 0$,

$$\int_{B_r(0)} f(x)\,dx \geq \int_{B_r(0)} g(x)\,dx$$

or equivalently,

$$\int_0^r \tilde{f}(t)t^{N-1}\,dt \geq \int_0^r \tilde{g}(t)t^{N-1}\,dt$$

where $\tilde{f}(t) := f(x)$ with $t = |x|$.

We denote by $\mathcal{R}(\mathbb{R}^N)$ the set of all functions in $L^1(\mathbb{R}^N)$ which are radially-symmetric and decreasing.

In order to show that the mild-solutions of (I) are nondecreasing functions in t respect to the order $\succ$ we first prove an elliptic version.

Lemma 2.2 (Monotonicity Lemma; Elliptic Version). *Let φ be a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \varphi(0)$ and $0 \in \mathrm{int}D(\varphi)$ if $N = 1$ or 2. Let $J_\lambda := (I + \lambda A_\varphi)^{-1}$ be the resolvent of A_φ and $0 \leq f \in \mathcal{R}(\mathbb{R}^N)$. Then $J_\lambda f$ is radially-symmetric and $f \succ J_\lambda f$ for all $\lambda > 0$.*

Proof. For simplicity of the proof we suppose that φ is univalued. Since f is radially-symmetric and the problem

$$u - \lambda\Delta\varphi(u) = f \tag{3}$$

is invariant under rotations (see [5]) $J_\lambda f$ is radially-symmetric. Set

$$V(r) := \int_{B_r(0)} J_\lambda f = \omega_N \int_0^r \widetilde{J_\lambda f}(t)\ t^{N-1}\ dt \quad \text{and}$$

$$F(r) := \int_{B_r(0)} f = \omega_N \int_0^r \tilde{f}(t)\ t^{N-1}\ dt.$$

If $v_\lambda := \varphi(J_\lambda f)$, integrating in (3) we obtain

$$\begin{aligned}\int_{B_r(0)} J_\lambda f - f = \lambda \int_{B_r(0)} \Delta v_\lambda &= \lambda\omega_N \int_0^r t^{N-1}(\widetilde{v_\lambda}''(t) + \frac{N-1}{t}\widetilde{v_\lambda}'(t))dt \\ &= \lambda\omega_N \int_0^r \frac{d}{dt}(t^{N-1}\frac{d\widetilde{v_\lambda}}{dt})\ dt = \lambda\omega_N\ r^{N-1}\ \frac{d\widetilde{v_\lambda}}{dr}(r).\end{aligned}$$

Consequently,

$$V(r) - F(r) = \lambda\omega_N\ r^{N-1}\frac{d\widetilde{v_\lambda}}{dr}(r). \tag{4}$$

Assume that $f \succ J_\lambda f$ does not hold so that $G := \{r \geq 0 : V(r) > F(r)\}$ is a nonempty open set of $\mathbb{R}^+$. Then, there exist disjoint open intervals I_n such that $G = \cup_{n=1}^{\infty} I_n$.

By (4), if $r \in G$ we have $\frac{d\widetilde{v_\lambda}}{dr}(r) > 0$, from where it follows that $\widetilde{J_\lambda f}$ is nondecreasing in G since φ is nondecreasing. Hence $V - F$ is a convex function in each I_n. On the other hand, by [5], $V(\infty) \leq F(\infty)$, then since $F(0) = V(0) = 0$ there exists $r_n \in I_n$ such that

$$V(r_n) - F(r_n) = \max\{V(r) - F(r) : r \in I_n\},$$

which is a contradiction. Therefore, G is empty and the proof concludes.

We derive via the Crandall-Liggett exponential formula the following evolution version of the above result.

Lemma 2.3 (Monotonicity Lemma; Parabolic Version). *Let φ a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0\in\varphi(0)$ and $0\in \mathrm{int}D(\varphi)$ if $N=1$ or 2. Let $0\leq u_0\in\mathcal{R}(\mathbb{R}^N)$. If $u(x,t)$ is the mild-solution of problem (I) then $u(.,t)\prec u(.,s)\prec u_0$ for $t\geq s\geq 0$.*

Lemma 2.4. *Let $\varphi:\mathbb{R}\to\mathbb{R}$ a nondecreasing continuous function such that $\varphi(0)=0$ and $0\in\mathrm{int}D(\varphi)$ if $N=1$ or 2. Let $0\leq u_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and $u(t,.)=S(t)u_0$ the mild-solution of problem (I) with initial data u_0. If there exists $w=L^1_{loc}-\lim_{t\to\infty}u(t,.)$, then $\Delta\varphi(w)=0$ in $\mathcal{D}'(\mathbb{R}^N)$.*

Proof. Let $\phi\in\mathcal{D}(\mathbb{R}^N)$. Consider

$$\psi(s):=\int_{\mathbb{R}^N}u(s,x)\phi(x)\,dx.$$

Since $u(t,x)$ is a weak solution of $u_t=\Delta\varphi(u)$ (see [7]), one has

$$\frac{d}{ds}\psi(s)=\int_{\mathbb{R}^N}\varphi(u(s,x))\Delta\phi(x)\,dx.$$

Hence, ψ is absolutely continuous, and consequently, for every $t>0$

$$\begin{aligned}\frac{1}{t}\int_{\mathbb{R}^N}[u(2t,x)-u(t,x)]\phi(x)\,dx&=\frac{1}{t}\int_t^{2t}\Big(\frac{d}{ds}\int_{\mathbb{R}^N}u(s,x)\phi(x)\,dx\Big)\,ds\\&=\frac{1}{t}\int_t^{2t}\Big(\int_{\mathbb{R}^N}\varphi(u(s,x))\Delta\phi(x)\,dx\Big)\,ds.\end{aligned}$$

Now, since $u(s,.)\to w$ in $L^1_{loc}(\mathbb{R}^N)$, the left hand side of the above expression goes to zero when $t\to\infty$. Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\int_t^{2t}\Big(\int_{\mathbb{R}^N}\varphi(u(s,x))\Delta\phi(x)\,dx\Big)\,ds=0. \tag{5}$$

On the other hand, since $u\in C\big([0,\infty[;L^1(\mathbb{R}^N)\big)$, $\|u(t,.)\|_\infty\leq\|u_0\|_\infty$ and φ is continuous, by the dominated convergence theorem we have that the mapping

$$s\to\int_{\mathbb{R}^N}\varphi(u(s,x))\Delta\phi(x)\,dx$$

is continuous. Hence, there exists s_t with $t \leq s_t \leq 2t$ such that

$$(6)\qquad \frac{1}{t}\int_t^{2t} \Big(\int_{\mathbb{R}^N} \varphi(u(s,x))\Delta\phi(x)\ dx \Big)\ ds = \int_{\mathbb{R}^N} \varphi(u(s_t,x))\Delta\phi(x)\ dx.$$

Applying the dominated convergence theorem, using (5) and (6)

$$\int_{\mathbb{R}^N} \varphi(w(x))\Delta\phi(x)\ dx = 0.$$

Therefore, $\Delta\varphi(w) = 0$ in $\mathcal{D}'(\mathbb{R}^N)$.

Given $p \in \mathbb{P}$ we denote by $\omega_p(u_0)$ the ω-limit set of u_0 respect to the $\|.\|_p$-norm, i.e.,

$$\omega_p(u_0) := \{w \in L^1(p) : w = \|.\|_p\!-\lim_{n\to\infty} S(t_n)u_0 \text{ for some sequence } t_n \to \infty\}.$$

Also define $\omega_{loc}(u_0) :=$

$$\{w \in L^1_{loc}(\mathbb{R}^N) : w = L^1_{loc} - \lim_{n\to\infty} S(t_n)u_0 \text{ for some sequence } t_n \to \infty\}.$$

Evidently, $\omega_p(u_0) \subset \omega_{loc}(u_0)$, so, by Lemma 2.1, $\omega_{loc}(u_0) \neq \emptyset$.

In the next theorem we show that the dynamical system $\big(S(t)\big)_{t\geq 0}$ is *gradient-like* in the sense of [17]. More concretely we have:

Theorem 2.5. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ a nondecreasing continuous function such that $\varphi(0) = 0$ and $0 \in \mathrm{int}D(\varphi)$ if $N = 1$ or 2. Let $u_0 \in L^1(\mathbb{R}^N)$ and $u(t,.) = S(t)u_0$ the mild-solution of problem (I) with initial data u_0. Then, $\omega_{loc}(u_0) \subset \{w \in L^1(\mathbb{R}^N) : \varphi(w) = 0\}$.*

Proof. Firstly, we suppose that $0 \leq u_0 \in \mathcal{R}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Define

$$V(r,t) := \int_{B_r(0)} u(x,t) = \omega_N \int_0^r \tilde{u}(\tau,t)\tau^{N-1}\ d\tau,$$

then from Lemma 2.3

$$0 \leq V(r,t) \leq V(r,s) \leq V_0(r) = \int_{B_r(0)} u_0(x)\ dx,$$

for all $r > 0$, $t \geq s \geq 0$. Define $V_\infty(r) := \lim_{t\to\infty} V(r,t)$ for $r > 0$.

On the other hand, by Lemma 2.1, the ω-limit set $\omega_{loc}(u_0)$ is nonempty. Let w_1, $w_2 \in \omega_{loc}(u_0)$, then there exist $t_n \to \infty$ and $s_n \to \infty$ such that

$$w_1 = \lim_{n\to\infty} u(.,t_n) \text{ in } L^1_{loc}(\mathbb{R}^N) \text{ and } w_2 = \lim_{n\to\infty} u(.,s_n) \text{ in } L^1_{loc}(\mathbb{R}^N).$$

Hence, for every $r > 0$ we have

$$\int_{B_r(0)} w_1(x)\, dx = \lim_{n\to\infty} \int_{B_r(0)} u(x,t_n)\, dx = \lim_{n\to\infty} V(r,t_n) = V_\infty(r) \text{ and}$$
$$\int_{B_r(0)} w_2(x)\, dx = \lim_{n\to\infty} \int_{B_r(0)} u(x,s_n)\, dx = \lim_{n\to\infty} V(r,s_n) = V_\infty(r).$$

Thus, $\int_{B_r(0)} w_1(x)\, dx = \int_{B_r(0)} w_2(x)\, dx$, from where it follows, having in mind that w_1 and w_2 are radially-symmetric, $w_1 = w_2$. Therefore, $\omega_{loc}(u_0) = \{w\}$ and consequently

$$\lim_{t\to\infty} u(.,t) = w \text{ in } L^1_{loc}(\mathbb{R}^N).$$

Then, for every $r > 0$ we have

$$\int_{B_r(0)} w(x)\, dx = \lim_{t\to\infty} \int_{B_r(0)} u(x,t)\, dx \leq \int_{B_r(0)} u_0(x)\, dx.$$

Hence, $w \in L^1(\mathbb{R}^N)$.

By Lemma 2.4, $\Delta\varphi(w) = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. Moreover, since $\|S(t)u_0\|_\infty \leq \|u_0\|_\infty$, $\varphi(w)$ is bounded and consequently constant. Hence, since $w \in L^1(\mathbb{R}^N)$, it follows that $\varphi(w) = 0$.

In the following step we suppose that $0 \leq u_0 \in \mathcal{D}(\mathbb{R}^N)$. Then, there exists $v_0 \in \mathcal{R}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, such that $0 \leq u_0 \leq v_0$. Since every $S(t)$ is order-preserving, $0 \leq S(t)u_0 \leq S(t)v_0$. Thus, if $v \in \omega_{loc}(u_0)$, by the first step of the proof we have $0 \leq v \leq w$ with $\omega_{loc}(v_0) = \{w\}$. This shows that $v \in L^1(\mathbb{R}^N)$ and $\varphi(v) = 0$.

In the following step we suppose that $0 \leq u_0 \in L^1(\mathbb{R}^N)$. For every $n \in \mathbb{N}$ let $0 \leq u_{0n} \in \mathcal{D}(\mathbb{R}^N)$ be such that $\|u_0 - u_{0n}\|_1 \leq \frac{1}{n}$. Given $w \in \omega_{loc}(u_0)$ there exist $t_k \to \infty$ such that $S(t_k)u_0 \to w$ in $L^1_{loc}(\mathbb{R}^N)$. Now, by Lemma 2.1 and the above step, for every $n \in \mathbb{N}$, there exists $t_k^n \to \infty$ with $\{t_k^{n+1}\}_k$ a subsequence of $\{t_k^n\}_k$ such that $w_n = \lim_{k\to\infty} S(t_k^n)u_{0n} \in L^1(\mathbb{R}^N)$ and $\varphi(w_n) = 0$. Now, for every $r > 0$ we have

$$\int_{B_r(0)} |w(x) - w_n(x)|\, dx \leq \int_{B_r(0)} |w(x) - S(t_n^n)u_0(x)|\, dx$$

$$+ \int_{B_r(0)} |S(t_n^n)u_0(x) - S(t_n^n)u_{0n}(x)|\, dx + \int_{B_r(0)} |S(t_n^n)u_{0n}(x) - w_n(x)|\, dx$$

$$\leq \int_{B_r(0)} |w(x) - S(t_n^n)u_0(x)|\, dx + \int_{B_r(0)} |u_0(x) - u_{0n}(x)|\, dx$$

$$+ \int_{B_r(0)} |S(t_n^n)u_{0n}(x) - w_n(x)|\, dx.$$

This shows that $w_n \to w$ a.e. in $\mathbb{R}^N$ and consequently, $\varphi(w) = 0$. On the other hand, since $u_{0n} \to u_0$ in $L^1(\mathbb{R}^N)$ there exists $h \in L^1(\mathbb{R}^N)$ such that $0 \leq u_{on} \leq h$ a.e. in $\mathbb{R}^N$. Then,

$$\int_{\mathbb{R}^N} S(t_k^n)u_0 \leq \int_{\mathbb{R}^N} u_0 \leq \int_{\mathbb{R}^N} h.$$

Hence, by Fatou's Lemma we have

$$\int_{\mathbb{R}^N} w_n \leq \int_{\mathbb{R}^N} h \text{ for all } n \in \mathbb{N}.$$

Again applying Fatou's Lemma it follows that $w \in L^1(\mathbb{R}^N)$. The order preseving property of every $S(t)$ now yields the general theorem when $u_0 \in L^1(\mathbb{R}^N)$.

Corollary 2.6. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ a nondecreasing continuous function such that $\varphi^{-1}(0) = \{0\}$ and $0 \in \operatorname{int}D(\varphi)$ if $N = 1$ or 2. Let $u_0 \in L^1(\mathbb{R}^N)$ and $u(t,.) = S(t)u_0$ the mild-solution of problem (I) with initial data u_0. Then,*

$$\lim_{t\to\infty} u(t,.) = 0 \text{ in } L^1(p) \text{ for all } p \in \mathbb{P}.$$

Proof. As consequence of the above Theorem $\omega_p(u_0) = \{0\}$. Then, by the compactness of the orbits in $L^1(p)$, the result follows.

When $N \geq 3$, the convergence of the orbits without the assumptions $\varphi^{-1}(0) = \{0\}$ and φ continuous can be obtained by using the fact that the solutions form a contraction semigroup in $L^1(\rho_\alpha)$ (Theorem 1.1).

Theorem 2.7. *Suppose $N \geq 3$ and let $0 < \alpha \leq (N-2)/2$. Let φ be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \varphi(0)$. Let $u_0 \in L^1(\rho_\alpha)$ and $u(t,.) = S(t)u_0$ the mild-solution of problem (I) with initial data u_0. Then,*

$$\lim_{t\to\infty} u(t,.) = w \text{ in } L^1(\rho_\alpha) \text{ with } \varphi(w) = 0.$$

Proof. Firstly, we suppose that $0 \leq u_0 \in \mathcal{R}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. With the same proof as in the first part of Theorem 2.5 we obtain that $\omega_{\rho_\alpha}(u_0) = \{w\}$. Now, since $\left(S(t)\right)_{t\geq 0}$ is a contraction semigroup in $L^1(\rho_\alpha)$ it follows that w is an equilibrium point. Consequently, $\varphi(w) = 0$.

Suppose now that $0 \leq u_0 \in \mathcal{D}(\mathbb{R}^N)$. Then, there exists $v_0 \in \mathcal{R}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $0 \leq u_0 \leq v_0$. Since every $S(t)$ is order-preserving, $0 \leq S(t)u_0 \leq S(t)v_0$. Thus, if $v \in \omega_{\rho_\alpha}(u_0)$, by the first step of the proof we have $0 \leq v \leq w$ with $\omega_{\rho_\alpha}(v_0) = \{w\}$. Hence, $0 \leq \varphi(v) \leq \varphi(w)$ and consequently $\varphi(v) = 0$. Therefore, $\omega_{\rho_\alpha}(u_0) = \{v\}$ with $\varphi(v) = 0$

From here, since $\mathcal{D}(\mathbb{R}^N)$ is dense in $L^1(\rho_\alpha)$ and every $S(t)$ is an order-preserving contraction in $L^1(\rho_\alpha)$, it is easy to finish the proof.

Remark 2.8. As we said in the introduction the uniform bounds obtained in [4] are interesting in the study of the asymptotic behaviour of solutions of problem (I). In fact, as consequence of these bounds and our compactness result (Lemma 2.1) it is possible to obtain the above theorem for the case in which φ is a nondecreasing continuous function.

We do not know if this theorem is true when $N = 1$ or 2.

As consequence of the above theorem we obtain the following inhomogeneous version.

Corollary 2.9. *Suppose $N \geq 3$ and $0 < \alpha \leq (N-2)/2$. Let φ be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \varphi^{-1}(0)$ and $f \in L^1(0,\infty; L^1(\rho_\alpha))$. If $u(x,t)$ is the mild-solution of the problem*

$$(III) \qquad \begin{cases} u_t = \Delta\varphi(u) + f & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

with $u_0 \in L^1(\rho_\alpha)$, then there exists $w \in L^1(\rho_\alpha)$ such that

$$\lim_{t\to\infty} u(.,t) = w \text{ in } L^1(\rho_\alpha).$$

Proof. For every $n \in \mathbb{N}$, set

$$f_n(t) = \begin{cases} f(t) & \text{if } t \leq n \\ 0 & \text{if } t > n. \end{cases}$$

Then, if $u_n(t)$ is the mild-solution of

$$\begin{cases} u_n' + A_\varphi u_n \ni f_n \\ u_n(0) = u_0 \end{cases}$$

and v_n is the mild-solution of

$$\begin{cases} v_n' + A_\varphi v_n \ni 0 \\ v_n(0) = u_n(n), \end{cases}$$

it follows from the translation property of the mild-solutions that $v_n(t) = u_n(t+n)$. Now, by the above theorem

$$\lim_{t \to \infty} v_n(t) = w_n \quad \text{in} \quad L^1(\rho_\alpha), \quad \text{with} \quad \varphi(w_n) = 0.$$

Hence, $\lim_{t\to\infty} u_n(t) = w_n$ in $L^1(\rho_\alpha)$, with $\varphi(w_n) = 0$.

Let $h, g \in \omega_{\rho_\alpha}(u_0)$ with $h = \lim_{k\to\infty} u(t_k)$ and $g = \lim_{k\to\infty} u(s_k)$. Then,

$$\|h-g\|_{\rho_\alpha} \leq \|h-u(t_k)\|_{\rho_\alpha} + \|u(t_k)-u_n(t_k)\|_{\rho_\alpha} + \|u_n(t_k)-w_n\|_{\rho_\alpha} +$$

$$+\|w_n-u_n(s_k)\|_{\rho_\alpha} + \|u_n(s_k)-u(s_k)\|_{\rho_\alpha} + \|u(s_k)-g\|_{\rho_\alpha}.$$

On the other hand, since u and u_n are integral solutions we have that

$$\|u(t)-u_n(t)\|_{\rho_\alpha} \leq \|u_0-u_n(0)\|_{\rho_\alpha} + \int_0^t \|f(\tau)-f_n(\tau)\|_{\rho_\alpha}\, d\tau \leq \int_n^\infty \|f(\tau)\|_{\rho_\alpha}\, d\tau.$$

Then, given $\epsilon > 0$, let $n \in \mathbb{N}$ such that

$$\int_n^\infty \|f(\tau)\|_{\rho_\alpha}\, d\tau \leq \epsilon/6.$$

For n fixed, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ we have

$$\|h - u(t_k)\|_{\rho_\alpha} \leq \epsilon/6, \ \|u_n(t_k) - w_n\|_{\rho_\alpha} \leq \epsilon/6,$$

$$\|u_n(s_k) - w_n\|_{\rho_\alpha} \leq \epsilon/6, \quad \|g - u(s_k)\|_{\rho_\alpha} \leq \epsilon/6.$$

Hence, $\|h - g\|_{\rho_\alpha} \leq \epsilon$. Consequently, $\omega_{\rho_\alpha}(u_0) = \{w\}$ and by the compactness of the orbits we obtain

$$\lim_{t\to\infty} u(.,t) = w \quad \text{in} \quad L^1(\rho_\alpha).$$

Remark 2.10. The simplest heat transfer phenomenon involving phase change can be modelled by the equation (I) with φ a nondecreasing continuos function, whose graph has a flat part. For instance, in the classical two-phase Stefan problem (see [15]), φ is given by

$$\varphi(r) = \begin{cases} r & \text{if } r \leq 0 \\ (r-a)^+ & \text{if } r \geq 0. \end{cases}$$

Suppose we have a function φ of the above type, i.e., $\varphi : \mathbb{R} \to \mathbb{R}$ a nondecreasing continuous function such that $\sup\{s \in \mathbb{R} : \varphi(s) = 0\} = a > 0$. Let $0 \leq u_0 \in L^1(\mathbb{R}^N)$ and $u(t,.) = S(t)u_0$ the mild-solution of problem (I) with initial data u_0. Then, by Theorem 2.5, if $w \in \omega_{loc}(u_0)$ we have that $w \in L^1(\mathbb{R}^N)$ and $0 \leq w \leq a$. Moreover, since $0 \leq \inf\{u_0, a\} = S(t)\big(\inf\{u_0, a\}\big) \leq S(t)u_0$, it follows that $\inf\{u_0, a\} \leq w \leq a$. Consequently, the functions in the ω-limit set of u_0 develops "mesas" on the set where u_0 is greater than a. For the porous medium equation this phenomena was noticed in [14]

To finish we are going to see that for φ's as before the solutions of problem (I) have compact support when the initial data has compact support. More concretely we have the following result.

Theorem 2.11. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *a nondecreasing continuous function such that* $\sup\{s \in \mathbb{R} : \varphi(s) = 0\} = a > 0$ *and* $0 \in \text{int}D(\varphi)$ *if* $N = 1$ *or* 2. *Let* $0 \leq u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $u(t,.) = S(t)u_0$ *the mild-solution of* *problem (I) with initial data u_0. Then, $u(t,.)$ has compact support when u_0 has compact support.*

Proof. Obviously, it is enough to prove the theorem in the case $u_0 = b\ \mathcal{X}_{B_{r_0}(0)}$ where $b > a$ and $\mathcal{X}_{B_{r_0}(0)}$ is the characteristic function of the ball $B_{r_0}(0)$. Let $\alpha = r_0 \sqrt[N]{b/a}$. We are going to prove that $u(t,.) \equiv 0$ in $\Sigma := \mathbb{R}^N \sim \overline{B_\alpha(0)}$. First we claim

$$J_\lambda^n u_0 \leq a \quad \text{in } \Sigma \quad \text{for all } n \in \mathbb{N}, \quad \lambda > 0. \tag{7}$$

Let $v(r) := \varphi(J_\lambda^n u_0)(x)$ with $|x| = r$. In the proof of Lemma 2.2 we saw that

$$\frac{dv}{dr}(r) = \int_{B_r(0)} J_\lambda^n u_0 - \int_{B_r(0)} J_\lambda^{n-1} u_0 \leq 0.$$

Hence, v is nondecreasing. Thus, if (7) is false, there exists $\beta > \alpha$ such that $v(r) \geq v(\beta) > 0$ for $0 \leq r \leq \beta$, from where it follows that $J_\lambda^n u_0(x) > a$ if $|x| \leq \beta$. Consequently,

$$a\mu\big(B_\beta(0)\big) < \int_{B_\beta(0)} J_\lambda^n u_0 \leq \int_{B_\beta(0)} u_0 = b\mu\big(B_{r_0}(0)\big) = a\mu\big(B_\alpha(0)\big),$$

which is a contradiction. Therefore the claim is true.

From (7) and the Crandall-Liggett exponential formula we obtain

$$u(t,.) \leq a \quad \text{in } \Sigma \quad \text{for all } t > 0. \tag{8}$$

Having in mind that $u(t,.)$ is a weak solution of $u_t = \Delta\varphi(u)$ (see [7]), by (8) it is easy to see that

$$u(t,x) = c(x) \quad \text{a.e. in } \Sigma \text{ for all } t > 0.$$

Then, since there exist $t_n \to 0$ such that $u(t_n,.) \to u_0$ a.e. in $\mathbb{R}^N$, it follows that $u(t,x) = u_0(x)$ a.e. in Σ for all $t > 0$. Consequently $u(t,x) = 0$ a.e. in Σ for all $t > 0$.

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