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Renormalized solutions for degenerate elliptic–parabolic problems with nonlinear dynamical boundary conditions and L^1-data

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Abstract

We consider a degenerate elliptic–parabolic problem with nonlinear dynamical boundary conditions. Assuming L^1-data, we prove existence and uniqueness in the framework of renormalized solutions. Particular instances of this problem appear in various phenomena with changes of phase like multiphase Stefan problems and in the weak formulation of the mathematical model of the so-called Hele–Shaw problem. Also, the problem with non-homogeneous Neumann boundary condition is included.

Keywords: Degenerate elliptic–parabolic problems; Renormalized solutions; Multiphase Stefan problems; Hele–Shaw problem

1. Introduction

In this paper we obtain existence and uniqueness of renormalized solutions for a degenerate elliptic–parabolic problem with nonlinear dynamical boundary condition of the form

$$P_{\gamma,\beta}(f,g,z_0,w_0) \qquad \begin{cases} z_t - \operatorname{div} \mathbf{a}(x,Du) = f, & z \in \gamma(u) \text{ in } Q_T :=]0,T[\times \Omega, \\ w_t + \mathbf{a}(x,Du)\cdot\eta = g, & w \in \beta(u) \text{ on } S_T :=]0,T[\times \partial\Omega, \\ z(0) = z_0 \quad \text{in } \Omega, & w(0) = w_0 \quad \text{in } \partial\Omega, \end{cases}$$

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where $T > 0$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $v_0 \in L^1(\Omega)$, $w_0 \in L^1(\partial\Omega)$, $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$ and η is the unit outward normal on $\partial\Omega$. Here the function $\mathbf{a}: \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray–Lions conditions. The nonlinearities γ and β are maximal monotone graphs in $\mathbb{R}^2$ (see [20]) such that $0 \in \gamma(0)$, $\mathrm{Dom}(\gamma) = \mathbb{R}$, and $0 \in \beta(0)$. In particular, γ and β may be multivalued and this allows to include the Dirichlet boundary condition (taking β to be the monotone graph $\{0\} \times \mathbb{R}$), the *non-homogeneous* Neumann boundary condition (taking β to be the monotone graph $\beta(r) = 0$ for all $r \in \mathbb{R}$), as well as many other nonlinear fluxes on the boundary that occur in some problems in mechanics and physics (see [28] or [19]). Note also that, since γ may be multivalued, problems of type $P_{\gamma,\beta}(f, g, z_0, w_0)$ appear in various phenomena with changes of phase like multiphase Stefan problem (see [25]) and in the weak formulation of the mathematical model of the so-called Hele–Shaw problem (see [26] and [29]), for which γ is the Heaviside maximal monotone graph. Also, if $\gamma(r) = 0$ for all $r \in \mathbb{R}$, we consider an elliptic problem with nonlinear dynamical boundary condition.

The dynamical boundary conditions, although not too widely considered in the mathematical literature, are very natural in many mathematical models as heat transfer in a solid in contact with moving fluid, thermoelasticity, diffusion phenomena, problems in fluid dynamics, etc. (see [11,23,30,43] and the references therein). These dynamical boundary conditions also appear in the study of the Stefan problem when the boundary material has a large thermal conductivity and sufficiently small thickness. Hence, the boundary material is regarded as the boundary of the domain. For instance, this is the case if one considers an iron ball in which water and ice coexist. For more details about these physical considerations one can see for instance [1]. They also appear in the study of the Hele–Shaw problem. Recall that, in [26] the authors give the weak formulation of the problem in the form of a nonlinear degenerate parabolic problem, governed by the Laplace operator and the multivalued Heaviside function, with static boundary condition. From the physical point of view they assume that the prescribed value of the flux on the boundary is known, but in some practical situations, it may be not possible to prescribe or to control the exact value of the flux on the boundary. In [42] (see also [43]), the authors consider the case of nonlocal dynamical boundary conditions and use variational methods to solve the problem. In the present paper, we cover the case of general nonlinear diffusion and local dynamical boundary conditions. Notice, that general nonlinear diffusion operators of Leray–Lions type, different from the Laplacian, appear when one deals with non-Newtonian fluids (see, e.g., [9,37,38] and the references therein for the case of Hele–Shaw problem with non-Newtonian fluids). Another interesting application we have in mind concerns the filtration equation with dynamical boundary conditions (see, e.g., [44]), which appears for example in the study of rainfall infiltration through the soil, when the accumulation of the water on the ground surfaces caused by the saturation of the surface layer is taken into account. Observe that β may be such that $\mathrm{Ran}(\beta)$ is different from $\mathbb{R}$, so that we cover the case where the boundary conditions are either dynamical or static with respect to the values of w in the problem under consideration. This is the situation where the saturation happens only for values of w in a subinterval of $\mathbb{R}$.

There is an extensive literature for doubly nonlinear problems with homogeneous Dirichlet boundary conditions (see [2,3,10,15,17,21,34] and the references therein). Nevertheless, to our knowledge, there is little literature on problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ as we pointed out in [5], where existence and uniqueness of weak solutions of this problem have been obtained for $L^{p'}$-data. Our aim in this paper is to prove existence and uniqueness of solutions for L^1-data of $P_{\gamma,\beta}(f, g, z_0, w_0)$. There are mainly two types of difficulties in studying this kind of problems, the nonlinearities γ and β and the consideration of L^1-data so that finite energy solutions could not be expected. To solve this last difficulty, the framework of renormalized solutions, which was originally introduced in [27] for transport equations, has proved to be a powerful approach to study large class of second order PDEs (see [3,7,16,17,22] and the references therein).

Another main difficulty when dealing with doubly nonlinear parabolic problems is the uniqueness. For the Laplace operator, thanks to the linearity of the operator, the problem can be solved by using suitable test functions with respect to u (see for instance [33]). For nonlinear operators this kind of argument turns out to be nonuseful. In [15], for an elliptic–parabolic problem with Dirichlet boundary condition, it is shown that the notion of integral solution [12] is a very useful tool to prove uniqueness (see also [32] for non-homogeneous and time dependent Neumann boundary conditions). For general nonlinearities, even for homogeneous Dirichlet boundary condition, the question of uniqueness is more difficult and most of the arguments used in the literature are based on doubling variables methods (see for instance [8,17,21,22,34] and the references therein). In [5] we have shown that the notion of integral solution is a very useful tool to prove uniqueness of weak solutions of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ for $L^{p'}$-data. In this paper, we use the same method to prove uniqueness of renormalized solutions of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ for L^1-data.

We also want to point out that our existence and uniqueness proofs work without any continuity assumptions on γ^{-1} or β^{-1} and any hypothesis about the jumps of γ or β. For the existence of the renormalized solution, we use a monotone approximation of f, g, z_0 and w_0, by L^∞ functions $f_{m,n}$, $g_{m,n}$, $z_{0,m,n}$ and $w_{0,m,n}$. So that, by using the results of [5], the problem has a unique weak solution $(z_{m,n}, w_{m,n})$. Thanks to the nonlinear semigroup theory (see [14,45]), the results of [4] concerning the stationary problem associated with $P_{\gamma,\beta}(f, g, z_0, w_0)$, it is not difficult to get the L^1 convergence of $(z_{m,n}, w_{m,n})$. Nevertheless, the characterization of the limit of $(z_{m,n}, w_{m,n})$ in terms of the partial differential equation is very technical due to the fact that the problem is doubly nonlinear. For the convergence of $u_{m,n}$ (see the proof of Theorem 2.6), we use the monotonicity with respect to m and n, as it was used in [3], and for the identification of the limit equation we use Landes approximation (see [39]). Recall that this kind of arguments was also used in [3] for elliptic–parabolic problems and in [35] for degenerate parabolic problems of Stefan type. Here we extend these arguments to our general setting (other kind of arguments may be found in [17]). For the uniqueness, we show that renormalized solutions are *integral solutions*, concept due to Ph. Bénilan (see [12,14]). In other words, we show that renormalized solutions satisfy a contraction property with respect to stationary solutions. The main difficulties here are due to the nonlinear and non-homogeneous boundary conditions and to the jumps of γ and β. In [17], to obtain a contraction principle for a similar problem in the case of Dirichlet boundary condition ($\beta = \{0\} \times \mathbb{R}$), and for γ having a set of jumps without density points, the authors give an improvement of the "hole filling" argument of [21], using the doubling variable technique in time and a very useful choice of test functions. This technique can be adapted to our problem. Now, as in [5], by the nonlinear semigroup theory, we are able to simplify the proof of uniqueness without using the doubling variable technique in time and without imposing any condition on the jumps of γ and β.

Let us briefly summarize the contents of the paper. In Section 2 we fix the notation and give some preliminaries; we also give the concept of renormalized solution for the problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ and state the existence and uniqueness result for renormalized solutions of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$. In Section 3 we show the existence of renormalized solutions and finally in Section 4 we prove the uniqueness of renormalized solutions.

2. Preliminaries and main result

In this section, after some preliminaries, we introduce the concept of renormalized solution for problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ and we state the existence and uniqueness result for this type of solutions.

Throughout the paper, $\Omega \subset \mathbb{R}$ is a bounded domain with smooth boundary $\partial\Omega$, $p > 1$, γ and β are maximal monotone graphs in $\mathbb{R}^2$ such that $\mathrm{Dom}(\gamma) = \mathbb{R}$, $0 \in \gamma(0) \cap \beta(0)$ and the Carathéodory function $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies

(H$_1$) there exists $\lambda > 0$ such that $\mathbf{a}(x, \xi) \cdot \xi \geqslant \lambda |\xi|^p$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$,
(H$_2$) there exist $c > 0$ and $\varrho \in L^{p'}(\Omega)$ such that $|\mathbf{a}(x, \xi)| \leqslant c(\varrho(x) + |\xi|^{p-1})$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $p' = \frac{p}{p-1}$,
(H$_3$) $(\mathbf{a}(x, \xi) - \mathbf{a}(x, \eta)) \cdot (\xi - \eta) > 0$ for a.e. $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$, $\xi \neq \eta$.

The hypotheses (H$_1$)–(H$_3$) are classical in the study of nonlinear operators in divergence form (see [41] or [13]). The model example of function $\mathbf{a}$ satisfying these hypotheses is $\mathbf{a}(x, \xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p-Laplacian operator $\Delta_p(u) = \mathrm{div}(|Du|^{p-2}Du)$.

In [13], the authors introduce the set

$$\mathcal{T}^{1,p}(\Omega) = \{u\colon\ \Omega \to \mathbb{R} \text{ measurable such that } T_k(u) \in W^{1,p}(\Omega)\ \forall k > 0\},$$

where $T_k(s) = \sup(-k, \inf(s, k))$. They also prove that given $u \in \mathcal{T}^{1,p}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$DT_k(u) = v\chi_{\{|u|<k\}} \quad \forall k > 0.$$

This function v will be denoted by Du. It is clear that if $u \in W^{1,p}(\Omega)$, then $v \in L^p(\Omega)$ and $v = Du$ in the usual sense.

We denote $\mathcal{T}_\tau^{1,p}(\Omega)$ the set of functions u in $\mathcal{T}^{1,p}(\Omega)$ such that there exists a measurable function w on $\partial\Omega$ with $T_k(w) = \mathrm{tr}(T_k(u))$ a.e. on $\partial\Omega$ for all $k > 0$, where tr is the usual $W^{1,p}$-trace. The function w is the trace of u in a generalized sense. In the sequel, the trace of $u \in \mathcal{T}_\tau^{1,p}(\Omega)$ on $\partial\Omega$ will be denoted by u.

For a maximal monotone graph ϑ in $\mathbb{R} \times \mathbb{R}$, its *main section* ϑ^0 is defined by

$$\vartheta^0(s) := \begin{cases} \text{the element of minimal absolute value of } \vartheta(s) & \text{if } \vartheta(s) \neq \emptyset, \\ +\infty & \text{if } [s, +\infty) \cap \mathrm{Dom}(\vartheta) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap \mathrm{Dom}(\vartheta) = \emptyset. \end{cases}$$

We shall denote $\vartheta_- := \inf \mathrm{Ran}(\vartheta)$ and $\vartheta_+ := \sup \mathrm{Ran}(\vartheta)$. If $0 \in \mathrm{Dom}(\vartheta)$, $j_\vartheta(r) = \int_0^r \vartheta^0(s)\, ds$ defines a convex l.s.c. function such that $\vartheta = \partial j_\vartheta$. If j_ϑ^* is the Legendre transformation of j_ϑ then $\vartheta^{-1} = \partial j_\vartheta^*$.

For the maximal monotone graphs γ and β, we shall denote

$$\mathcal{R}^+_{\gamma,\beta} := \gamma_+|\Omega| + \beta_+|\partial\Omega|, \qquad \mathcal{R}^-_{\gamma,\beta} := \gamma_-|\Omega| + \beta_-|\partial\Omega|.$$

In the sequel, we suppose $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$ and we write $\mathcal{R}_{\gamma,\beta} :=]\mathcal{R}^-_{\gamma,\beta}, \mathcal{R}^+_{\gamma,\beta}[$.

It is said that **a** is *smooth* (see [6] and [4]) if, for any $\phi \in L^\infty(\Omega)$ such that there exists a bounded weak solution u of the homogeneous Dirichlet problem

$$(D)\quad \begin{cases} -\operatorname{div} \mathbf{a}(x, Du) = \phi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

there exists $g \in L^1(\partial\Omega)$ such that u is also a weak solution of the Neumann problem

$$(N)\quad \begin{cases} -\operatorname{div} \mathbf{a}(x, Du) = \phi & \text{in } \Omega, \\ \mathbf{a}(x, Du) \cdot \eta = g & \text{on } \partial\Omega. \end{cases}$$

Functions **a** corresponding to linear operators with smooth coefficients and p-Laplacian type operators are smooth (see [19] and [40]).

The following integration by parts formula, which is a slight modification of [5, Lemma 4.1], will play an important role in our arguments. We denote by $(.,.)$ the pairing between $(W^{1,p}(\Omega))'$ and $W^{1,p}(\Omega)$.

Lemma 2.1. *(See [5].) Let ϑ and ϱ be maximal monotone graphs in $\mathbb{R}^2$. Let $z \in C([0,T] : L^1(\Omega))$, $w \in C([0,T] : L^1(\partial\Omega))$, $F \in L^{p'}(0,T; (W^{1,p}(\Omega))')$, $f \in L^1(0,T; L^1(\Omega))$ and $g \in L^1(0,T; L^1(\partial\Omega))$ such that*

$$\int_0^T\int_\Omega z\psi_t\,dx\,dt + \int_0^T\int_{\partial\Omega} w\psi_t\,d\sigma\,dt = \int_0^T \big(F(t), \psi(t)\big)\,dt + \int_0^T\int_\Omega f\psi\,dx\,dt + \int_0^T\int_{\partial\Omega} g\psi\,d\sigma\,dt$$

for any $\psi \in W^{1,1}(0,T; W^{1,1}(\Omega)\cap L^\infty(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega)) \cap L^\infty(0,T; L^\infty(\Omega))$, $\psi(0) = \psi(T) = 0$. Then,

$$\int_0^T\int_\Omega\left(\int_0^{z(t)} H\big(x, \big(\vartheta^{-1}\big)^0(s)\big)\,ds\right)\psi_t\,dx\,dt + \int_0^T\int_{\partial\Omega}\left(\int_0^{w(t)} H\big(x, \big(\varrho^{-1}\big)^0(s)\big)\,ds\right)\psi_t\,d\sigma\,dt$$
$$= \int_0^T \big(F(t), H\big(x, u(t)\big)\psi(t)\big)\,dt + \int_0^T\int_\Omega fH(x,u)\psi\,dx\,dt + \int_0^T\int_{\partial\Omega} gH(x,u)\psi\,d\sigma\,dt,$$

being $u \in L^p(0,T; W^{1,p}(\Omega))$ such that $z \in \vartheta(u)$ a.e. in Q_T and $w \in \varrho(u)$ a.e. in S_T, $H(x,r)$ a bounded Carathéodory function of bounded variation in r, such that $H(.,u(.,.)) \in L^p(0,T; W^{1,p}(\Omega))$, and $\psi \in \mathcal{D}(]0,T[\times\mathbb{R}^N)$.

We now recall the concept of weak solution for problem $P_{\gamma,\beta}(f,g,z_0,w_0)$ and state the existence and uniqueness result given in [5] for such solutions.

Definition 2.2. Given $f \in L^1(0,T; L^1(\Omega))$, $g \in L^1(0,T; L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$, a *weak solution* of $P_{\gamma,\beta}(f,g,z_0,w_0)$ in $[0,T]$ is a couple (z,w) such that $z \in C([0,T] : L^1(\Omega))$, $w \in C([0,T] : L^1(\partial\Omega))$, $z(0) = z_0$, $w(0) = w_0$ and there exists $u \in L^p(0,T; W^{1,p}(\Omega))$ such that $z \in \gamma(u)$ a.e. in Q_T, $w \in \beta(u)$ a.e. on S_T and

$$\frac{d}{dt}\int_{\Omega} z(t)\xi\,dx + \frac{d}{dt}\int_{\partial\Omega} w(t)\xi\,d\sigma + \int_{\Omega} \mathbf{a}\big(x, Du(t)\big)\cdot D\xi\,dx$$
$$= \int_{\Omega} f(t)\xi\,dx + \int_{\partial\Omega} g(t)\xi\,d\sigma \tag{1}$$

in $\mathcal{D}'(]0,T[)$ for any $\xi \in C^1(\bar{\Omega})$.

Theorem 2.3. *(See [5].) Assume* $\mathrm{Dom}(\gamma) = \mathbb{R}$ *and assume either* $\mathrm{Dom}(\beta) = \mathbb{R}$ *or* $\mathbf{a}$ *smooth. Let* $T > 0$. *Let* $f \in L^{p'}(0,T;L^{p'}(\Omega))$, $g \in L^{p'}(0,T;L^{p'}(\partial\Omega))$, $z_0 \in L^{p'}(\Omega)$ *and* $w_0 \in L^{p'}(\partial\Omega)$ *such that*

$$\gamma_- \leqslant z_0 \leqslant \gamma_+, \qquad \beta_- \leqslant w_0 \leqslant \beta_+, \tag{2}$$
$$\int_{\Omega} j_\gamma^*(z_0)\,dx + \int_{\partial\Omega} j_\beta^*(w_0)\,d\sigma < +\infty \tag{3}$$

and

$$\int_{\Omega} z_0\,dx + \int_{\partial\Omega} w_0\,d\sigma + \int_0^t \Big(\int_{\Omega} f(s)\,dx + \int_{\partial\Omega} g(s)\,d\sigma\Big)\,ds \in \mathcal{R}_{\gamma,\beta} \tag{4}$$

for all $t \in [0,T]$. *Then, there exists a unique weak solution* (z,w) *of problem* $P_{\gamma,\beta}(f,g,z_0,w_0)$ *in* $[0,T]$.

Moreover, the following L^1*-contraction principle holds. For* $i = 1,2$, *let* $f_i \in L^1(0,T; L^1(\Omega))$, $g_i \in L^1(0,T;L^1(\partial\Omega))$, $z_{i0} \in L^1(\Omega)$ *and* $w_{i0} \in L^1(\partial\Omega)$; *let* (z_i,w_i) *be a weak solution in* $[0,T]$ *of problem* $P_{\gamma,\beta}(f_i,g_i,z_{i0},w_{i0})$, $i=1,2$. *Then*

$$\int_{\Omega} \big(z_1(t)-z_2(t)\big)^+\,dx + \int_{\partial\Omega} \big(w_1(t)-w_2(t)\big)^+\,d\sigma$$
$$\leqslant \int_{\Omega} (z_{10}-z_{20})^+\,dx + \int_{\partial\Omega} (w_{10}-w_{20})^+\,d\sigma$$
$$+ \int_0^t\int_{\Omega} \big(f_1(s)-f_2(s)\big)^+\,dx\,ds + \int_0^t\int_{\partial\Omega} \big(g_1(s)-g_2(s)\big)^+\,d\sigma\,ds$$

for every $t \in [0,T]$.

Let us give the concept of renormalized solution.

Definition 2.4. Given $f \in L^1(0,T;L^1(\Omega))$, $g \in L^1(0,T;L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$, a *renormalized solution* of $P_{\gamma,\beta}(f,g,z_0,w_0)$ in $[0,T]$ is a couple (z,w), $z \in C([0,T]: L^1(\Omega))$, $w \in C([0,T]:L^1(\partial\Omega))$, $z(0) = z_0$, $w(0) = w_0$, for which there exists a measurable function u in $]0,T[\times\Omega$, $u(t)\in\mathcal{T}_\tau^{1,p}(\Omega)$ a.e. $t\in]0,T[$, such that $T_k(u)\in L^p(0,T;W^{1,p}(\Omega))$ for all $k>0$, $z\in\gamma(u)$ a.e. in Q_T, $w\in\beta(u)$ a.e. on S_T,

$$\frac{d}{dt}\int_\Omega\left(\int_0^{z(t)}H\big((\gamma^{-1})^0(s)\big)\,ds\right)\xi\,dx+\frac{d}{dt}\int_{\partial\Omega}\left(\int_0^{w(t)}H\big((\beta^{-1})^0(s)\big)\,ds\right)\xi\,d\sigma$$
$$+\int_\Omega\mathbf{a}\big(x,Du(t)\big)\cdot D\big(H\big(u(t)\big)\xi\big)\,dx$$
$$=\int_\Omega f(t)H\big(u(t)\big)\xi\,dx+\int_{\partial\Omega}g(t)H\big(u(t)\big)\xi\,d\sigma \tag{5}$$

in $\mathcal{D}'(]0,T[)$, for any $\xi\in C^1(\bar\Omega)$ and any Lipschitz continuous function $H:\mathbb{R}\to\mathbb{R}$ of compact support, and

$$\lim_{n\to+\infty}\int_{\{(t,x)\in Q_T:\ n\leqslant|u(t,x)|\leqslant n+1\}}\mathbf{a}(x,Du)\cdot Du\,dx\,dt=0. \tag{6}$$

Remark 2.5. (i) In (5) and (6) every term is well defined. Observe that the third term of the left-hand side of (5) has to be understood as

$$\int_\Omega\mathbf{a}(x,Du)\cdot D\big(H(u)\xi\big)\,dx=\int_\Omega\mathbf{a}\big(x,DT_M(u)\big)\cdot D\big(H\big(T_M(u)\big)\xi\big)\,dx,$$

where $M>0$ is such that $\operatorname{supp}(H)\subset[-M,M]$. Similarly, the integral in (6) has to be understood as

$$\int_{\{(t,x)\in Q_T:\ n\leqslant|u(t,x)|\leqslant n+1\}}\mathbf{a}\big(x,DT_{n+1}(u)\big)\cdot DT_{n+1}(u)\,dx\,dt.$$

(ii) A renormalized solution satisfies

$$\int_\Omega z(t)\,dx+\int_{\partial\Omega}w(t)\,d\sigma=\int_\Omega z_0\,dx+\int_{\partial\Omega}w_0\,d\sigma+\int_0^t\left(\int_\Omega f(s)\,dx+\int_{\partial\Omega}g(s)\,d\sigma\right)ds \tag{7}$$

for all $t\in[0,T]$.

(iii) A weak solution in the sense of Definition 2.2 is a renormalized solution. In fact, if (z,w) is a weak solution of $P_{\gamma,\beta}(f,g,z_0,w_0)$ in $[0,T]$, then there exists $u\in L^p(0,T;W^{1,p}(\Omega))$ such that $z\in\gamma(u)$ a.e. in Q_T, $w\in\beta(u)$ a.e. on S_T and

$$\int_0^T \int_\Omega \mathbf{a}(x, Du) \cdot D\psi \, dx \, dt = \int_0^T \int_\Omega z\psi_t \, dx \, dt + \int_0^T \int_{\partial\Omega} w\psi_t \, d\sigma \, dt + \int_0^T \int_\Omega f\psi \, dx \, dt + \int_0^T \int_{\partial\Omega} g\psi \, d\sigma \, dt \tag{8}$$

for any $\psi \in W^{1,1}(0, T; W^{1,1}(\Omega) \cap L^\infty(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$, $\psi(0) = \psi(T) = 0$. Then, by Lemma 2.1, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \int_0^{z(t)} H\big((\gamma^{-1})^0(s)\big)\, ds \, \psi_t \, dx \, dt + \int_0^T \int_{\partial\Omega} \int_0^{w(t)} H\big((\beta^{-1})^0(s)\big)\, ds \, \psi_t \, d\sigma \, dt \\
&\quad = \int_0^T \int_\Omega \mathbf{a}(x, Du) \cdot D\big(H(u)\psi\big)\, dx \, dt \\
&\quad\quad - \int_0^T \int_\Omega f H(u)\psi \, dx \, dt - \int_0^T \int_{\partial\Omega} g H(u)\psi \, d\sigma \, dt
\end{aligned} \tag{9}$$

for any $H : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous of compact support and $\psi(t, x) = \varphi(t)\xi(x)$, with $\varphi \in \mathcal{D}(]0, T[)$ and $\xi \in C^1(\bar{\Omega})$. Hence (5) holds. Moreover, since $u \in L^p(0, T; W^{1,p}(\Omega))$, (6) also holds, and consequently (z, w) is a renormalized solution of $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0, T]$.

(iv) If u is a renormalized solution such that $u \in L^p(0, T; W^{1,p}(\Omega))$, u is a weak solution in the sense of Definition 2.2.

The main result of this paper is the following existence and uniqueness theorem.

Theorem 2.6. *Assume* $\mathrm{Dom}(\gamma) = \mathbb{R}$ *and assume either* $\mathrm{Dom}(\beta) = \mathbb{R}$ *or* $\mathbf{a}$ *smooth. Let* $T > 0$.

(i) *Let* $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ *and* $w_0 \in L^1(\partial\Omega)$ *such that*

$$\gamma_- \leqslant z_0 \leqslant \gamma_+, \qquad \beta_- \leqslant w_0 \leqslant \beta_+ \tag{10}$$

and

$$\int_\Omega z_0 \, dx + \int_{\partial\Omega} w_0 \, d\sigma + \int_0^t \bigg(\int_\Omega f(s)\, dx + \int_{\partial\Omega} g(s)\, d\sigma \bigg) ds \in \mathcal{R}_{\gamma,\beta} \tag{11}$$

for all $t \in [0, T]$. *Then, there exists a unique renormalized solution* (z, w) *of* $P_{\gamma,\beta}(f, g, z_0, w_0)$ *in* $[0, T]$.

(ii) *Moreover, the following* L^1*-contraction principle holds. For* $i = 1, 2$, *let* $f_i \in L^1(0, T; L^1(\Omega))$, $g_i \in L^1(0, T; L^1(\partial\Omega))$, $z_{i0} \in L^1(\Omega)$ *and* $w_{i0} \in L^1(\partial\Omega)$; *and let* (z_i, w_i) *be a renormalized solution in* $[0, T]$ *of* $P_{\gamma,\beta}(f_i, g_i, z_{i0}, w_{i0})$, $i = 1, 2$. *Then*

$$\begin{aligned}
&\int_{\Omega} \big(z_1(t) - z_2(t)\big)^+ \, dx + \int_{\partial\Omega} \big(w_1(t) - w_2(t)\big)^+ \, d\sigma \\
&\quad \leqslant \int_{\Omega} (z_{10} - z_{20})^+ \, dx + \int_{\partial\Omega} (w_{10} - w_{20})^+ \, d\sigma \\
&\qquad + \int_0^t \int_{\Omega} \big(f_1(s) - f_2(s)\big)^+ \, dx \, ds + \int_0^t \int_{\partial\Omega} \big(g_1(s) - g_2(s)\big)^+ \, d\sigma \, ds
\end{aligned} \tag{12}$$

for every $t \in [0, T]$.

To prove the above theorem we use the nonlinear semigroup theory (see [12,14] or [24]).

Remark 2.7. We recall that in the case $\beta = 0$, for the Laplacian operator and γ the multivalued Heaviside function (i.e., for the Hele–Shaw problem), existence and uniqueness of weak solutions for this problem is known to be true only if

$$\int_{\Omega} z_0 \, dx + \int_0^t \left(\int_{\Omega} f(s) \, dx + \int_{\partial\Omega} g(s) \, d\sigma \right) ds \in \big(0, |\Omega|\big) \quad \text{for any } t \in [0, T]$$

(see [36, Theorem 3.1 and Example 8.1], see also [31]). The same example works for renormalized solutions, so condition (11) is necessary.

3. Existence of renormalized solutions

In this section we prove the existence part of Theorem 2.6. We use the following lemma proved in [5, Lemma 4.2].

Lemma 3.1. *(See [5].) Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\Omega)$, $\{z_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$, $\{w_n\}_{n\in\mathbb{N}} \subset L^1(\partial\Omega)$ such that, for every $n \in \mathbb{N}$, $z_n \in \gamma(u_n)$ a.e. in Ω and $w_n \in \beta(u_n)$ a.e. in $\partial\Omega$. Let us suppose that*

(i) *if $\mathcal{R}^+_{\gamma,\beta} = +\infty$, there exists $M > 0$ such that*

$$\int_{\Omega} z_n^+ \, dx + \int_{\partial\Omega} w_n^+ \, d\sigma < M \quad \forall n \in \mathbb{N};$$

(ii) *if $\mathcal{R}^+_{\gamma,\beta} < +\infty$, there exists $M \in \mathbb{R}$ such that*

$$\int_{\Omega} z_n \, dx + \int_{\partial\Omega} w_n \, d\sigma < M < \mathcal{R}^+_{\gamma,\beta}$$

and

$$\lim_{L\to+\infty} \left(\int_{\{x\in\Omega:\ z_n(x) < -L\}} |z_n| \, dx + \int_{\{x\in\partial\Omega:\ w_n(x) < -L\}} |w_n| \, d\sigma \right) = 0$$

uniformly in $n \in \mathbb{N}$.

Then, there exists a constant $C = C(M)$ such that

$$\|u_n^+\|_{L^p(\Omega)} \leqslant C\big(\|Du_n^+\|_{L^p(\Omega)} + 1\big) \quad \forall n \in \mathbb{N}.$$

Proof of Theorem 2.6. *(Existence)* We divide the proof in several steps.

Step 1: Approximate problems. For $f \in L^1(0,T;L^1(\Omega))$, $g \in L^1(0,T;L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$ satisfying (10) and (11), let

$$f_{m,n} = \sup\big\{\inf\{m, f\}, -n\big\}, \qquad g_{m,n} = \sup\big\{\inf\{m, g\}, -n\big\},$$
$$z_{0m,n} = \sup\big\{\inf\{m, z_0\}, -n\big\} \quad \text{and} \quad w_{0m,n} = \sup\big\{\inf\{m, w_0\}, -n\big\},$$

where $m, n \in \mathbb{N}$, and consider the approximate problems

$$P_{\gamma,\beta}(f_{m,n}, g_{m,n}, z_{0m,n}, w_{0m,n}).$$

It is clear that for m, n large enough, $f_{m,n}, g_{m,n}, z_{0m,n}, w_{0m,n}$ satisfy (2)–(4), in fact, there exist $r_1, r_2 \in \mathbb{R}$ such that, for any m, n large enough and any $t \in [0,T]$,

$$\mathcal{R}^-_{\gamma,\beta} < r_1 \leqslant \int_\Omega z_{0m,n}\,dx + \int_{\partial\Omega} w_{0m,n}\,d\sigma + \int_0^t \left(\int_\Omega f_{m,n}(s)\,dx + \int_{\partial\Omega} g_{m,n}(s)\,d\sigma\right) ds \leqslant r_2 < \mathcal{R}^+_{\gamma,\beta}. \tag{13}$$

Therefore, by Theorem 2.3, there exists a unique weak solution $(z_{m,n}, w_{m,n})$ of problem $P_{\gamma,\beta}(f_{m,n}, g_{m,n}, z_{0m,n}, w_{0m,n})$, so there exists $u_{m,n} \in L^p(0,T,W^{1,p}(\Omega))$ such that $z_{m,n} \in \gamma(u_{m,n})$ a.e. in $\Omega \times]0,T[$, $w_{m,n} \in \beta(u_{m,n})$ a.e. in $\partial\Omega \times]0,T[$, and

$$\int_0^T\int_\Omega \mathbf{a}(x, Du_{m,n}) \cdot D\psi\,dx\,dt = \int_0^T\int_\Omega z_{m,n}\psi_t\,dx\,dt + \int_0^T\int_{\partial\Omega} w_{m,n}\psi_t\,d\sigma\,dt + \int_0^T\int_\Omega f_{m,n}\psi\,dx\,dt + \int_0^T\int_{\partial\Omega} g_{m,n}\psi\,d\sigma\,dt \tag{14}$$

for any $\psi \in W^{1,1}(0,T;W^{1,1}(\Omega) \cap L^\infty(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$, $\psi(0) = \psi(T) = 0$. Since $f_{m,n}$, $g_{m,n}$, $z_{0m,n}$ and $w_{0m,n}$ are monotone nondecreasing in m and monotone nonincreasing in n, by results of [4] and [5], we can also consider that so are $u_{m,n}$, $z_{m,n}$ and $w_{m,n}$. Therefore, there exists a subsequence $\{n(m)\}_m$ such that

$$\lim_m (z_{m,n(m)}, w_{m,n(m)}) = (z, w) \quad \text{a.e. in } Q_T \times S_T, \tag{15}$$

$$\lim_m u_{m,n(m)} = u \quad \text{a.e. in } Q_T, \tag{16}$$

and

$$\lim_m u_{m,n(m)} = v \quad \text{a.e. on } S_T, \tag{17}$$

where $z(t,x), w(t,x), u(t,x), v(t,x) \in \bar{\mathbb{R}}$. Let us write

$$\begin{aligned} z_m &= z_{m,n(m)}, \qquad w_m = w_{m,n(m)}, \\ u_m &= u_{m,n(m)}, \\ f_m &= f_{m,n(m)}, \qquad g_m = g_{m,n(m)}, \\ z_{0m} &= z_{0m,n(m)} \quad \text{and} \quad w_{0m} = w_{0m,n(m)}. \end{aligned} \tag{18}$$

Step 2: Convergence of z_m, w_m. Let us see that

$$\lim_m (z_m, w_m) = (z, w) \quad \text{in } C\big([0,T]; X\big), \tag{19}$$

where $X = L^1(\Omega) \times L^1(\partial\Omega)$ provided with the natural norm

$$\big\|(f,g)\big\| := \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}, \quad (f,g) \in X.$$

Observe that then $z(0) = z_0$ and $w(0) = w_0$ also hold.

Consider the operator $\mathcal{B}^{\gamma,\beta}$ defined in X by $(\hat{z}, \hat{w}) \in \mathcal{B}^{\gamma,\beta}(z,w)$ if and only if there exists $u \in W^{1,p}(\Omega)$ such that $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv\, dx = \int_\Omega \hat{z} v\, dx + \int_{\partial\Omega} \hat{w} v\, d\sigma \tag{20}$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. By results in [4] and [5], we know that the abstract Cauchy problem in X,

$$\begin{cases} V'(t) + \mathcal{B}^{\gamma,\beta}\big(V(t)\big) \ni (f,g), & t \in (0,T), \\ V(0) = (z_0, w_0), \end{cases} \tag{21}$$

has a unique mild solution for any $f \in L^1(0,T; L^1(\Omega))$, $g \in L^1(0,T; L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$ satisfying (10) and (11). Moreover, under the hypothesis of Theorem 2.3, in [5] it is proved that the mild solution of problem (21) is the unique weak solution of $P_{\gamma,\beta}(f, g, z_0, w_0)$.

Therefore, (z_m, w_m) is the mild solution of problem (21) for data (f_m, g_m) and (z_{0m}, w_{0m}). Since $(f_m, g_m) \to (f,g)$ in $L^1(0,T; X)$ and $(z_{0m}, w_{0m}) \to (z_0, w_0)$ in X, by the nonlinear semigroup theory, there exists $\lim_m (z_m, w_m)$ in $C([0,T]; X)$ and by (15), (19) holds, being (z,w) the mild solution of (21) for data (f,g) and (z_0, w_0). We shall see that (z,w) is, in fact, a renormalized solution of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$.

Step 3: Boundedness of $T_k(u_m)$. Let us see there exists $C_1 > 0$ such that, for any $k > 0$,

$$\int_0^T \int_\Omega \big|DT_k(u_m)\big|^p\, dx\, dt \leqslant \frac{k}{\lambda}\big(\big\|(f,g)\big\|_{L^1(0,T;X)} + \big\|(z_0, w_0)\big\|_X\big) \tag{22}$$

and

$$\left\| T_k\left(\left(u_m(t)\right)^{\pm}\right)\right\|_{L^p(\Omega)} \leqslant C_1\left(\left\| DT_k\left(\left(u_m(t)\right)^{\pm}\right)\right\|_{L^p(\Omega)} + 1\right) \quad \forall t \in [0, T]. \tag{23}$$

By Lemma 2.1, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \int_0^{z_m(t)} G\left(\left(\gamma^{-1}\right)^0(s)\right) ds\, \psi_t\, dx\, dt + \int_0^T \int_{\partial\Omega} \int_0^{w_m(t)} G\left(\left(\beta^{-1}\right)^0(s)\right) ds\, \psi_t\, d\sigma\, dt \\
&\quad = \int_0^T \int_\Omega \mathbf{a}(x, Du_m) \cdot D\left(G(u_m)\psi\right) dx\, dt \\
&\qquad - \int_0^T \int_\Omega f_m G(u_m)\psi\, dx\, dt - \int_0^T \int_{\partial\Omega} g_m G(u_m)\psi\, d\sigma\, dt
\end{aligned} \tag{24}$$

for any bounded function of bounded variation $G(r)$ such that $G(u_m) \in L^p(0, T; W^{1,p}(\Omega))$ and for any $\psi \in \mathcal{D}(]0, T[\times\mathbb{R}^N)$. Taking in (24) $\psi(t, x) = \varphi(t)$, $\varphi \in \mathcal{D}(]0, T[)$, and $G(r) = T_k(r)$, $k \geqslant 0$, we get

$$\begin{aligned}
&\int_0^T \varphi_t \int_\Omega \int_0^{z_m(t)} T_k\left(\left(\gamma^{-1}\right)^0(s)\right) ds\, dx\, dt + \int_0^T \varphi_t \int_{\partial\Omega} \int_0^{w_m(t)} T_k\left(\left(\beta^{-1}\right)^0(s)\right) ds\, d\sigma\, dt \\
&\quad = \int_0^T \varphi \int_\Omega \mathbf{a}(x, Du_m) \cdot DT_k(u_m)\, dx\, dt \\
&\qquad - \int_0^T \varphi \int_\Omega f_m T_k(u_m)\, dx\, dt - \int_0^T \varphi \int_{\partial\Omega} g_m T_k(u_m)\, d\sigma\, dt.
\end{aligned} \tag{25}$$

Therefore

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \int_0^{z_m(t)} T_k\left(\left(\gamma^{-1}\right)^0(s)\right) ds\, dx + \frac{d}{dt} \int_{\partial\Omega} \int_0^{w_m(t)} T_k\left(\left(\beta^{-1}\right)^0(s)\right) ds\, d\sigma \\
&\qquad + \int_\Omega \mathbf{a}\left(x, Du_m(t)\right) \cdot DT_k\left(u_m(t)\right) dx \\
&\quad = \int_\Omega f_m(t) T_k\left(u_m(t)\right) dx + \int_{\partial\Omega} g_m(t) T_k\left(u_m(t)\right) d\sigma
\end{aligned} \tag{26}$$

in $\mathcal{D}'(]0, T[)$. Integrating (26) from 0 to T and using (H_1), we get (22).

In order to prove (23) we need to treat separately different cases. In the case $\mathcal{R}^+_{\gamma,\beta} = +\infty$, let

$$M = \sup_{t\in[0,T]} \left(\int_\Omega z^+(t)\,dx + \int_{\partial\Omega} w^+(t)\,d\sigma \right) + 1.$$

Then, by (19) there exists $m_0 \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \left(\int_\Omega (z_m)^+(t)\,dx + \int_{\partial\Omega} (w_m)^+(t)\,d\sigma \right) < M \quad \forall m \geqslant m_0.$$

In the case $\mathcal{R}^+_{\gamma,\beta} < +\infty$, by (13) (see Remark 2.5(ii) and (iii)), there exist $M \in \mathbb{R}$ and $m_0 \in \mathbb{N}$ such that, for all $m \geqslant m_0$,

$$\sup_{t\in[0,T]} \left(\int_\Omega z_m(t)\,dx + \int_{\partial\Omega} w_m(t)\,d\sigma \right) < M < \mathcal{R}^+_{\gamma,\beta}.$$

Moreover, by (19),

$$\lim_{L\to+\infty} \left(\int_{\{x\in\Omega:\ z_m(t)(x)<-L\}} \left|z_m(t)\right|dx + \int_{\{x\in\partial\Omega:\ w_m(t)(x)<-L\}} \left|w_m(t)\right|d\sigma \right) = 0$$

uniformly in $m \in \mathbb{N}$ and $t \in [0,T]$.

Let us define

$$z_m^k = \begin{cases} z_m & \text{if } |u_m| < k, \\ \gamma^0(k) & \text{if } u_m \geqslant k, \\ \gamma^0(-k) & \text{if } u_m \leqslant -k, \end{cases}$$

and

$$w_m^k = \begin{cases} w_m & \text{if } |u_m| < k, \\ \beta^0(k) & \text{if } u_m \geqslant k, \\ \beta^0(-k) & \text{if } u_m \leqslant -k. \end{cases}$$

Then $z_m^k \in \gamma(T_k(u_m))$ a.e. in Q_T and $w_m^k \in \beta(T_k(u_m))$ a.e. in S_T. Now, in the case $\mathcal{R}^+_{\gamma,\beta} = +\infty$, there exists $M \in \mathbb{R}$ such that, for all $k > 0$,

$$\sup_{t\in[0,T]} \left(\int_\Omega \left(z_m^k\right)^+(t)\,dx + \int_{\partial\Omega} \left(w_m^k\right)^+(t)\,d\sigma \right) < M \quad \forall m \geqslant m_0.$$

And in the case $\mathcal{R}^+_{\gamma,\beta} < +\infty$, there exist $M < \mathcal{R}^+_{\gamma,\beta}$ and k_0 such that, for all $k \geqslant k_0$ and for all $m \geqslant m_0$,

$$\sup_{t\in[0,T]} \left(\int_\Omega z_m^k(t)\,dx + \int_{\partial\Omega} w_m^k(t)\,d\sigma \right) < M$$

and

$$\lim_{L\to+\infty}\left(\int_{\{x\in\Omega:\ z_m^k(t)(x)<-L\}}\left|z_m^k(t)\right|dx+\int_{\{x\in\partial\Omega:\ w_m^k(t)(x)<-L\}}\left|w_m^k(t)\right|d\sigma\right)=0$$

uniformly in $m,k\in\mathbb{N}$ and $t\in[0,T]$. Therefore, by Lemma 3.1, (23) follows for the positive part of u_m. For the negative part of u_m we use again Lemma 3.1 for $\hat{u}_m=-u_m$, $\hat{z}_m=-z_m$, $\hat{w}_m=-w_m$ and the graphs $\hat{\gamma}(r)=-\gamma(-r)$ and $\hat{\beta}(r)=-\beta(-r)$.

Step 4: Convergence of $T_k(u_m)$. In this step we show that

$$u\quad\text{is finite a.e. in } Q_T, \tag{27}$$

$$u(t)\in\mathcal{T}_\tau^{1,p}(\Omega)\quad\text{a.e. } t\in\,]0,T[, \tag{28}$$

$$z\in\gamma(u)\quad\text{a.e. in } Q_T, \tag{29}$$

$$w\in\beta(u)\quad\text{a.e. in } S_T, \tag{30}$$

and, for any $k\in\mathbb{N}$,

$$\begin{aligned} T_k(u_m)\text{ converges to } T_k(u)\qquad &\text{weakly in } L^p\big(0,T;W^{1,p}(\Omega)\big),\\ &\text{strongly in } L^p\big(0,T;L^p(\Omega)\big) \end{aligned} \tag{31}$$

and

$$T_k(u_m)\text{ converges to } T_k(u)\quad\text{in } L^p\big(0,T;L^p(\partial\Omega)\big). \tag{32}$$

Indeed, having in mind (22) and (23),

$$\begin{aligned}\mathcal{L}^{N+1}\big(\big\{(t,x)\in Q_T:\ u_m^{\pm}(t,x)\geqslant k\big\}\big)&\leqslant\int_0^T\int_\Omega\frac{|T_k((u_m(t))^{\pm})|^p}{k^p}\,dx\,dt\\ &\leqslant\frac{C_2}{k^p}\int_0^T\big(1+\big\|DT_k\big((u_m(t))^{\pm}\big)\big\|_{L^p(\Omega)}^p\big)\,dt\leqslant\frac{C_3}{k^p}(1+k).\end{aligned}$$

This implies, taking limits first as m goes to $+\infty$ and after as k goes to $+\infty$, that (27) holds. Hence, again by (22), (31) and (32) hold for any $k>0$, and consequently $u(t)\in\mathcal{T}^{1,p}(\Omega)$ a.e. $t\in\,]0,T[$.

Similarly, since

$$
\begin{aligned}
&\big(\mathcal{L}^1 \times \mathcal{H}^{N-1}\big)\big(\big\{(t,x) \in S_T\colon u_m^{\pm}(t,x) \geqslant k\big\}\big)\\
&\quad\leqslant \int_0^T \int_{\partial\Omega} \frac{|T_k((u_m)^{\pm})|^p}{k^p}\, d\sigma\, dt\\
&\quad\leqslant \frac{C_4}{k^p} \int_0^T \big(\big\| T_k\big((u_m)^{\pm}\big)\big\|_{L^p(\Omega)} + \big\| DT_k\big((u_m)^{\pm}\big)\big\|_{L^p(\Omega)}\big)^p\, dt \leqslant \frac{C_5}{k^p}(1+k),
\end{aligned}
$$

v is measurable in S_T, and (28) holds.

Finally, by (15)–(17), (27) and (28) and the facts that

$$
\begin{aligned}
&z_m \in \gamma(u_m) \quad \text{a.e. in } Q_T,\\
&w_m \in \beta(u_m) \quad \text{a.e. in } S_T,
\end{aligned}
$$

and γ and β are maximal monotone graphs, (29) and (30) hold.

Step 5: Uniform renormalized condition for u_m. Let us define

$$
\nu(n) := \sup_m \int_{\{(t,x)\in Q_T\colon n<|u_m(t,x)|<n+1\}} \mathbf{a}(x, Du_m)\cdot Du_m\, dx\, dt.
$$

Then

$$
\lim_n \nu(n) = 0. \tag{33}
$$

In order to prove (33) we take in (26) $k = n+1$ and after $k = n$. Subtracting the corresponding equalities and integrating from 0 to T, we get

$$
\begin{aligned}
0 &\leqslant \int_{\{(t,x)\in Q_T\colon n<|u_m(t,x)|<n+1\}} \mathbf{a}(x, Du_m)\cdot Du_m\, dx\, dt\\
&= -\int_\Omega \int_{z_{0m}}^{z_m(T)} G_n\big(\big(\gamma^{-1}\big)^0(s)\big)\, ds\, dx - \int_{\partial\Omega} \int_{w_{0m}}^{w_m(T)} G_n\big(\big(\beta^{-1}\big)^0(s)\big)\, ds\, d\sigma\\
&\quad + \int_0^T \int_\Omega f_m G_n(u_m)\, dx\, dt + \int_0^T \int_{\partial\Omega} g_m G_n(u_m)\, d\sigma\, dt,
\end{aligned} \tag{34}
$$

where $G_n(r) := T_{n+1}(r) - T_n(r)$. Therefore, since

$$
\lim_{n\to+\infty} \mathcal{L}^{N+1}\big(\big\{(t,x) \in Q_T\colon \big|u_m(t,x)\big| \geqslant n\big\}\big) = 0
$$

and

$$\lim_{n\to+\infty}\left(\mathcal{L}^1\times\mathcal{H}^{N-1}\right)\left(\left\{(t,x)\in S_T\colon\ \left|u_m(t,x)\right|\geqslant n\right\}\right)=0$$

uniformly in m, by equiintegrability, the two last terms on the right-hand side of equality (34) go to zero as n goes to $+\infty$. For the first term on the right-hand side of (34) we have

$$-\int_\Omega\int_{z_{0m}}^{z_m(T)} G_n\left(\left(\gamma^{-1}\right)^0(s)\right)ds\,dx$$
$$\leqslant\int_\Omega\left(z_0^+-\sup\gamma(n)\right)^+dx+\int_\Omega\left(\inf\gamma(-n)-z_0^-\right)^-d\sigma,$$

which converges to zero by (10). Similarly, we can handle with the second term on the right-hand side of (34) and the proof of (33) is concluded.

Step 6: Convergence of $\mathbf{a}(x,DT_k(u_m))$. Let us see that

$$\mathbf{a}\left(x,DT_k(u_m)\right)\rightharpoonup\mathbf{a}\left(x,DT_k(u)\right)\quad\text{weakly in } L^{p'}(Q_T)\text{ as } m\to+\infty. \tag{35}$$

Let $n\in\mathbb{N}$, $n>k$. Given any subsequence of u_m, by (22) and (H_2), there exists a subsequence, still denoted by u_m, such that

$$\mathbf{a}\left(x,DT_k(u_m)\right)\rightharpoonup\Phi_k\quad\text{weakly in }\left(L^{p'}(Q_T)\right)^N, \tag{36}$$

$$\mathbf{a}\left(x,DT_{n+1}(u_m)\right)\rightharpoonup\Phi_{n+1}\quad\text{weakly in }\left(L^{p'}(Q_T)\right)^N \tag{37}$$

and

$$\mathbf{a}\left(x,DT_{n+1}(u_m)\right)\chi_{\{|u_m|>k\}}\rightharpoonup\Psi_{n+1,k}\quad\text{weakly in }\left(L^{p'}(Q_T)\right)^N \tag{38}$$

as $m\to+\infty$. Let us prove that, for any $\varphi\in\mathcal{D}(]0,T[)$, $0\leqslant\varphi\leqslant1$,

$$\lim_{m\to+\infty}\int_0^T\varphi\int_\Omega\mathbf{a}(x,Du_m)\cdot DT_k(u_m)\,dx\,dt\leqslant\int_0^T\varphi\int_\Omega\Phi_k\cdot DT_k(u)\,dx\,dt. \tag{39}$$

Then, by Minty–Browder's method, it is easy to see that

$$\Phi_k=\mathbf{a}\left(x,DT_k(u)\right), \tag{40}$$

and (35) is proved.

Now, in order to get (39), we take limit in (25) to obtain

$$\lim_{m\to+\infty}\int_0^T \varphi \int_\Omega \mathbf{a}(x, Du_m)\cdot DT_k(u_m)\,dx\,dt$$
$$= \int_0^T \varphi_t \int_\Omega \int_0^{z(t)} T_k\big(\big(\gamma^{-1}\big)^0(s)\big)\,ds\,dx\,dt + \int_0^T \varphi_t \int_{\partial\Omega} \int_0^{w(t)} T_k\big(\big(\beta^{-1}\big)^0(s)\big)\,ds\,d\sigma\,dt$$
$$+ \int_0^T \varphi \int_\Omega f T_k(u)\,dx\,dt + \int_0^T \varphi \int_{\partial\Omega} g T_k(u)\,d\sigma\,dt. \tag{41}$$

Consequently, it is enough to prove that

$$\int_0^T \varphi \int_\Omega \Phi_k\cdot DT_k(u)\,dx\,dt$$
$$\geqslant \int_0^T \varphi_t \int_\Omega \int_0^{z(t)} T_k\big(\big(\gamma^{-1}\big)^0(s)\big)\,ds\,dx\,dt + \int_0^T \varphi_t \int_{\partial\Omega} \int_0^{w(t)} T_k\big(\big(\beta^{-1}\big)^0(s)\big)\,ds\,d\sigma\,dt$$
$$+ \int_0^T \varphi \int_\Omega f T_k(u)\,dx\,dt + \int_0^T \varphi \int_{\partial\Omega} g T_k(u)\,d\sigma\,dt. \tag{42}$$

To this end we use the regularization method of Landes [39]. For $k, \nu \in \mathbb{N}$, we define the regularization in time of the function $T_k(u)$ given by

$$\big(T_k(u)\big)_\nu(t,x) := \nu \int_{-\infty}^t e^{\nu(s-t)} T_k\big(u(s,x)\big)\,ds,$$

extending $T_k(u)$ by 0 for $s<0$. Observe that $(T_k(u))_\nu \in L^p(0,T;W^{1,p}(\Omega))\cap L^\infty(Q)$, it is differentiable for a.e. $t\in(0,T)$ with

$$\big|\big(T_k(u)\big)_\nu(t,x)\big| \leqslant k\big(1-e^{-\nu t}\big) < k \quad \text{a.e.},$$
$$\frac{\partial (T_k(u))_\nu}{\partial t} = \nu\big(T_k(u) - \big(T_k(u)\big)_\nu\big) \in L^p\big(0,T;W^{1,p}(\Omega)\big)\cap L^\infty(Q),$$
$$\big(T_k(u)\big)_\nu(0,x) = 0 \quad \forall x\in\Omega,$$
$$\lim_{\nu\to\infty}\big(T_k(u)\big)_\nu = T_k(u) \quad \text{in } L^p\big(0,T;W^{1,p}(\Omega)\big) \text{ and in } L^p\big(]0,T[\times\partial\Omega\big)$$

and, moreover,

$$\int_0^T \varphi \int_\Omega \Phi_k\cdot DT_k(u)\,dx\,dt = \lim_{\nu\to\infty}\lim_{m\to\infty}\int_0^T \varphi \int_\Omega \mathbf{a}\big(x, DT_k(u_m)\big)\cdot D\big(T_k(u)\big)_\nu\,dx\,dt.$$

By considering $H_n(r) = \inf(1, (n+1-|r|)^+)$, for any $n > k$, which implies $H_n(u_m) = 1$ if $|u_m| \leqslant k$, we have

$$
\begin{aligned}
&\int_0^T \varphi \int_\Omega \Phi_k \cdot DT_k(u)\,dx\,dt \\
&\quad = \lim_{\nu\to\infty}\lim_{m\to\infty} \int_0^T \varphi \int_\Omega \mathbf{a}\big(x, DT_k(u_m)\big) \cdot D\big(H_n(u_m)\big(T_k(u)\big)_\nu\big)\,dx\,dt \\
&\quad = \lim_{\nu\to\infty}\lim_{m\to\infty} \left(\int_0^T \varphi \int_\Omega \mathbf{a}\big(x, DT_{n+1}(u_m)\big) \cdot D\big(H_n(u_m)\big(T_k(u)\big)_\nu\big)\,dx\,dt \right. \\
&\qquad \left. - \int_{\{(t,x)\in Q_T:\ k<|u_m(t,x)|\leqslant n+1\}} \varphi \mathbf{a}\big(x, DT_{n+1}(u_m)\big) \cdot D\big(H_n(u_m)\big(T_k(u)\big)_\nu\big)\,dx\,dt \right) \\
&\quad = \lim_{\nu\to\infty}\lim_{m\to\infty} \left(\int_0^T \varphi \int_\Omega \mathbf{a}\big(x, DT_{n+1}(u_m)\big) \cdot D\big(H_n(u_m)\big(T_k(u)\big)_\nu\big)\,dx\,dt \right. \\
&\qquad - \int_{\{(t,x)\in Q_T:\ k<|u_m(t,x)|\leqslant n+1\}} \varphi \mathbf{a}(x, Du_m) \cdot D\big(\big(T_k(u)\big)_\nu\big) H_n(u_m)\,dx\,dt \\
&\qquad \left. - \int_{\{(t,x)\in Q_T:\ k<|u_m(t,x)|\leqslant n+1\}} \varphi \mathbf{a}(x, Du_m) \cdot Du_m H_n'(u_m)\big(T_k(u)\big)_\nu\,dx\,dt \right) \\
&\quad = \lim_{\nu\to\infty}\lim_{m\to\infty} \left(\int_0^T \varphi \int_\Omega \mathbf{a}\big(x, DT_{n+1}(u_m)\big) \cdot D\big(H_n(u_m)\big(T_k(u)\big)_\nu\big)\,dx\,dt \right. \\
&\qquad - \int_0^T \varphi \int_\Omega \mathbf{a}\big(x, DT_{n+1}(u_m)\big) \cdot D\big(\big(T_k(u)\big)_\nu\big) H_n(u_m) \chi_{\{(t,x)\in Q_T:\ |u_m(t,x)|>k\}}\,dx\,dt \\
&\qquad \left. - \int_{\{(t,x)\in Q_T:\ k<|u_m(t,x)|\leqslant n+1\}} \varphi \mathbf{a}(x, Du_m) \cdot Du_m H_n'(u_m)\big(T_k(u)\big)_\nu\,dx\,dt \right).
\end{aligned}
$$

Since $|(T_k(u))_\nu| = k(1 - e^{-\nu t})$ in $\{(t,x) \in Q_T:\ |u(t,x)| \geqslant k\}$, having in mind (38), we get

$$\lim_{\nu\to\infty}\lim_{m\to\infty}\int_0^T\varphi\int_\Omega \mathbf{a}\big(x, DT_{n+1}(u_m)\big)\cdot D\big(T_k(u)\big)_\nu H_n(u_m)\chi_{\{(t,x)\in Q_T:\ |u_m(t,x)|>k\}}\,dx\,dt$$

$$=\lim_{\nu\to\infty}\lim_{m\to\infty}\int_0^T\varphi\bigg[\int_\Omega \mathbf{a}\big(x, DT_{n+1}(u_m)\big)\cdot D\big(T_k(u)\big)_\nu$$

$$\times H_n(u_m)\chi_{\{(t,x)\in Q_T:\ |u_m(t,x)|>k\}}\chi_{\{(t,x)\in Q_T:\ |u(t,x)|<k\}}\,dx\bigg]dt=0.$$

Hence, for any $n>k$, we have

$$\int_0^T\varphi\int_\Omega \Phi_k\cdot DT_k(u)\,dx\,dt$$

$$=\lim_{\nu\to\infty}\lim_{m\to\infty}\Bigg(\int_0^T\varphi\int_\Omega \mathbf{a}\big(x, DT_{n+1}(u_m)\big)\cdot D\big(H_n(u_m)\big(T_k(u)\big)_\nu\big)\,dx\,dt$$

$$-\int_{\{(t,x)\in Q_T:\ k<|u_m(t,x)|\leqslant n+1\}}\varphi\mathbf{a}(x, Du_m)\cdot Du_m H_n'(u_m)\big(T_k(u)\big)_\nu\,dx\,dt\Bigg). \tag{43}$$

Now,

$$\left|\int_{\{(t,x)\in Q_T:\, k<|u_m(t,x)|\leqslant n+1\}}\varphi\mathbf{a}(x, Du_m)\cdot Du_m H_n'(u_m)\big(T_k(u)\big)_\nu\,dx\,dt\right|\leqslant k\nu(n),$$

thus

$$-\int_{\{(t,x):\ k<|u_m(t,x)|\leqslant n+1\}}\varphi\mathbf{a}(x, Du_m)\cdot Du_m H_n'(u_m)\big(T_k(u)\big)_\nu\,dx\,dt\geqslant -k\nu(n),$$

so that, by (43) and (33), we get

$$\int_0^T\varphi\int_\Omega \Phi_k\cdot DT_k(u)\,dx\,dt$$

$$\geqslant \liminf_{n\to\infty}\liminf_{\nu\to\infty}\liminf_{m\to\infty}\int_0^T\varphi\int_\Omega \mathbf{a}\big(x, DT_{n+1}(u_m)\big)\cdot D\big(H_n(u_m)\big(T_k(u)\big)_\nu\big)\,dx\,dt. \tag{44}$$

Since H_n is a bounded function of bounded variation, from (24), by approximation of $(T_k(u))_\nu\varphi$, we deduce

$$\int_0^T \varphi \int_\Omega \mathbf{a}\big(x, DT_{n+1}(u_m)\big) \cdot D\big(H_n(u_m)\big(T_k(u)\big)_\nu\big)\, dx\, dt$$

$$= \int_0^T \int_\Omega \mathbf{a}(x, Du_m) \cdot D\big(H_n(u_m)\big(T_k(u)\big)_\nu \varphi\big)\, dx\, dt$$

$$= \int_0^T \int_\Omega b_n^\gamma(z_m)\big(\varphi\big(T_k(u)\big)_\nu\big)_t\, dx\, dt + \int_0^T \int_{\partial\Omega} b_n^\beta(w_m)\big(\varphi\big(T_k(u)\big)_\nu\big)_t\, d\sigma\, dt$$

$$+ \int_0^T \int_\Omega f_m H_n(u_m)\varphi\big(T_k(u)\big)_\nu\, dx\, dt + \int_0^T \int_{\partial\Omega} g_m H_n(u_m)\varphi\big(T_k(u)\big)_\nu\, d\sigma\, dt, \tag{45}$$

where

$$b_n^\gamma(r) = \int_0^r H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\, ds \quad \text{for } r \in \mathrm{Ran}(\gamma), \tag{46}$$

and

$$b_n^\beta(r) = \int_0^r H_n\big(\big(\beta^{-1}\big)^0(s)\big)\, ds \quad \text{for } r \in \mathrm{Ran}(\beta). \tag{47}$$

Letting $m \to \infty$ in (45), we have

$$\lim_{m\to\infty} \int_0^T \int_\Omega \mathbf{a}(x, Du_m) \cdot D\big(H_n(u_m)\big(T_k(u)\big)_\nu \varphi\big)\, dx\, dt$$

$$= \int_0^T \int_\Omega b_n^\gamma(z)\big(\varphi\big(T_k(u)\big)_\nu\big)_t\, dx\, dt + \int_0^T \int_{\partial\Omega} b_n^\beta(w)\big(\varphi\big(T_k(u)\big)_\nu\big)_t\, d\sigma\, dt$$

$$+ \int_0^T \int_\Omega f H_n(u)\varphi\big(T_k(u)\big)_\nu\, dx\, dt + \int_0^T \int_{\partial\Omega} g H_n(u)\varphi\big(T_k(u)\big)_\nu\, d\sigma\, dt.$$

For the first term on the right-hand side, using the fact that $(T_k(u))_\nu = T_k((T_k(u))_\nu)$, $z \in \gamma(u)$, the monotonicity of b_n^γ and the integration by parts formula, we get

$$
\begin{aligned}
&\int_0^T \int_\Omega b_n^\gamma(z)\big(\varphi\big(T_k(u)\big)_\nu\big)_t \, dx\, dt \\
&\quad = \int_0^T \int_\Omega b_n^\gamma(z)\varphi_t\big(T_k(u)\big)_\nu \, dx\, dt + \nu \int_0^T \int_\Omega b_n^\gamma(z)\varphi\big(T_k(u) - \big(T_k(u)\big)_\nu\big)\, dx\, dt \\
&\quad \geqslant \int_0^T \int_\Omega b_n^\gamma(z)\varphi_t\big(T_k(u)\big)_\nu \, dx\, dt + \nu \int_0^T \int_\Omega b_n^\gamma\big(\gamma^0\big(\big(T_k(u)\big)_\nu\big)\big)\varphi\big(T_k(u) - \big(T_k(u)\big)_\nu\big)\, dx\, dt \\
&\quad = \int_0^T \int_\Omega b_n^\gamma(z)\varphi_t\big(T_k(u)\big)_\nu \, dx\, dt + \int_0^T \int_\Omega b_n^\gamma\big(\gamma^0\big(\big(T_k(u)\big)_\nu\big)\big)\big(\big(T_k(u)\big)_\nu\big)_t \varphi \, dx\, dt \\
&\quad = \int_0^T \int_\Omega b_n^\gamma(z)\varphi_t\big(T_k(u)\big)_\nu \, dx\, dt - \int_0^T \int_\Omega \int_0^{(T_k(u))_\nu} b_n^\gamma\big(\gamma^0(s)\big)\, ds\, \varphi_t \, dx\, dt.
\end{aligned}
$$

Now, letting $\nu \to \infty$, we deduce that

$$
\begin{aligned}
&\liminf_{\nu\to\infty} \int_0^T \int_\Omega b_n^\gamma(z)\big(\varphi\big(T_k(u)\big)_\nu\big)_t \, dx\, dt \\
&\quad \geqslant \int_0^T \int_\Omega b_n^\gamma(z)\varphi_t T_k(u)\, dx\, dt - \int_0^T \int_\Omega \int_0^{T_k(u)} b_n^\gamma\big(\gamma^0(s)\big)\, ds\, \varphi_t \, dx\, dt,
\end{aligned}
$$

so that,

$$
\begin{aligned}
&\liminf_{n\to\infty} \liminf_{\nu\to\infty} \liminf_{m\to\infty} \int_0^T \int_\Omega b_n^\gamma(z_m)\big(\varphi\big(T_k(u)\big)_\nu\big)_t \, dx\, dt \\
&\quad \geqslant \int_0^T \int_\Omega \varphi_t z T_k(u)\, dx\, dt - \int_0^T \int_\Omega \int_0^{T_k(u)} \gamma^0(s)\, ds\, \varphi_t \, dx\, dt.
\end{aligned}
$$

Using the fact that, since $z \in \gamma(u)$,

$$
zT_k(u) - \int_0^{T_k(u)} \gamma^0(s)\, ds = \int_0^z T_k\big(\big(\gamma^{-1}\big)^0(s)\big)\, ds,
$$

we obtain that

$$\liminf_{n\to\infty}\liminf_{\nu\to\infty}\liminf_{m\to\infty}\int_0^T\int_\Omega b_n^\gamma(z_m)\big(\varphi\big(T_k(u)\big)_\nu\big)_t\,dx\,dt$$
$$\geqslant\int_0^T\varphi_t\int_\Omega\int_0^{z(t)}T_k\big(\big(\gamma^{-1}\big)^0(s)\big)\,ds\,dx\,dt.$$

In the same way, we get that

$$\liminf_{n\to\infty}\liminf_{\nu\to\infty}\liminf_{m\to\infty}\int_0^T\int_{\partial\Omega} b_n^\beta(w_m)\big(\varphi\big(T_k(u)\big)_\nu\big)_t\,d\sigma\,dt$$
$$\geqslant\int_0^T\varphi_t\int_{\partial\Omega}\left(\int_0^{w(t)}T_k\big(\big(\beta^{-1}\big)^0(s)\big)\,ds\right)d\sigma\,dt.$$

Then, passing to the limit in (45), by (44), (42) follows.

Step 7: Passing to the limit. In this step we see that

$$\frac{d}{dt}\int_\Omega\left(\int_0^{z(t)}H\big(\big(\gamma^{-1}\big)^0(s)\big)\,ds\right)\xi\,dx+\frac{d}{dt}\int_{\partial\Omega}\left(\int_0^{w(t)}H\big(\big(\beta^{-1}\big)^0(s)\big)\,ds\right)\xi\,d\sigma$$
$$+\int_\Omega\mathbf{a}\big(x,Du(t)\big)\cdot D\big(H\big(u(t)\big)\xi\big)\,dx$$
$$=\int_\Omega f(t)H\big(u(t)\big)\xi\,dx+\int_{\partial\Omega}g(t)H\big(u(t)\big)\xi\,d\sigma \tag{48}$$

in $\mathcal{D}'(]0,T[)$.

By Step 6, for any $\varphi\in\mathcal{D}(]0,T[)$, $0\leqslant\varphi\leqslant1$, we have

$$\lim_{m\to+\infty}\int_0^T\varphi\int_\Omega\big(\mathbf{a}\big(x,DT_k(u_m)\big)-\mathbf{a}\big(x,DT_k(u)\big)\big)\cdot\big(DT_k(u_m)-DT_k(u)\big)\,dx\,dt=0.$$

Then, we can suppose, extracting a subsequence if necessary, that

$$\varphi\big(\mathbf{a}\big(x,DT_k(u_m)\big)-\mathbf{a}\big(x,DT_k(u)\big)\big)\cdot\big(DT_k(u_m)-DT_k(u)\big)\to0 \tag{49}$$

in $L^1(Q_T)$, a.e. in Q_T, and is dominated in $L^1(Q_T)$.

Taking in (24) $G(r)=H(r)$, being $H:\mathbb{R}\to\mathbb{R}$ a Lipschitz continuous function of compact support, and $\psi(t,x)=\varphi(t)\xi(x)$, $\varphi\in\mathcal{D}(]0,T[)$ and $\xi\in C^1(\bar\Omega)$, we have

$$\int_0^T \int_\Omega \left(\int_0^{z_m(t)} H\big((\gamma^{-1})^0(s)\big)\,ds \right) \xi \varphi_t \,dx\,dt + \int_0^T \int_{\partial\Omega} \left(\int_0^{w_m(t)} H\big((\beta^{-1})^0(s)\big)\,ds \right) \xi \varphi_t \,d\sigma\,dt$$
$$= \int_0^T \varphi \int_\Omega \mathbf{a}(x, Du_m) \cdot D\big(H(u_m)\xi\big)\,dx\,dt$$
$$- \int_0^T \varphi \int_\Omega f_m H(u_m)\xi \,dx\,dt - \int_0^T \varphi \int_{\partial\Omega} g_m(t) H(u_m)\xi \,d\sigma\,dt. \tag{50}$$

Now, if $\operatorname{supp}(H) \subset [-M, M]$,

$$\int_0^T \varphi \int_\Omega \mathbf{a}(x, Du_m) \cdot D\big(H(u_m\xi)\big)\,dx\,dt$$
$$= \int_0^T \varphi \int_\Omega \mathbf{a}\big(x, DT_M(u_m)\big) \cdot D\big(H\big(T_M(u_m)\big)\xi\big)\,dx\,dt$$
$$= \int_0^T \varphi \int_\Omega H\big(T_M(u_m)\big)\mathbf{a}\big(x, DT_M(u_m)\big) \cdot D\xi \,dx\,dt$$
$$+ \int_0^T \varphi \int_\Omega \xi H'\big(T_M(u_m)\big)\mathbf{a}\big(x, DT_M(u_m)\big) \cdot DT_M(u_m)\,dx\,dt$$
$$= \int_0^T \varphi \int_\Omega H\big(T_M(u_m)\big)\mathbf{a}\big(x, DT_M(u_m)\big) \cdot D\xi \,dx\,dt$$
$$+ \int_0^T \varphi \bigg[\int_\Omega \xi H'\big(T_M(u_m)\big)\big(\mathbf{a}\big(x, DT_M(u_m)\big) - \mathbf{a}\big(x, DT_M(u)\big)\big)$$
$$\times \big(DT_M(u_m) - DT_M(u)\big)\,dx \bigg]\,dt$$
$$+ \int_0^T \varphi \int_\Omega \xi H'\big(T_M(u_m)\big)\mathbf{a}\big(x, DT_M(u_m)\big) \cdot DT_M(u)\,dx\,dt$$
$$+ \int_0^T \varphi \int_\Omega \xi H'\big(T_M(u_m)\big)\mathbf{a}\big(x, DT_M(u)\big) \cdot DT_M(u_m)\,dx\,dt$$
$$- \int_0^T \varphi \int_\Omega \xi H'\big(T_M(u_m)\big)\mathbf{a}\big(x, DT_M(u)\big) \cdot DT_M(u)\,dx\,dt.$$

Since by approximation we can assume H to be smooth, by (35) and (49), we get

$$\begin{aligned}
&\lim_{m\to\infty} \int_0^T \varphi \int_\Omega \mathbf{a}(x, Du_m) \cdot D\big(H(u_m)\xi\big)\, dx\, dt \\
&= \int_0^T \varphi \int_\Omega H\big(T_M(u)\big)\mathbf{a}\big(x, DT_M(u)\big) \cdot D\xi\, dx\, dt \\
&\quad + \int_0^T \varphi \int_\Omega \xi H'\big(T_M(u)\big)\mathbf{a}\big(x, DT_M(u)\big) \cdot DT_M(u)\, dx\, dt \\
&= \int_0^T \varphi \int_\Omega \mathbf{a}(x, Du) \cdot D\big(H(u)\xi\big)\, dx\, dt.
\end{aligned}$$

Consequently, taking limit in (50) as $m \to \infty$, (48) follows.

Step 8: Renormalized condition. Let us see finally that

$$\lim_{n\to+\infty} \int_{\{(t,x)\in Q_T:\ n\leqslant|u(t,x)|\leqslant n+1\}} \mathbf{a}(x, Du) \cdot Du\, dx\, dt = 0. \tag{51}$$

By (49), we have

$$\begin{aligned}
&\lim_{m\to+\infty} \int_s^{T-s} \int_\Omega \mathbf{a}\big(x, DT_k(u_m)\big) \cdot DT_k(u_m)\, dx\, dt \\
&\quad = \int_s^{T-s} \int_\Omega \mathbf{a}\big(x, DT_k(u)\big) \cdot DT_k(u)\, dx\, dt
\end{aligned} \tag{52}$$

for any $0 < s < T/2$. Taking now in (52) $k = n+1$, $k = n$ and subtracting the resulting equalities, for any $0 < s < T/2$,

$$\begin{aligned}
&\lim_m \int_s^{T-s} \int_{\{x\in\Omega:\ n\leqslant|u_m(t,x)|\leqslant n+1\}} \mathbf{a}\big(x, Du_m(t)\big) \cdot Du_m(t)\, dx\, dt \\
&= \int_s^{T-s} \int_{\{x\in\Omega:\ n\leqslant|u(t,x)|\leqslant n+1\}} \mathbf{a}\big(x, Du(t)\big) \cdot Du(t)\, dx\, dt.
\end{aligned}$$

Then, by the definition of $\nu(n)$,

$$\int_s^{T-s} \int_{\{x\in\Omega:\ n\leqslant|u(t,x)|\leqslant n+1\}} \mathbf{a}\big(x, Du(t)\big) \cdot Du(t)\,dx\,dt \leqslant \nu(n).$$

Therefore, taking limits as s goes to 0, and taking into account (33), (51) is proved.

With this last step the proof of the existence part of Theorem 2.6 is concluded. $\square$

Remark 3.2. Using (49), we can get, as in [18, Lemma 5], the strong convergence of $\{DT_k(u_m)\}_m$.

4. Uniqueness of renormalized solution

In this section we prove the uniqueness part of Theorem 2.6 using as main tool the concept of *integral solution* due to Ph. Bénilan (see [12,14]).

Definition 4.1. A function $V = (z, w) \in C([0, T]; X)$ is an *integral solution* of (21) in $[0, T]$, if for every $(\hat{f}, \hat{g}) \in \mathcal{B}^{\gamma,\beta}(\hat{z}, \hat{w})$ we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \big|z(t) - \hat{z}\big|\,dx + \frac{d}{dt}\int_{\partial\Omega} \big|w(t) - \hat{w}\big|\,d\sigma \\
&\quad\leqslant \int_\Omega \big(f(t) - \hat{f}\big)\operatorname{sign}_0\big(z(t) - \hat{z}\big)\,dx + \int_{\{x\in\Omega:\ z(t,x)=\hat{z}(x)\}} \big|f(t) - \hat{f}\big|\,dx \\
&\qquad + \int_{\partial\Omega} \big(g(t) - \hat{g}\big)\operatorname{sign}_0\big(w(t) - \hat{w}\big)\,d\sigma + \int_{\{x\in\partial\Omega:\ w(t,x)=\hat{w}(x)\}} \big|g(t) - \hat{g}\big|\,d\sigma
\end{aligned}$$

in $\mathcal{D}'(]0, T[)$, and $V(0) = (z_0, w_0)$.

Under the hypothesis $\mathrm{Dom}(\gamma) = \mathbb{R}$ and either $\mathrm{Dom}(\beta) = \mathbb{R}$ or $\mathbf{a}$ smooth, the operator $\mathcal{B}^{\gamma,\beta}$ (see Section 3) is accretive in X (see [4] and [5]). In [5, Theorem 3.6] the existence of mild solutions of problem (21) is proved under conditions (11) and (10). Now, mild solutions and integral solutions of problem (21) coincide (see [12,14]). In Theorem 4.3, we shall prove that a renormalized solution of $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0, T]$ is an integral solution of (21). Consequently, since in fact $\mathcal{B}^{\gamma,\beta}$ is T-accretive in X (see [4] and [5]), the contraction principle (12) follows by the nonlinear semigroup theory. Finally, under the assumptions of Theorem 2.6(i), the mild solution of (21) in $[0, T]$ is the unique renormalized solution of $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0, T]$.

We shall use the following integration by parts formula.

Lemma 4.2. *Let (z, w) be a renormalized solution of $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0, T]$. Let $k > 0$, $n \in \mathbb{N}$, $H_n(r) = \inf(1, (n + 1 - |r|)^+)$, and $h \in W^{1,p}(\Omega)$. Then,*

$$\frac{d}{dt}\int_{\Omega}\left(\int_0^{z(t)} H_n\big((\gamma^{-1})^0(s)\big)T_k\big((\gamma^{-1})^0(s)-h\big)\,ds\right)\psi\,dx$$
$$+\frac{d}{dt}\int_{\partial\Omega}\left(\int_0^{w(t)} H_n\big((\beta^{-1})^0(s)\big)T_k\big((\beta^{-1})^0(s)-h\big)\,ds\right)\psi\,d\sigma$$
$$+\int_{\Omega}\mathbf{a}\big(x,Du(t)\big)D\big(H_n\big(u(t)\big)T_k\big(u(t)-h\big)\psi\big)\,dx$$
$$=\int_{\Omega} f(t)H_n\big(u(t)\big)T_k\big(u(t)-h\big)\psi\,dx+\int_{\partial\Omega} g(t)H_n\big(u(t)\big)T_k\big(u(t)-h\big)\psi\,d\sigma$$

in $\mathcal{D}'(]0,T[)$, for any $\psi\in\mathcal{D}(\mathbb{R}^{\mathbb{N}})$, being u the function given in the definition of (z,w) as renormalized solution.

Proof. Let b_n^γ and b_n^β be defined as in (46) and (47), respectively. Since (z,w) is a renormalized solution of $P_{\gamma,\beta}(f,g,z_0,w_0)$ in $[0,T]$, for $\xi\in C^1(\bar{\Omega})$,

$$-\frac{d}{dt}\int_{\Omega} b_n^\gamma\big(z(t)\big)\xi\,dx-\frac{d}{dt}\int_{\partial\Omega} b_n^\beta\big(w(t)\big)\xi\,d\sigma$$
$$=\int_{\Omega}\mathbf{a}\big(x,Du(t)\big)\cdot D\big(H_n\big(u(t)\big)\xi\big)\,dx-\int_{\Omega} f(t)H_n\big(u(t)\big)\xi\,dx-\int_{\partial\Omega} g(t)H_n\big(u(t)\big)\xi\,d\sigma$$

in $\mathcal{D}'(]0,T[)$. Therefore, since $b_n^\gamma(z(t))\in(b_n^\gamma\circ\gamma)(u(t))$, $b_n^\beta(z(t))\in(b_n^\beta\circ\beta)(u(t))$, by Lemma 2.1, applied with $H(x,r)=T_k(r-T_m(h))$ and

$$\big(F(t),\xi\big)=\int_{\Omega}\mathbf{a}\big(x,T_{k+m+n}\big(u(t)\big)\big)D\xi\,dx,$$

$m\in\mathbb{N}$, it follows that

$$\int_0^T\int_{\Omega}\left(\int_0^{b_n^\gamma(z(t))} T_k\big(\big((b_n^\gamma\circ\gamma)^{-1}\big)^0(s)-T_m(h)\big)\,ds\right)\psi_t\,dx\,dt$$
$$+\int_0^T\int_{\partial\Omega}\left(\int_0^{b_n^\beta(w(t))} T_k\big(\big((b_n^\beta\circ\beta)^{-1}\big)^0(s)-T_m(h)\big)\,ds\right)\psi_t\,d\sigma\,dt$$
$$=\int_0^T\int_{\Omega}\mathbf{a}(x,Du)\cdot D\big(H_n(u)T_k\big(u-T_m(h)\big)\psi\big)\,dx\,dt$$

$$- \int_0^T \int_\Omega f H_n(u) T_k\big(u - T_m(h)\big)\psi \, dx \, dt$$
$$- \int_0^T \int_{\partial\Omega} g H_n(u) T_k\big(u - T_m(h)\big)\psi \, d\sigma \, dt$$

for any $\psi \in \mathcal{D}(]0, T[\times \mathbb{R}^N)$. Therefore, by the change of variables formula,

$$\begin{aligned}
&\int_0^T \int_\Omega \left(\int_0^{z(t)} H_n\big((\gamma^{-1})^0(s)\big) T_k\big(\big((b_n^\gamma \circ \gamma)^{-1}\big)^0\big(b_n^\gamma(s)\big) - T_m(h)\big)\, ds \right) \psi_t \, dx \, dt \\
&\quad + \int_0^T \int_{\partial\Omega} \left(\int_0^{w(t)} H_n\big((\beta^{-1})^0(s)\big) T_k\big(\big((b_n^\beta \circ \beta)^{-1}\big)^0\big(b_n^\beta(s)\big) - T_m(h)\big)\, ds \right) \psi_t \, d\sigma \, dt \\
&= \int_0^T \int_\Omega \mathbf{a}(x, Du) \cdot D\big(H_n(u) T_k\big(u - T_m(h)\big)\psi\big)\, dx \, dt \\
&\quad - \int_0^T \int_\Omega f H_n(u) T_k\big(u - T_m(h)\big)\psi \, dx \, dt - \int_0^T \int_{\partial\Omega} g H_n(u) T_k\big(u - T_m(h)\big)\psi \, d\sigma \, dt \qquad (53)
\end{aligned}$$

for any $\psi \in \mathcal{D}(]0, T[\times \mathbb{R}^N)$. Observe that

$$\begin{aligned}
&\int_0^{z(t)} H_n\big((\gamma^{-1})^0(s)\big) T_k\big(\big((b_n^\gamma \circ \gamma)^{-1}\big)^0\big(b_n^\gamma(s)\big) - T_m(h)\big)\, ds \\
&\quad = \int_0^{z(t)} H_n\big((\gamma^{-1})^0(s)\big) T_k\big((\gamma^{-1})^0(s) - T_m(h)\big)\, ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{w(t)} H_n\big((\beta^{-1})^0(s)\big) T_k\big(\big((b_n^\beta \circ \beta)^{-1}\big)^0\big(b_n^\beta(s)\big) - T_m(h)\big)\, ds \\
&\quad = \int_0^{w(t)} H_n\big((\beta^{-1})^0(s)\big) T_k\big((\beta^{-1})^0(s) - T_m(h)\big)\, ds.
\end{aligned}$$

Indeed, let us see, for example, that

$$H_n\big((\gamma^{-1})^0(s)\big)T_k\big(\big((b_n^\gamma\circ\gamma)^{-1}\big)^0\big(b_n^\gamma(s)\big)-T_m(h)\big)$$
$$=H_n\big((\gamma^{-1})^0(s)\big)T_k\big((\gamma^{-1})^0(s)-T_m(h)\big). \tag{54}$$

If $s=0$ then $(\gamma^{-1})^0(s)=0=((b_n^\gamma\circ\gamma)^{-1})^0(b_n^\gamma(s))$ and (54) holds. If $b_n^\gamma(s)=0$ and $s\neq 0$ then $H_n((\gamma^{-1})^0(s))=0$ and (54) also holds. If $b_n^\gamma(s)>0$ then $0\leqslant(\gamma^{-1})^0(s)\in(b_n^\gamma\circ\gamma)^{-1}(b_n^\gamma(s))$, and if $\alpha\in(b_n^\gamma\circ\gamma)^{-1}(b_n^\gamma(s))$ then there exists $c\in\gamma(\alpha)$ such that $b_n^\gamma(s)=b_n^\gamma(c)$; now, if $s\leqslant c$ it is easy to see that $(\gamma^{-1})^0(s)\leqslant\alpha$, so $(\gamma^{-1})^0(s)=((b_n^\gamma\circ\gamma)^{-1})^0(b_n^\gamma(s))$, and, if $s>c$ then $H_n((\gamma^{-1})^0(s))=0$, therefore in any case (54) holds. Similarly, if $b_n^\gamma(s)<0$, (54) is true.

Therefore, taking limit as m goes to $+\infty$ in (53) we finish the proof. □

To prove the following theorem we use a similar scheme to that used in the proof of Theorem 5.3 in [5]. Now here, we have to overcome the added difficulties due to the fact that for u we only know that its truncations are in $L^p(0,T;W^{1,p}(\Omega))$. In this sense the renormalized condition (6) plays a role.

Theorem 4.3. *Let (z,w) be a renormalized solution of $P_{\gamma,\beta}(f,g,z_0,w_0)$ in $[0,T]$. Let $(\hat f,\hat g)\in\mathcal{B}^{\gamma,\beta}(\hat z,\hat w)$. Then,*

$$\frac{d}{dt}\int_\Omega\big|z(t)-\hat z\big|\,dx+\frac{d}{dt}\int_{\partial\Omega}\big|w(t)-\hat w\big|\,d\sigma$$
$$\leqslant\int_\Omega\big(f(t)-\hat f\big)\operatorname{sign}_0\big(z(t)-\hat z\big)\,dx+\int_{\{x\in\Omega:\ z(t,x)=\hat z(x)\}}\big|f(t)-\hat f\big|\,dx$$
$$+\int_{\partial\Omega}\big(g(t)-\hat g\big)\operatorname{sign}_0\big(w(t)-\hat w\big)\,d\sigma+\int_{\{x\in\partial\Omega:\ w(t,x)=\hat w(x)\}}\big|g(t)-\hat g\big|\,d\sigma$$

in $\mathcal{D}'(]0,T[)$, that is, since $(z(0),w(0))=(z_0,w_0)$, (z,w) is an integral solution of (21) in $[0,T]$.

Proof. We divide the proof in three steps.

Step 1: Inequality inside Ω. Let H_n be as in Lemma 4.2, $\psi\in\mathcal{D}(\Omega)$, $0\leqslant\psi\leqslant 1$, $\rho\in W^{1,p}(\Omega)$, $-1\leqslant\rho\leqslant 1$. Given $(\hat f,\hat g)\in\mathcal{B}^{\gamma,\beta}(\hat z,\hat w)$ there exists $\hat u\in W^{1,p}(\Omega)$ such that $\hat z(x)\in\gamma(\hat u(x))$ a.e. in Ω, $\hat w(x)\in\beta(\hat u(x))$ a.e. in $\partial\Omega$, and

$$\int_\Omega\mathbf{a}(x,D\hat u)\cdot Dv\,dx=\int_\Omega\hat f v\,dx+\int_{\partial\Omega}\hat g v\,d\sigma$$

for all $v\in L^\infty(\Omega)\cap W^{1,p}(\Omega)$. Then, if u is the function given in the definition of (z,w) as renormalized solution, we have, for $0<\hat t\leqslant t<T$,

$$\begin{aligned}
&\int_\Omega \int_{z(\hat t)}^{z(t)} H_n\big((\gamma^{-1})^0(s)\big)\,\mathrm{sign}_0(s-\hat z)\psi\,ds\,dx \\
&\quad + \int_\Omega \int_{z(\hat t)}^{z(t)} H_n\big((\gamma^{-1})^0(s)\big)\big(\rho - \mathrm{sign}_0(s-\hat z)\big)\chi_{\{s:\,(\gamma^{-1})^0(s)=\hat u\}}\psi\,ds\,dx \\
&\quad + \int_{\hat t}^{t}\int_\Omega \big(\mathbf{a}\big(x,Du(s)\big)-\mathbf{a}(x,D\hat u)\big)\cdot Du(s)H_n'\big(u(s)\big)\,\mathrm{sign}_0\big(u(s)-\hat u\big)\psi\,dx\,ds \\
&\quad + \int_{\hat t}^{t}\int_\Omega \big(\mathbf{a}\big(x,Du(s)\big)-\mathbf{a}(x,D\hat u)\big)\cdot D\psi\, H_n\big(u(s)\big)\,\mathrm{sign}_0\big(u(s)-\hat u\big)\,dx\,ds \\
&\leqslant \int_{\hat t}^{t}\bigg[\int_\Omega \big(f(s)-\hat f\big)H_n\big(u(s)\big) \\
&\quad \times\big(\mathrm{sign}_0\big(z(s)-\hat z\big)+\mathrm{sign}_0\big(u(s)-\hat u\big)\chi_{\{x\in\Omega:\,z(s,x)=\hat z(x)\}}\big)\psi\,dx\bigg]ds \\
&\quad + \int_{\hat t}^{t}\int_\Omega \big(f(s)-\hat f\big)H_n\big(u(s)\big)\big(\rho-\mathrm{sign}_0\big(z(s)-\hat z\big)\big)\chi_{\{x\in\Omega:\,u(s,x)=\hat u(x)\}}\psi\,dx\,ds. \qquad (55)
\end{aligned}$$

In order to prove (55), let us take in Lemma 4.2, $h(x)=\hat u(x)-k\rho(x)$, $\rho\in W^{1,p}(\Omega)$, $-1\leqslant \rho\leqslant 1$, $k>0$. Then, for any $\psi\in\mathcal{D}(\Omega)$, $0\leqslant\psi\leqslant 1$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(\int_{\hat z}^{z(t)} H_n\big((\gamma^{-1})^0(s)\big)\frac{1}{k}T_k\big((\gamma^{-1})^0(s)-\hat u+k\rho\big)\,ds\right)\psi\,dx \\
&\qquad + \int_\Omega \big(\mathbf{a}\big(x,Du(t)\big)-\mathbf{a}(x,D\hat u)\big)\cdot D\left(H_n\big(u(t)\big)\frac{1}{k}T_k\big(u(t)-\hat u+k\rho\big)\psi\right)dx \\
&\quad = \int_\Omega \big(f(t)-\hat f\big)H_n\big(u(t)\big)\frac{1}{k}T_k\big(u(t)-\hat u+k\rho\big)\psi\,dx \qquad (56)
\end{aligned}$$

in $\mathcal{D}'(]0,T[)$. Integrating from $\hat t$ to t, $0<\hat t\leqslant t<T$,

$$\begin{aligned}
&\int_\Omega\int_{\hat z}^{z(t)} H_n\big((\gamma^{-1})^0(s)\big)\frac{1}{k}T_k\big((\gamma^{-1})^0(s)-\hat u+k\rho\big)\psi\,ds\,dx \\
&\quad - \int_\Omega\int_{\hat z}^{z(\hat t)} H_n\big((\gamma^{-1})^0(s)\big)\frac{1}{k}T_k\big((\gamma^{-1})^0(s)-\hat u+k\rho\big)\psi\,ds\,dx
\end{aligned}$$

$$+\int_{\hat{t}}^{t}\left[\int_{\Omega}\big(\mathbf{a}\big(x, Du(s)\big)-\mathbf{a}(x, D\hat{u})\big)D\bigg(H_n\big(u(s)\big)\frac{1}{k}T_k\big(u(s)-\hat{u}+k\rho\big)\psi\bigg)\,dx\right]ds$$

$$=\int_{\hat{t}}^{t}\int_{\Omega}\big(f(s)-\hat{f}\big)H_n\big(u(s)\big)\frac{1}{k}T_k\big(u(s)-\hat{u}+k\rho\big)\psi\,dx\,ds. \tag{57}$$

For the first term in (57), we take limit in k and use that

$$\lim_{k\to 0}\frac{1}{k}T_k(r-q+k\rho)=\operatorname{sign}_0(r-q)+\rho\chi_{\{r=q\}}\quad \forall -1\leqslant\rho\leqslant 1, \tag{58}$$

and

$$\begin{aligned}&\operatorname{sign}_0(r-q)+\operatorname{sign}_0(\hat{r}-\hat{q})\chi_{\{r=q\}}\\&\quad=\operatorname{sign}_0(\hat{r}-\hat{q})+\operatorname{sign}_0(r-q)\chi_{\{\hat{r}=\hat{q}\}}\quad \forall\hat{r}\in\gamma(r),\ \hat{q}\in\gamma(q),\end{aligned} \tag{59}$$

to obtain

$$\begin{aligned}
&\lim_{k\to 0}\int_{\Omega}\int_{\hat{z}}^{z(t)}H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\frac{1}{k}T_k\big(\big(\gamma^{-1}\big)^0(s)-\hat{u}+k\rho\big)\psi\,ds\,dx\\
&=\int_{\Omega}\int_{\hat{z}}^{z(t)}H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\big(\operatorname{sign}_0\big(\big(\gamma^{-1}\big)^0(s)-\hat{u}\big)+\rho\chi_{\{s:\,(\gamma^{-1})^0(s)=\hat{u}\}}\big)\psi\,ds\,dx\\
&=\int_{\Omega}\bigg[\int_{\hat{z}}^{z(t)}H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\big(\operatorname{sign}_0(s-\hat{z})+\big(\rho-\operatorname{sign}_0(s-\hat{z})\big)\chi_{\{s:\,(\gamma^{-1})^0(s)=\hat{u}\}}\\
&\quad+\operatorname{sign}_0\big(\big(\gamma^{-1}\big)^0(s)-\hat{u}\big)\chi_{\{s:\,s=\hat{z}\}}\big)\psi\,ds\bigg]dx\\
&=\int_{\Omega}\int_{\hat{z}}^{z(t)}H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\big(\operatorname{sign}_0(s-\hat{z})+\big(\rho-\operatorname{sign}_0(s-\hat{z})\big)\chi_{\{s:\,(\gamma^{-1})^0(s)=\hat{u}\}}\big)\psi\,ds\,dx\\
&=\int_{\Omega}\int_{\hat{z}}^{z(t)}H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\operatorname{sign}_0(s-\hat{z})\psi\,ds\,dx\\
&\quad+\int_{\Omega}\int_{\hat{z}}^{z(t)}H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\big(\rho-\operatorname{sign}_0(s-\hat{z})\big)\chi_{\{s:\,(\gamma^{-1})^0(s)=\hat{u}\}}\psi\,ds\,dx.
\end{aligned}$$

Similarly, for the second term in (57),

$$\lim_{k\to 0}\left(-\int_{\Omega}\int_{\hat z}^{z(\hat t)} H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\frac{1}{k}T_k\big(\big(\gamma^{-1}\big)^0(s)-\hat u+k\rho\big)\psi\,ds\,dx\right)$$

$$= -\int_{\Omega}\int_{\hat z}^{z(\hat t)} H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\operatorname{sign}_0(s-\hat z)\psi\,ds\,dx$$

$$-\int_{\Omega}\int_{\hat z}^{z(\hat t)} H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\big(\rho-\operatorname{sign}_0(s-\hat z)\big)\chi_{\{s:\,(\gamma^{-1})^0(s)=\hat u\}}\psi\,ds\,dx.$$

Let us now decompose the third term in (57) as $D_1(k,n)+D_2(k,n)+D_3(k,n)+D_4(k,n)$, where

$$D_1(k,n)=\int_{\hat t}^{t}\int_{\Omega}\big(\mathbf{a}\big(x,Du(s)\big)-\mathbf{a}(x,D\hat u)\big)\cdot Du(s)H_n'\big(u(s)\big)\frac{1}{k}T_k\big(u(s)-\hat u+k\rho\big)\psi\,dx\,ds,$$

$$D_2(k,n)=\int_{\hat t}^{t}\int_{\Omega}\big(\mathbf{a}\big(x,Du(s)\big)-\mathbf{a}(x,D\hat u)\big)\cdot D\psi\,H_n\big(u(s)\big)\frac{1}{k}T_k\big(u(s)-\hat u+k\rho\big)\,dx\,ds,$$

$$D_3(k,n)=\int_{\hat t}^{t}\left[\int_{\Omega}\big(\mathbf{a}\big(x,Du(s)\big)-\mathbf{a}(x,D\hat u)\big)\cdot\big(Du(s)-D\hat u\big)\right.$$
$$\left.\times H_n\big(u(s)\big)\frac{1}{k}T_k'\big(u(s)-\hat u+k\rho\big)\psi\,dx\right]ds$$

and

$$D_4(k,n)=\int_{\hat t}^{t}\int_{\Omega}\big(\mathbf{a}\big(x,Du(s)\big)-\mathbf{a}(x,D\hat u)\big)\cdot D\rho\,H_n\big(u(s)\big)T_k'\big(u(s)-\hat u+k\rho\big)\psi\,dx\,ds.$$

Now, by the Dominated Convergence's Theorem, and using that $Du(s)=D\hat u$ when $u(s)=\hat u$,

$$\lim_{k\to 0}D_1(k,n)=\int_{\hat t}^{t}\int_{\Omega}\big(\mathbf{a}\big(x,Du(s)\big)-\mathbf{a}(x,D\hat u)\big)\cdot Du(s)H_n'\big(u(s)\big)\operatorname{sign}_0\big(u(s)-\hat u\big)\psi\,dx\,ds,$$

$$\lim_{k\to 0}D_2(k,n)=\int_{\hat t}^{t}\int_{\Omega}\big(\mathbf{a}\big(x,Du(s)\big)-\mathbf{a}(x,D\hat u)\big)\cdot D\psi\,H_n\big(u(s)\big)\operatorname{sign}_0\big(u(s)-\hat u\big)\,dx\,ds,$$

$$\lim_{k\to 0}D_4(k,n)=0$$

and, using (H_3), since $\hat{t} \leqslant t$,

$$D_3(k,n) \geqslant 0.$$

Finally, for the fourth term in (57), using (58) and (59), we have that

$$\lim_{k\to 0}\int_{\hat{t}}^{t}\int_{\Omega}\big(f(s)-\hat{f}\big)H_n\big(u(s)\big)\frac{1}{k}T_k\big(u(s)-\hat{u}+k\rho\big)\psi\,dx\,ds$$
$$=\int_{\hat{t}}^{t}\int_{\Omega}\big(f(s)-\hat{f}\big)H_n\big(u(s)\big)\big(\operatorname{sign}_0\big(z(s)-\hat{z}\big)+\operatorname{sign}_0\big(u(s)-\hat{u}\big)\chi_{\{x\in\Omega:\ z(s,x)=\hat{z}(x)\}}\big)\psi\,dx\,ds$$
$$+\int_{\hat{t}}^{t}\int_{\Omega}\big(f(s)-\hat{f}\big)H_n\big(u(s)\big)\big(\rho-\operatorname{sign}_0\big(z(s)-\hat{z}\big)\big)\chi_{\{x\in\Omega:\ u(s,x)=\hat{u}(x)\}}\psi\,dx\,ds.$$

Hence, taking limit in (57) as k goes to 0, we get (55).

Step 2: Inequality up to $\partial\Omega$. For any $0<\hat{t}\leqslant t<T$,

$$\int_{\Omega}\big|z(t)-\hat{z}\big|\,dx-\int_{\Omega}\big|z(\hat{t})-\hat{z}\big|\,dx+\int_{\partial\Omega}\big|w(t)-\hat{w}\big|\,d\sigma-\int_{\partial\Omega}\big|w(\hat{t})-\hat{w}\big|\,d\sigma$$
$$\leqslant\int_{\hat{t}}^{t}\bigg[\int_{\Omega}\big(f(s)-\hat{f}\big)\big(\operatorname{sign}_0\big(z(s)-\hat{z}\big)+\operatorname{sign}_0\big(u(s)-\hat{u}\big)\chi_{\{x\in\Omega:\ z(s,x)=\hat{z}(x)\}}\big)\,dx\bigg]\,ds$$
$$+\int_{\hat{t}}^{t}\bigg[\int_{\Omega}\big(f(s)-\hat{f}\big)\big(\operatorname{sign}_0\big(z(t)-\hat{z}\big)-\operatorname{sign}_0\big(z(s)-\hat{z}\big)\big)\chi_{\{x\in\Omega:\ u(s,x)=\hat{u}(x)\}}\,dx\bigg]\,ds$$
$$+\int_{\hat{t}}^{t}\bigg[\int_{\partial\Omega}\big(g(s)-\hat{g}\big)\big(\operatorname{sign}_0\big(w(s)-\hat{w}\big)+\operatorname{sign}_0\big(u(s)-\hat{u}\big)\chi_{\{x\in\partial\Omega:\ w(s,x)=\hat{w}(x)\}}\big)\,d\sigma\bigg]\,ds$$
$$+\int_{\hat{t}}^{t}\bigg[\int_{\partial\Omega}\big(g(s)-\hat{g}\big)\big(\operatorname{sign}_0\big(w(t)-\hat{w}\big)-\operatorname{sign}_0\big(w(s)-\hat{w}\big)\big)\chi_{\{x\in\partial\Omega:\ u(s,x)=\hat{u}(x)\}}\,d\sigma\bigg]\,ds. \tag{60}$$

In fact, since in (55) there are no space derivatives of ρ, by approximation, we can take, for each t fixed, $\rho=\operatorname{sign}_0(z(t)-\hat{z})$. Then, by monotonicity of sign_0, the second term in (55) is positive and so, for any $0<\hat{t}\leqslant t<T$,

$$\begin{aligned}
&\int_\Omega \int_{z(\hat t)}^{z(t)} H_n\big((\gamma^{-1})^0(s)\big)\operatorname{sign}_0(s-\hat z)\psi\,ds\,dx \\
&\quad + \int_{\hat t}^{t} \bigg[\int_\Omega \big(\mathbf{a}\big(x, Du(s)\big) - \mathbf{a}(x, D\hat u)\big)\cdot Du(s) \\
&\quad \times H_n'\big(u(s)\big)\operatorname{sign}_0\big(u(s)-\hat u\big)\psi\,dx\bigg]\,ds + I \\
&\leqslant \int_{\hat t}^{t} \bigg[\int_\Omega \big(f(s)-\hat f\big)H_n\big(u(s)\big)\big(\operatorname{sign}_0\big(z(s)-\hat z\big) \\
&\quad + \operatorname{sign}_0\big(u(s)-\hat u\big)\chi_{\{x\in\Omega:\ z(s,x)=\hat z(x)\}}\big)\psi\,dx\bigg]\,ds \\
&\quad + \int_{\hat t}^{t} \bigg[\int_\Omega \big(f(s)-\hat f\big)H_n\big(u(s)\big)\big(\operatorname{sign}_0\big(z(t)-\hat z\big) \\
&\quad - \operatorname{sign}_0\big(z(s)-\hat z\big)\big)\chi_{\{x\in\Omega:\ u(s,x)=\hat u(x)\}}\psi\,dx\bigg]\,ds, \qquad (61)
\end{aligned}$$

where

$$\begin{aligned}
I &= \int_{\hat t}^{t} \int_\Omega \big(\mathbf{a}\big(x, Du(s)\big) - \mathbf{a}(x, D\hat u)\big)\cdot D\psi\, H_n\big(u(s)\big)\operatorname{sign}_0\big(u(s)-\hat u\big)\,dx\,ds \\
&= \int_{\hat t}^{t} \int_\Omega \big(\mathbf{a}\big(x, Du(s)\big) - \mathbf{a}(x, D\hat u)\big)\cdot D(\psi-1)H_n\big(u(s)\big)\operatorname{sign}_0\big(u(s)-\hat u\big)\,dx\,ds.
\end{aligned}$$

Now, for $\hat\rho \in W^{1,p}(\Omega)$, $-1 \leqslant \hat\rho \leqslant 1$, proceeding as in Step 1 and using the fact that $\psi - 1 = -1$ on $\partial\Omega$, we obtain that

$$\begin{aligned}
I \geqslant &-\int_\Omega \int_{z(\hat t)}^{z(t)} H_n\big((\gamma^{-1})^0(s)\big)\operatorname{sign}_0(s-\hat z)(\psi-1)\,ds\,dx \\
&- \int_\Omega \int_{z(\hat t)}^{z(t)} H_n\big((\gamma^{-1})^0(s)\big)\big(\hat\rho - \operatorname{sign}_0(s-\hat z)\big)\chi_{\{s:\ (\gamma^{-1})^0(s)=\hat u\}}(\psi-1)\,ds\,dx \\
&+ \int_{\partial\Omega} \int_{w(\hat t)}^{w(t)} H_n\big((\beta^{-1})^0(s)\big)\operatorname{sign}_0(s-\hat w)\,ds\,d\sigma
\end{aligned}$$

$$+\int_{\partial\Omega}\int_{w(\hat t)}^{w(t)} H_n\big(\big(\beta^{-1}\big)^0(s)\big)\big(\hat\rho-\operatorname{sign}_0(s-\hat w)\big)\chi_{\{s:\,(\beta^{-1})^0(s)=\hat u\}}\,ds\,d\sigma$$

$$+\int_{\hat t}^{t}\bigg[\int_{\Omega}\big(f(s)-\hat f\big)H_n\big(u(s)\big)$$

$$\times\big(\operatorname{sign}_0\big(z(s)-\hat z\big)+\operatorname{sign}_0\big(u(s)-\hat u\big)\chi_{\{x\in\Omega:\,z(s,x)=\hat z(x)\}}\big)(\psi-1)\,dx\bigg]ds$$

$$+\int_{\hat t}^{t}\int_{\Omega}\big(f(s)-\hat f\big)H_n\big(u(s)\big)\big(\hat\rho-\operatorname{sign}_0\big(z(s)-\hat z\big)\big)\chi_{\{x\in\Omega:\,u(s,x)=\hat u(x)\}}(\psi-1)\,dx\,ds$$

$$-\int_{\hat t}^{t}\bigg[\int_{\partial\Omega}\big(g(s)-\hat g\big)H_n\big(u(s)\big)$$

$$\times\big(\operatorname{sign}_0\big(w(s)-\hat w\big)+\operatorname{sign}_0\big(u(s)-\hat u\big)\chi_{\{x\in\partial\Omega:\,w(s,x)=\hat w(x)\}}\big)\,d\sigma\bigg]ds$$

$$-\int_{\hat t}^{t}\int_{\partial\Omega}\big(g(s)-\hat g\big)H_n\big(u(s)\big)\big(\hat\rho-\operatorname{sign}_0\big(w(s)-\hat w\big)\big)\chi_{\{x\in\partial\Omega:\,u(s,x)=\hat u(x)\}}\,d\sigma\,ds$$

$$-\int_{\hat t}^{t}\int_{\Omega}\big(\mathbf{a}\big(x,Du(s)\big)-\mathbf{a}(x,D\hat u)\big)\cdot Du(s)H_n'\big(u(s)\big)\operatorname{sign}_0\big(u(s)-\hat u\big)(\psi-1)\,dx\,ds.$$

Therefore, from (61) we get

$$\int_{\Omega}\int_{z(\hat t)}^{z(t)} H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\operatorname{sign}_0(s-\hat z)\,ds\,dx$$

$$-\int_{\Omega}\int_{z(\hat t)}^{z(t)} H_n\big(\big(\gamma^{-1}\big)^0(s)\big)\big(\hat\rho-\operatorname{sign}_0(s-\hat z)\big)\chi_{\{s:\,(\gamma^{-1})^0(s)=\hat u\}}(\psi-1)\,ds\,dx$$

$$+\int_{\partial\Omega}\int_{w(\hat t)}^{w(t)} H_n\big(\big(\beta^{-1}\big)^0(s)\big)\operatorname{sign}_0(s-\hat w)\,ds\,d\sigma$$

$$+\int_{\partial\Omega}\int_{w(\hat t)}^{w(t)} H_n\big(\big(\beta^{-1}\big)^0(s)\big)\big(\hat\rho-\operatorname{sign}_0(s-\hat w)\big)\chi_{\{s:\,(\beta^{-1})^0(s)=\hat u\}}\,ds\,d\sigma$$

$$\leqslant \int_{\hat{t}}^{t} \int_{\Omega} \big(f(s) - \hat{f}\big) H_n\big(u(s)\big)\big(\operatorname{sign}_0\big(z(s) - \hat{z}\big) + \operatorname{sign}_0\big(u(s) - \hat{u}\big)\chi_{\{x\in\Omega:\ z(s,x)=\hat{z}(x)\}}\big)\,dx\,ds$$

$$+ \int_{\hat{t}}^{t} \int_{\Omega} \big(f(s) - \hat{f}\big) H_n\big(u(s)\big)\big(\operatorname{sign}_0\big(z(t) - \hat{z}\big) - \operatorname{sign}_0\big(z(s) - \hat{z}\big)\big)\chi_{\{x\in\Omega:\ u(s,x)=\hat{u}(x)\}}\psi\,dx\,ds$$

$$- \int_{\hat{t}}^{t} \int_{\Omega} \big(f(s) - \hat{f}\big) H_n\big(u(s)\big)\big(\hat{\rho} - \operatorname{sign}_0\big(z(s) - \hat{z}\big)\big)\chi_{\{x\in\Omega:\ u(s,x)=\hat{u}(x)\}}(\psi - 1)\,dx\,ds$$

$$+ \int_{\hat{t}}^{t} \bigg[\int_{\partial\Omega} \big(g(s) - \hat{g}\big) H_n\big(u(s)\big)$$

$$\times \big(\operatorname{sign}_0\big(w(s) - \hat{w}\big) + \operatorname{sign}_0\big(u(s) - \hat{u}\big)\chi_{\{x\in\partial\Omega:\ w(s,x)=\hat{w}(x)\}}\big)\,d\sigma\bigg]\,ds$$

$$+ \int_{\hat{t}}^{t} \int_{\partial\Omega} \big(g(s) - \hat{g}\big) H_n\big(u(s)\big)\big(\hat{\rho} - \operatorname{sign}_0\big(w(s) - \hat{w}\big)\big)\chi_{\{x\in\partial\Omega:\ u(s,x)=\hat{u}(x)\}}\,d\sigma\,ds$$

$$+ \int_{\hat{t}}^{t} \int_{\Omega} \big(\mathbf{a}\big(x, Du(s)\big) - \mathbf{a}(x, D\hat{u})\big)\cdot Du(s) H_n'\big(u(s)\big)\operatorname{sign}_0\big(u(s) - \hat{u}\big)\,dx\,ds.$$

Letting now n go to $+\infty$, on account of (6), we obtain

$$\int_{\Omega} \big|z(t) - \hat{z}\big|\,dx - \int_{\Omega} \big|z(\hat{t}) - \hat{z}\big|\,dx - \int_{\Omega} \int_{z(\hat{t})}^{z(t)} \big(\hat{\rho} - \operatorname{sign}_0(s - \hat{z})\big)\chi_{\{s:\ (\gamma^{-1})^0(s)=\hat{u}\}}(\psi - 1)\,ds\,dx$$

$$+ \int_{\partial\Omega} \big|w(t) - \hat{w}\big|\,d\sigma - \int_{\partial\Omega} \big|w(\hat{t}) - \hat{w}\big|\,d\sigma$$

$$+ \int_{\partial\Omega} \int_{w(\hat{t})}^{w(t)} \big(\hat{\rho} - \operatorname{sign}_0(s - \hat{w})\big)\chi_{\{s:\ (\beta^{-1})^0(s)=\hat{u}\}}\,ds\,d\sigma$$

$$\leqslant \int_{\hat{t}}^{t} \bigg[\int_{\Omega} \big(f(s) - \hat{f}\big)\big(\operatorname{sign}_0\big(z(s) - \hat{z}\big) + \operatorname{sign}_0\big(u(s) - \hat{u}\big)\chi_{\{x\in\Omega:\ z(s,x)=\hat{z}(x)\}}\big)\,dx\bigg]\,ds$$

$$+ \int_{\hat{t}}^{t} \bigg[\int_{\Omega} \big(f(s) - \hat{f}\big)\big(\operatorname{sign}_0\big(z(t) - \hat{z}\big) - \operatorname{sign}_0\big(z(s) - \hat{z}\big)\big)\chi_{\{x\in\Omega:\ u(s,x)=\hat{u}(x)\}}\psi\,dx\bigg]\,ds$$

$$
\begin{aligned}
&-\int_{\hat t}^{t}\int_{\Omega}\big(f(s)-\hat f\big)\big(\hat\rho-\operatorname{sign}_0\big(z(s)-\hat z\big)\big)\chi_{\{x\in\Omega:\ u(s,x)=\hat u(x)\}}(\psi-1)\,dx\,ds\\
&+\int_{\hat t}^{t}\bigg[\int_{\partial\Omega}\big(g(s)-\hat g\big)\big(\operatorname{sign}_0\big(w(s)-\hat w\big)+\operatorname{sign}_0\big(u(s)-\hat u\big)\chi_{\{x\in\partial\Omega:\ w(s,x)=\hat w(x)\}}\big)\,d\sigma\bigg]\,ds\\
&+\int_{\hat t}^{t}\int_{\partial\Omega}\big(g(s)-\hat g\big)\big(\hat\rho-\operatorname{sign}_0\big(w(s)-\hat w\big)\big)\chi_{\{x\in\partial\Omega:\ u(s,x)=\hat u(x)\}}\,d\sigma\,ds. \qquad (62)
\end{aligned}
$$

Taking into (62) ψ_m instead of ψ such that $L^1(\Omega)\text{-}\lim_m \psi_m = 1$ and letting m go to $+\infty$, we have

$$
\begin{aligned}
&\int_{\Omega}\big|z(t)-\hat z\big|\,dx-\int_{\Omega}\big|z(\hat t)-\hat z\big|\,dx+\int_{\partial\Omega}\big|w(t)-\hat w\big|\,d\sigma-\int_{\partial\Omega}\big|w(\hat t)-\hat w\big|\,d\sigma\\
&\quad+\int_{\partial\Omega}\int_{w(\hat t)}^{w(t)}\big(\hat\rho-\operatorname{sign}_0(s-\hat w)\big)\chi_{\{s:\ (\beta^{-1})^0(s)=\hat u\}}\,ds\,d\sigma\\
&\leqslant\int_{\hat t}^{t}\bigg[\int_{\Omega}\big(f(s)-\hat f\big)\big(\operatorname{sign}_0\big(z(s)-\hat z\big)+\operatorname{sign}_0\big(u(s)-\hat u\big)\chi_{\{x\in\Omega:\ z(s,x)=\hat z(x)\}}\big)\,dx\bigg]\,ds\\
&\quad+\int_{\hat t}^{t}\bigg[\int_{\Omega}\big(f(s)-\hat f\big)\big(\operatorname{sign}_0\big(z(t)-\hat z\big)-\operatorname{sign}_0\big(z(s)-\hat z\big)\big)\chi_{\{x\in\Omega:\ u(s,x)=\hat u(x)\}}\,dx\bigg]\,ds\\
&\quad+\int_{\hat t}^{t}\bigg[\int_{\partial\Omega}\big(g(s)-\hat g\big)\big(\operatorname{sign}_0\big(w(s)-\hat w\big)+\operatorname{sign}_0\big(u(s)-\hat u\big)\chi_{\{x\in\partial\Omega:\ w(s,x)=\hat w(x)\}}\big)\,dx\bigg]\,ds\\
&\quad+\int_{\hat t}^{t}\int_{\partial\Omega}\big(g(s)-\hat g\big)\big(\hat\rho-\operatorname{sign}_0\big(w(s)-\hat w\big)\big)\chi_{\{x\in\partial\Omega:\ u(s,x)=\hat u(x)\}}\,d\sigma\,ds. \qquad (63)
\end{aligned}
$$

Now, by approximation, we can take, for each t fixed, $\hat\rho$ such that its trace is equal to $\operatorname{sign}_0(w(t)-\hat w)$. Then the fifth term in the above expression is positive and (60) follows.

Step 3: Integral solution. Let

$$
\varphi_1(t):=\int_{\Omega}\big|z(t)-\hat z\big|\,dx+\int_{\partial\Omega}\big|w(t)-\hat w\big|\,d\sigma,
$$

$$\varphi_2(s) := \int_\Omega \big(f(s) - \hat{f}\big)\big(\operatorname{sign}_0\big(z(s) - \hat{z}\big) + \operatorname{sign}_0\big(u(s) - \hat{u}\big)\chi_{\{x\in\Omega:\ z(s,x)=\hat{z}(x)\}}\big)\,dx$$
$$+ \int_{\partial\Omega} \big(g(s) - \hat{g}\big)\big(\operatorname{sign}_0\big(w(s) - \hat{w}\big) + \operatorname{sign}_0\big(u(s) - \hat{u}\big)\chi_{\{x\in\partial\Omega:\ w(s,x)=\hat{w}(x)\}}\big)\,d\sigma$$

and

$$\varphi_3(t,s) := \int_\Omega \big(f(s) - \hat{f}\big)\big(\operatorname{sign}_0\big(z(t) - \hat{z}\big) - \operatorname{sign}_0\big(z(s) - \hat{z}\big)\big)\chi_{\{x\in\Omega:\ u(s,x)=\hat{u}(x)\}}\,dx$$
$$+ \int_{\partial\Omega} \big(g(s) - \hat{g}\big)\big(\operatorname{sign}_0\big(w(t) - \hat{w}\big) - \operatorname{sign}_0\big(w(s) - \hat{w}\big)\big)\chi_{\{x\in\partial\Omega:\ u(s,x)=\hat{u}(x)\}}\,d\sigma.$$

Taking in (60) $\hat{t} = t - h$, $h > 0$, dividing by h and letting h go to 0, we get for any $\eta \in \mathcal{D}(]0,T[)$, $\eta \geqslant 0$,

$$\begin{aligned}
-\int_0^T \varphi_1(t)\eta_t(t)\,dt &= -\lim_{h\to 0^+}\int_0^T \varphi_1(t)\frac{\eta(t+h) - \eta(t)}{h}\,dt \\
&= \lim_{h\to 0^+}\int_0^T \frac{\varphi_1(t) - \varphi_1(t-h)}{h}\eta(t)\,dt \\
&\leqslant \lim_{h\to 0^+}\left(\int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_2(s)\,ds\right)\eta(t)\,dt + \int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_3(t,s)\,ds\right)\eta(t)\,dt\right).
\end{aligned} \tag{64}$$

By the Dominated Convergence Theorem,

$$\begin{aligned}
\lim_{h\to 0^+}\int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_2(s)\,ds\right)\eta(t)\,dt &= -\lim_{h\to 0^+}\int_0^T \left(\int_0^t \varphi_2(s)\,ds\right)\frac{\eta(t+h) - \eta(t)}{h}\,dt \\
&= -\int_0^T \left(\int_0^t \varphi_2(s)\,ds\right)\eta_t(t)\,dt = \int_0^T \varphi_2(t)\eta(t)\,dt.
\end{aligned}$$

On the other hand, for h small enough,

$$\int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_3(t,s)\,ds\right)\eta(t)\,dt = \int_0^T \frac{1}{h}\left(\int_s^{s+h} \varphi_3(t,s)\eta(t)\,dt\right)ds$$

and

$$\begin{aligned}
&\left| \int_0^T \frac{1}{h}\left(\int_s^{s+h} \varphi_3(t,s)\eta(t)\,dt \right) ds \right| \\
&\quad \leqslant \int_0^T \frac{1}{h}\left(\int_s^{s+h}\int_\Omega \left|f(s)-\hat{f}\right|\left|\operatorname{sign}_0\big(z(t)-\hat{z}\big)-\operatorname{sign}_0\big(z(s)-\hat{z}\big)\right|\eta(t)\,dx\,dt \right) ds \\
&\qquad + \int_0^T \frac{1}{h}\left(\int_s^{s+h}\int_{\partial\Omega} \left|g(s)-\hat{g}\right|\left|\operatorname{sign}_0\big(w(t)-\hat{w}\big)-\operatorname{sign}_0\big(w(s)-\hat{w}\big)\right|\eta(t)\,d\sigma\,dt \right) ds \\
&\quad \leqslant \|\eta\|_{L^\infty(0,T)} \int_0^T\int_\Omega \left|f(s)-\hat{f}\right| \frac{1}{h}\int_s^{s+h} \left|\operatorname{sign}_0\big(z(t)-\hat{z}\big)-\operatorname{sign}_0\big(z(s)-\hat{z}\big)\right|\,dt\,dx\,ds \\
&\qquad + \|\eta\|_{L^\infty(0,T)} \int_0^T \left[\int_{\partial\Omega} \left|g(s)-\hat{g}\right| \frac{1}{h}\int_s^{s+h} \left|\operatorname{sign}_0\big(w(t)-\hat{w}\big)-\operatorname{sign}_0\big(w(s)-\hat{w}\big)\right|\,dt\,d\sigma \right] ds.
\end{aligned}$$

Now, since $(t,x)\mapsto \operatorname{sign}_0(z(t,x)-\hat{z}(x)) \in L^1(Q_T)$, if we set

$$\varrho_h(s,x) = \frac{1}{h}\int_s^{s+h} \left|\operatorname{sign}_0\big(z(t,x)-\hat{z}(x)\big)-\operatorname{sign}_0\big(z(s,x)-\hat{z}(x)\big)\right|\,dt,$$

we have that

$$\lim_{h\to 0^+} \varrho_h(s,.) = 0 \quad \text{in } L^1(\Omega) \text{ a.e. } s\in[0,T].$$

Moreover,

$$\varrho_h(s,x) \leqslant 2 \quad \text{a.e. in } Q_T.$$

Consequently, applying twice the Dominated Convergence Theorem, we get

$$\lim_{h\to 0^+}\int_0^T\int_\Omega \left|f(s)-\hat{f}\right| \frac{1}{h}\int_s^{s+h} \left|\operatorname{sign}_0\big(z(t)-\hat{z}\big)-\operatorname{sign}_0\big(z(s)-\hat{z}\big)\right|\,dt\,dx\,ds = 0.$$

Similarly,

$$\lim_{h\to 0^+}\int_0^T\int_{\partial\Omega} \left|g(s)-\hat{g}\right| \frac{1}{h}\int_s^{s+h} \left|\operatorname{sign}_0\big(w(t)-\hat{w}\big)-\operatorname{sign}_0\big(w(s)-\hat{w}\big)\right|\,dt\,d\sigma\,ds = 0.$$

Therefore, from (64) we obtain that

$$\begin{aligned}&\frac{d}{dt}\int\limits_{\Omega}\left|z(t)-\hat{z}\right|dx+\frac{d}{dt}\int\limits_{\partial\Omega}\left|w(t)-\hat{w}\right|d\sigma\\&\quad\leqslant\int\limits_{\Omega}\bigl(f(t)-\hat{f}\bigr)\operatorname{sign}_0\bigl(z(t)-\hat{z}\bigr)\,dx+\int\limits_{\{x\in\Omega:\ z(t)=\hat{z}\}}\left|f(t)-\hat{f}\right|dx\\&\qquad+\int\limits_{\partial\Omega}\bigl(g(t)-\hat{g}\bigr)\operatorname{sign}_0\bigl(w(t)-\hat{w}\bigr)\,d\sigma+\int\limits_{\{x\in\partial\Omega:\ w(t)=\hat{w}\}}\left|g(t)-\hat{g}\right|d\sigma\end{aligned}$$

in $\mathcal{D}'(]0,T[)$, and the proof of Theorem 4.3 is finished. □

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