

# On the best Lipschitz extension problem for a discrete distance and the discrete $\infty$ -Laplacian

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## Abstract

This paper is concerned with the best Lipschitz extension problem for a discrete distance that counts the number of steps. We relate this absolutely minimizing Lipschitz extension with a discrete  $\infty$ -Laplacian problem, which arise as the dynamic programming formula for the value function of some  $\varepsilon$ -tug-of-war games. As in the classical case, we obtain the absolutely minimizing Lipschitz extension of a datum  $f$  by taking the limit as  $p \rightarrow \infty$  in a nonlocal  $p$ -Laplacian problem.

## Résumé

Dans cet article nous étudions le problème de la meilleur extension Lipschitz pour une distance discrète qui compte le nombre de pas. Nous rapportons cette extension Lipschitz absolument minimisante avec un problème  $\infty$ -Laplacien discrète qui découle de la formule de programmation dynamique pour la fonction valeur de certains  $\varepsilon$ -jeux de tir à la corde. Comme dans le cas classique nous obtenons l'extension Lipschitz absolument minimisante d'une donnée  $f$  en prenant la limite quand  $p \rightarrow +\infty$  dans un problème non local du type  $p$ -Laplacien.

*Key words:* Lipschitz extension, Nonlocal  $p$ -Laplacian problem, Infinity Laplacian, Tug-of-war games.

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## 1 Introduction

Since the classical work of Aronsson [6], in which he introduced the concept of absolutely minimizing Lipschitz extension and showed its relation with the infinity Laplace equation, a large amount of literature has appeared in this direction. For a systematic treatment of the theory of absolute minimizers see the recent survey [7] by Aronson, Cradall and Juutinen, and the references therein. A new insight has come in with the work of Peres, Schramm, Sheffield and Wilson [20] where it has been shown an interesting connection between absolutely minimizing Lipschitz extension and Game Theory. More precisely, the authors of [20] proved that if  $u_\varepsilon$  is the value function for a certain  $\varepsilon$ -tug-of-war game with final payoff function  $f$ , then the uniform limit  $u$  of  $u_\varepsilon$ , as  $\varepsilon$  goes to zero, is the absolutely minimizing Lipschitz extension of  $f$ .

In this work our aim is twofold, first we characterize the value function  $u_\varepsilon$  as the absolutely minimizing Lipschitz extension with respect to a discrete distance in a proper way, and next we show that  $u_\varepsilon$  can be obtained by taking the limit as  $p \rightarrow \infty$  in a nonlocal  $p$ -Laplacian Dirichlet problem with boundary data  $f$ .

Let  $(X, d)$  be an arbitrary metric space and let  $f : A \subset X \rightarrow \mathbb{R}$ . We denote by  $L_d(f, A)$  the smallest Lipschitz constant of  $f$  in  $A$ , i.e.,

$$L_d(f, A) := \sup_{x, y \in A} \frac{|f(x) - f(y)|}{d(x, y)}.$$

If we are given a Lipschitz function  $f : A \subset X \rightarrow \mathbb{R}$ , i.e.,  $L_d(f, A) < +\infty$ , then it is well-known that there exists a *minimal Lipschitz extension* (MLE for short) of  $f$  to  $X$ , that is, a function  $h : X \rightarrow \mathbb{R}$  such that  $h|_A = f$  and  $L_d(h, X) = L_d(f, A)$ . We will denote the space of such extensions as  $\text{MLE}(f, X)$ .

Extremal extensions were explicitly constructed by McShane [17] and Whitney [21],

$$\Psi(f)(x) := \inf_{y \in A} (f(y) + L_d(f, A)d(x, y)), \quad x \in X,$$

and

$$\Lambda(f)(x) := \sup_{y \in A} (f(y) - L_d(f, A)d(x, y)), \quad x \in X,$$

belong to  $\text{MLE}(f, X)$ , and if  $u \in \text{MLE}(f, X)$  then  $\Lambda(f) \leq u \leq \Psi(f)$ .

The notion of minimal Lipschitz extension is not completely satisfactory since it involves only the global Lipschitz constant of the extension and ignore what may happen locally. To solve this problem, in the particular case of the euclidean space  $\mathbb{R}^N$ , Aronsson [6] introduced the concept of *absolutely minimizing Lipschitz extension* (AMLE for short) and proved the existence of AMLE by means of a variant of the Perron's method. An extension of this concept to the case of a general metric space is due to Juutinen [11] (see also [18]). In [11], Juutinen gave the following definition.

**Definition 1.1** Let  $A$  be any nonempty subset of the metric spaces  $(X, d)$  and let  $f : A \subset X \rightarrow \mathbb{R}$  be a Lipschitz function. A function  $h : X \rightarrow \mathbb{R}$  is an *absolutely minimizing Lipschitz extension* of  $f$  to  $X$  if

- (i)  $h \in \text{MLE}(f, X)$ ,
- (ii) whenever  $B \subset X$  and  $g \in \text{MLE}(f, X)$  such that  $g = h$  in  $X \setminus B$ , then  $L_d(h, B) \leq L_d(g, B)$ .

Also in [11] it is proved the existence of an AMLE under the assumption that the metric space  $(X, d)$  is a separable length space.

Aronsson's original definition in  $\mathbb{R}^N$  was formulated in a slightly different way. He assumed that  $A$  is a compact set and required that  $L_d(h, D) = L_d(h, \partial D)$  for every bounded open set  $D$  in  $\mathbb{R}^N \setminus A$ . As remarked by Juutinen in [11], for a general metric space "this kind of definition would be somewhat ambiguous because the boundary of an open subset of a metric space may very well be empty", and the issue of [11] was to find a right way to interpret the "boundary condition".

Moreover, in [6], Aronsson proposed an approach to obtain the AMLE extension of a datum  $f$  by taking the limit as  $p \rightarrow \infty$  in the  $p$ -Laplacian problem

$$\begin{cases} -\Delta_p u_p = 0 & \text{in } \Omega, \\ u_p = f & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

This approach was made completely rigorous by Jensen in [10] (see also [9]). In [7] you can find the following result: the limit as  $p \rightarrow \infty$  of  $u_p$ ,  $u_\infty$ , is the best Lipschitz extension (AMLE) of  $f$  in  $\Omega$  and moreover it is characterized as the unique viscosity solution to

$$\begin{cases} -\Delta_\infty u_\infty = 0, & \Omega, \\ u_\infty = f, & \partial\Omega, \end{cases} \quad (1.2)$$

where  $\Delta_\infty$  is the infinity Laplace operator, that is, the degenerate elliptic operator given by

$$\Delta_\infty u := \sum_{i,j=1}^N u_{x_i} u_{x_j} u_{x_i x_j}.$$

Recently, Peres, Schramm, Sheffield and Wilson [20] have shown that the infinity Laplace equation (1.2) is solved by the continuous value function for a random turn tug-of-war game, in which the players, at each step, flip a fair coin to determine which player plays.

Given a bounded domain  $\Omega$  in  $\mathbb{R}^N$  and a function defined outside  $\Omega$  (this will be properly stated afterward), our aim is to study the Lipschitz extension problem to  $\Omega$  respect to the discrete distance that counts the number of steps,

$$d_\varepsilon(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \varepsilon \left( \left\lceil \frac{|x-y|}{\varepsilon} \right\rceil + 1 \right) & \text{if } x \neq y, \end{cases} \quad (1.3)$$

where  $|\cdot|$  is the Euclidean norm and  $\lceil r \rceil$  is defined for  $r > 0$  by  $\lceil r \rceil := n$ , if  $n < r \leq n+1$ ,  $n = 0, 1, 2, \dots$ , that is,

$$d_\varepsilon(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \varepsilon & \text{if } 0 < |x - y| \leq \varepsilon, \\ 2\varepsilon & \text{if } \varepsilon < |x - y| \leq 2\varepsilon, \\ \vdots & \end{cases}$$

The distance  $d_\varepsilon$  was used in [20] in relation with  $\varepsilon$ -tug-of-war games. It was also used in [2] to give a mass transport interpretation of a nonlocal model of sandpiles.

### 1.1 Description of the main results

Since  $(\mathbb{R}^N, d_\varepsilon)$  is not a separable length space, the general concept of AMLE due to Juutinen does not work on it. We give a concept of AMLE respect to the distance  $d_\varepsilon$ , which we name as  $\text{AMLE}_\varepsilon(f, \Omega)$ , in an slight different way that finds the right manner to interpret the “boundary condition” (observe that for the metric  $d_\varepsilon$  the boundary of  $\Omega$  is empty).

In addition, we relate this absolutely minimizing Lipschitz extension problem with a discrete  $\infty$ -Laplacian problem, which arise as the dynamic programming formula for the value function of some  $\varepsilon$ -tug-of-war game. More precisely,

we characterize the value function for the  $\varepsilon$ -tug-of-war game with payoff function  $f$  as the  $\text{AMLE}_\varepsilon(f, \Omega)$ . Therefore, as consequence of the results in [20] we have existence and uniqueness of  $\text{AMLE}_\varepsilon(f, \Omega)$ .

Finally, we also obtain the nonlocal version of the approximation by the  $p$ -Laplacian, that is, we get the  $\text{AMLE}_\varepsilon(f, \Omega)$  by taking the limit as  $p \rightarrow \infty$  in a nonlocal  $p$ -Laplacian problem.

## 2 Definition and characterization of $\text{AMLE}_\varepsilon$

Given a set  $A \subset \mathbb{R}^N$  and  $\varepsilon > 0$ , we denote

$$A_\varepsilon := \left\{ x \in \mathbb{R}^N : \text{dist}(x, A) := \inf_{y \in A} |x - y| < \varepsilon \right\}.$$

The euclidean open ball centred at  $x$  with radius  $r$  will be denoted by  $B_r(x)$ , and with  $\overline{B}_r(x)$  its closure. Throughout the paper, we assume that  $\Omega$  is a bounded domain of  $\mathbb{R}^N$ .

Given  $u : \Omega_\varepsilon \rightarrow \mathbb{R}$  and  $D \subset \Omega$ , we define

$$L_\varepsilon(u, D) := \sup_{\substack{x \in D, y \in D_\varepsilon \\ |x - y| \leq \varepsilon}} \frac{|u(x) - u(y)|}{\varepsilon}.$$

Observe that

$$L_\varepsilon(u, D) = \sup_{\substack{x \in D, y \in D_\varepsilon \\ |x - y| \leq \varepsilon}} \frac{|u(x) - u(y)|}{d_\varepsilon(x, y)} \leq \sup_{x \in D, y \in D_\varepsilon} \frac{|u(x) - u(y)|}{d_\varepsilon(x, y)}.$$

And that, if  $D$  is convex, the above inequality is an equality. Indeed, for  $x_0 = x \in D$ ,  $x_n = y \in D_\varepsilon$  and  $x_1, x_2, \dots, x_{n-1}$  in the segment between  $x$  and  $y$  such that  $|x_i - x_{i-1}| = \varepsilon$  for  $i = 1, \dots, n-1$ , and  $|x_n - x_{n-1}| \leq \varepsilon$ , we have that

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{i=1}^n |u(x_i) - u(x_{i-1})| \leq \varepsilon \sum_{i=1}^n L_\varepsilon(u, D) \\ &= \varepsilon n L_\varepsilon(u, D) = d_\varepsilon(x, y) L_\varepsilon(u, D). \end{aligned}$$

Also, for any convex  $D$ ,

$$L_{d_\varepsilon}(u, D) = \sup_{x, y \in D, |x - y| \leq \varepsilon} \frac{|u(x) - u(y)|}{\varepsilon}.$$

Therefore, the constant  $L_\varepsilon(u, D)$  is not genuinely the Lipschitz constant associated to  $d_\varepsilon$  in  $D$ ,  $L_{d_\varepsilon}(u, D)$ , even if  $D$  is convex, but, as we will see, it is the

right one to treat the absolutely minimizing Lipschitz extensions when  $d_\varepsilon$  is considered.

**Definition 2.1** Let  $f : \Omega_\varepsilon \setminus \Omega \rightarrow \mathbb{R}$  be bounded. We say that a function  $u : \Omega_\varepsilon \rightarrow \mathbb{R}$  is an *Absolutely Minimizing Lipschitz Extension for  $L_\varepsilon$  of  $f$  into  $\Omega$*  ( $u$  is  $\text{AMLE}_\varepsilon(f, \Omega)$  for shortness) if

- (i)  $u = f$  in  $\Omega_\varepsilon \setminus \Omega$ ,
- (ii) for every  $D \subset \Omega$  and  $v : \Omega_\varepsilon \rightarrow \mathbb{R}$  with  $v = u$  in  $\Omega_\varepsilon \setminus D$ , then  $L_\varepsilon(u, D) \leq L_\varepsilon(v, D)$ .

**Lemma 2.2** *When  $\Omega$  is convex the above definition is equivalent to the following two conditions, that match better the idea of Definition 1.1,*

- (i')  $u \in \text{MLE}(f, \Omega_\varepsilon)$ ,
- (ii') for every  $D \subset \Omega$  and  $v \in \text{MLE}(f, \Omega_\varepsilon)$  with  $v = u$  in  $\Omega_\varepsilon \setminus D$ , then  $L_\varepsilon(u, D) \leq L_\varepsilon(v, D)$ .

**Proof.** Let us first see that (i), (ii) implies (i') ((ii') is immediate): take  $v \in \text{MLE}(f, \Omega_\varepsilon)$ , then

$$L_\varepsilon(u, \Omega) \leq L_\varepsilon(v, \Omega) \leq L_{d_\varepsilon}(v, \Omega_\varepsilon) = L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega).$$

Therefore, since  $\Omega$  is convex,

$$\sup_{x \in \Omega, y \in \Omega_\varepsilon} \frac{|u(x) - u(y)|}{d_\varepsilon(x, y)} = L_\varepsilon(u, \Omega) \leq L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega). \quad (2.1)$$

On the other hand, since  $u = f$  in  $\Omega_\varepsilon \setminus \Omega$ ,

$$L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega) \leq L_{d_\varepsilon}(u, \Omega_\varepsilon) = \sup_{x \in \Omega_\varepsilon, y \in \Omega_\varepsilon} \frac{|u(x) - u(y)|}{d_\varepsilon(x, y)}. \quad (2.2)$$

Consequently, from (2.1) and (2.2),  $L_{d_\varepsilon}(u, \Omega_\varepsilon) = L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega)$ .

Let us now see that (i'), (ii') implies (ii) ((i) is immediate). Let us argue by contradiction and suppose that there exist  $D$  and  $v$  such that  $v = u$  in  $\Omega_\varepsilon \setminus D$  and  $L_\varepsilon(v, D) < L_\varepsilon(u, D)$ . Then, by (ii'),  $v$  can not be in  $\text{MLE}(f, \Omega_\varepsilon)$ . But also, the above strict inequality implies that, on account that  $\Omega_\varepsilon$  is convex,

$$\begin{aligned} L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega) &\leq L_{d_\varepsilon}(v, \Omega_\varepsilon) = \sup_{\substack{x \notin D, y \notin D \\ |x - y| \leq \varepsilon}} \frac{|v(x) - v(y)|}{\varepsilon} \\ &= \sup_{\substack{x \notin D, y \notin D \\ |x - y| \leq \varepsilon}} \frac{|u(x) - u(y)|}{\varepsilon} \leq L_{d_\varepsilon}(u, \Omega_\varepsilon) = L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega) \end{aligned}$$

and consequently  $v \in \text{MLE}(f, \Omega_\varepsilon)$ , which is a contradiction.  $\square$

Remark that independently of the convexity of  $\Omega$ , if  $u$  is  $\text{AMLE}_\varepsilon(f, \Omega)$ , it always holds that

$$L_\varepsilon(u, \Omega) \leq L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega).$$

In the next result we obtain the characterization of the  $\text{AMLE}_\varepsilon(f, \Omega)$  by means of a discrete  $\infty$ -Laplacian problem.

**Theorem 2.3** *Let  $f : \Omega_\varepsilon \setminus \Omega \rightarrow \mathbb{R}$  be bounded. Then,  $u : \Omega_\varepsilon \rightarrow \mathbb{R}$  is  $\text{AMLE}_\varepsilon(f, \Omega)$  if and only if  $u$  is a solution of*

$$\begin{cases} -\Delta_\infty^\varepsilon u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Omega_\varepsilon \setminus \Omega, \end{cases} \quad (2.3)$$

where

$$\Delta_\infty^\varepsilon u(x) := \sup_{y \in \overline{B}_\varepsilon(x)} u(y) + \inf_{y \in \overline{B}_\varepsilon(x)} u(y) - 2u(x) \quad (2.4)$$

is the discrete infinity Laplace operator.

**Proof.** Without loss of generality we will take  $\varepsilon = 1$  along the proof. Let us first take  $u$  a solution of (2.3) and suppose that  $u$  is not  $\text{AMLE}_1(f, \Omega)$ . Then, there exists  $D \subset \Omega$  and  $v : \Omega_1 \rightarrow \mathbb{R}$ ,  $v = u$  in  $\Omega_1 \setminus D$ , such that  $L_1(v, D) < L_1(u, D)$ . Set  $\delta := L_1(u, D) - L_1(v, D) > 0$ , and let  $n \in \mathbb{N}$ ,  $n > 3$ , such that

$$\sup_D u - \inf_D u \leq (n-1)L_1(u, D). \quad (2.5)$$

Take  $(x_0, y_0) \in D \times \Omega_1$ ,  $|x_0 - y_0| \leq 1$ , such that

$$L_1(u, D) - \frac{\delta}{n} \leq |u(x_0) - u(y_0)| \leq L_1(u, D).$$

We have that  $\Delta_\infty^1 u(x_0) = 0$  and  $\Delta_\infty^1 u(y_0) = 0$  if  $y_0 \in \Omega$ . Let us suppose that  $u(y_0) \geq u(x_0)$  (the other case being similar), which implies

$$L_1(u, D) - \frac{\delta}{n} \leq u(y_0) - u(x_0) \leq L_1(u, D). \quad (2.6)$$

If  $y_0 \notin D$ , set  $y_1 = y_0$ . If  $y_0 \in D$ , since  $\Delta_\infty^1 u(y_0) = 0$  and  $x_0 \in \overline{B}_1(y_0)$ , we have

$$\sup_{y \in \overline{B}_1(y_0)} u(y) - u(y_0) = u(y_0) - \inf_{y \in \overline{B}_1(y_0)} u(y) \geq u(y_0) - u(x_0) \geq L_1(u, D) - \frac{\delta}{n}.$$

Hence, there exists  $y_1 \in \overline{B}_1(y_0)$  such that

$$u(y_1) - u(y_0) \geq L_1(u, D) - \frac{2\delta}{n}.$$

Also, since  $\Delta_\infty^1 u(x_0) = 0$ , we have

$$u(x_0) - \inf_{x \in \overline{B}_1(x_0)} u(x) = \sup_{x \in \overline{B}_1(x_0)} u(x) - u(x_0) \geq u(y_0) - u(x_0) \geq L_1(u, D) - \frac{\delta}{n},$$

and consequently, there exists  $x_1 \in \overline{B}_1(x_0)$  such that

$$u(x_0) - u(x_1) \geq L_1(u, D) - \frac{2\delta}{n}.$$

Following this construction, and with the rule that in the case  $x_j \notin D$  or  $y_j \notin D$ , then  $x_i = x_j$  or  $y_i = y_j$  for all  $i \geq j$ , we claim that there exists  $m \leq n$  for which  $x_m \notin D$  and  $y_m \notin D$ . In fact, if not, then either  $\{x_i\}_{i=1, \dots, n} \subset D$ , either  $\{y_i\}_{i=1, \dots, n} \subset D$ , with  $\{x_i\}_{i=1, \dots, n}$  and  $\{y_i\}_{i=1, \dots, n}$  satisfying

$$u(y_i) - u(y_{i-1}) \geq L_1(u, D) - \frac{2\delta}{n}, \quad y_i \in \overline{B}_1(y_{i-1}), \quad i = 1, \dots, n, \quad (2.7)$$

and

$$u(x_i) - u(x_{i-1}) \geq L_1(u, D) - \frac{2\delta}{n}, \quad x_i \in \overline{B}_1(x_{i-1}), \quad i = 1, \dots, n. \quad (2.8)$$

Let us suppose the first of these two possibilities, that is,  $\{x_i\}_{i=1, \dots, n} \subset D$ . Then, having in mind (2.5), (2.6) and (2.8), we get

$$\begin{aligned} (n-1)L_1(u, D) &\geq u(y_0) - u(x_n) \\ &= u(y_0) - u(x_0) + u(x_0) - u(x_1) + \dots + u(x_{n-1}) - u(x_n) \\ &\geq L_1(u, D) - \frac{\delta}{n} + (n+1)(L_1(u, D) - \frac{2\delta}{n}), \end{aligned}$$

from where it follows that

$$\frac{2n+3}{n}\delta \geq 3L_1(u, D) \geq 3\delta,$$

which is a contradiction since  $n > 3$ . Now, for  $\{x_i, y_i\}_{i=1, \dots, m}$ , we have

$$v(y_m) - v(x_m) = u(y_m) - u(x_m) \geq 2m \left( L_1(u, D) - \frac{2\delta}{n} \right) + L_1(u, D) - \frac{\delta}{n},$$

$$v(y_m) - v(x_m) \leq (2m+1)L_1(v, D),$$

and therefore,

$$(2m+1)L_1(u, D) - (4m+1)\frac{\delta}{n} \leq (2m+1)L_1(v, D),$$



that is

$$\delta = L_1(u, D) - L_1(v, D) \leq \frac{4m+1}{2m+1} \frac{\delta}{n},$$

which implies  $n \leq \frac{4m+1}{2m+1} \leq 2$ , which is a contradiction since  $n > 3$ .

Let us now consider  $u$  an  $\text{AMLE}_1(f, \Omega)$  and suppose that  $u$  is not a solution of (2.3). Then,  $\{x \in \Omega : \Delta_\infty^1 u(x) \neq 0\} \neq \emptyset$ . Let us suppose without loss of generality, that,

$$\left\{ x \in \Omega : \sup_{y \in \overline{B}_1(x)} u(y) - u(x) > u(x) - \inf_{y \in \overline{B}_1(x)} u(y) \right\} \neq \emptyset.$$

Then, there exists  $\delta > 0$  and a nonempty set  $D \subset \Omega$  such that

$$\sup_{y \in \overline{B}_1(x)} u(y) - u(x) > u(x) - \inf_{y \in \overline{B}_1(x)} u(y) + \delta \quad \text{for all } x \in D. \quad (2.9)$$

Consider the function  $v : \Omega_1 \rightarrow \mathbb{R}$  defined by

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega_1 \setminus D, \\ u(x) + \frac{\delta}{2} & \text{if } x \in D. \end{cases}$$

Then, since  $u$  is an  $\text{AMLE}_1(f, \Omega)$ , we have  $L_1(u, D) \leq L_1(v, D)$ . Now, there exists  $x_0 \in D$  and  $y_0 \in \overline{B}_1(x_0)$  such that

$$L_1(v, D) \leq \frac{\delta}{4} + |v(x_0) - v(y_0)|.$$

Therefore, if  $v(x_0) \geq v(y_0)$ , by (2.9),

$$\begin{aligned} L_1(v, D) &\leq \frac{\delta}{4} + v(x_0) - v(y_0) \leq \frac{3\delta}{4} + u(x_0) - u(y_0) \\ &\leq \frac{3\delta}{4} + u(x_0) - \inf_{x \in \overline{B}_1(x_0)} u(x) < -\frac{\delta}{4} + \sup_{x \in \overline{B}_1(x_0)} u(x) - u(x_0) < L_1(u, D), \end{aligned}$$

which is a contradiction, and, if  $v(x_0) < v(y_0)$ ,

$$L_1(v, D) \leq \frac{\delta}{4} + v(y_0) - v(x_0) = -\frac{\delta}{4} + v(y_0) - u(x_0),$$

so, if  $y_0 \notin D$ ,

$$L_1(v, D) \leq -\frac{\delta}{4} + u(y_0) - u(x_0) < L_1(u, D),$$

also a contradiction, and if  $y_0 \in D$ , since also  $x_0 \in \overline{B}_1(y_0)$ , by (2.9),

$$L_1(v, D) \leq \frac{\delta}{4} + u(y_0) - u(x_0) \leq \frac{\delta}{4} + u(y_0) - \inf_{y \in \overline{B}_1(y_0)} u(y)$$

$$< -\frac{3\delta}{4} + \sup_{y \in \overline{B}_1(y_0)} u(y) - u(y_0) < L_1(u, D),$$

again a contradiction. Then, in any case we arrive to a contradiction and consequently  $u$  is a solution of (2.3).  $\square$

The first analysis of the interesting functional equation  $-\Delta_\infty^\varepsilon u = 0$  appeared in the article by Le Gruyer and Archer [13], but it also arises as the dynamic programming formula for the value function of some tug-of-war games (see for instance [8,15,16,20]). Let us briefly review the  $\varepsilon$ -tug-of-war game introduced by Peres, Schramm, Sheffield and Wilson in [20]. Fix a number  $\varepsilon > 0$ . The dynamic of the game is as follows. There are two players moving a token inside a set  $E_\Omega$  containing  $\Omega$ , a bounded domain in  $\mathbb{R}^N$ . The token is placed at an initial position  $x_0 \in \Omega$ . At the  $k$ th stage of the game, player I and player II select points  $x_k^I$  and  $x_k^{II}$ , respectively, both belonging to  $\overline{B}_\varepsilon(x_{k-1}) \cap E_\Omega$ . The token is then moved to  $x_k$ , where  $x_k$  is chosen randomly so that  $x_k = x_k^I$  or  $x_k = x_k^{II}$ , depending who was the winner of a flip of a fair coin. After the  $k$ th stage of the game, if  $x_k \in \Omega$  then the game continues to stage  $k+1$ . Otherwise, if  $x_k \in E_\Omega \setminus \Omega$ , the game ends and player II pays player I the amount  $f(x_k)$ , where  $f : E_\Omega \setminus \Omega \rightarrow \mathbb{R}$  is a final payoff function of the game. Of course, player I attempts to maximize the payoff, while player II attempts to minimize it.

Given a strategy for player I, that is a mapping  $S_I$  from the set of all possible partially played games  $(x_0, x_1, \dots, x_{k-1})$  to possible positions  $x_k \in \overline{B}_\varepsilon(x_{k-1})$ , and a strategy  $S_{II}$  for player II, we denote by  $E_I^{x_0}(S_I, S_{II})$  and  $E_{II}^{x_0}(S_I, S_{II})$  the expected value of  $f(x_k)$  ( $\mathbb{E}_{S_I, S_{II}}^{x_0}[f(x_k)]$ ), if the game terminates a.s.,  $-\infty$  and  $+\infty$ , respectively, otherwise (there is a severe penalization for both players if the game never ends). The value of the game for player I is the quantity

$$\sup_{S_I} \inf_{S_{II}} E_I^{x_0}(S_I, S_{II}),$$

where the supremum is taken over all possible strategies for player I and the infimum over all strategies of player II. Similarly, the value of the game for player II is

$$\inf_{S_{II}} \sup_{S_I} E_{II}^{x_0}(S_I, S_{II}).$$

We denote the value for player I as a function of the starting point  $x_0 \in \Omega$  by  $u_I^\varepsilon(x_0)$ , and similarly the value for player II by  $u_{II}^\varepsilon(x_0)$ . The game is said to have a value if  $u_I^\varepsilon = u_{II}^\varepsilon =: u^\varepsilon$ . According with the Dynamic Programming Principle, see [20], there is a value function for the  $\varepsilon$ -tug-of-war game,  $u^\varepsilon$ , that satisfies the functional equation

$$u^\varepsilon(x) = \frac{1}{2} \left( \sup_{y \in \overline{B}_\varepsilon(x) \cap E_\Omega} u^\varepsilon(y) + \inf_{y \in \overline{B}_\varepsilon(x) \cap E_\Omega} u^\varepsilon(y) \right) \quad \text{in } \Omega,$$

with  $u^\varepsilon = f$  in  $E_\Omega \setminus \Omega$ . Observe that this is (2.3) when  $E_\Omega = \Omega_\varepsilon$  (see [15,16])

for this problem). In [20], using martingale methods, it is proved that problem (2.3) has a unique solution; then, by Theorem 2.3, we get the following existence and uniqueness result.

**Theorem 2.4** *Let  $f : \Omega_\varepsilon \setminus \Omega \rightarrow \mathbb{R}$  be bounded. Then, there is a unique  $\text{AMLE}_\varepsilon(f, \Omega)$ .*

Some of the difficulties in the analysis of the  $\varepsilon$ -tug-of-war game are due to the fact that the value function  $u^\varepsilon$  can be discontinuous. When the limit  $u := \lim_{\varepsilon \rightarrow 0} u^\varepsilon$  exists pointwise, the function  $u$  is called the *continuum value of the game*. In [20], Peres, Schramm, Sheffield and Wilson proved that if  $E_\Omega = \overline{\Omega}$  and the terminal payoff function of the game  $f$  is Lipschitz continuous on  $\partial\Omega$  then the continuum value  $u$  exists and  $u^\varepsilon \rightarrow u$  uniformly in  $\overline{\Omega}$  as  $\varepsilon \rightarrow 0$ . Moreover,  $u$  is the unique AMLE extension of  $f$  to  $\Omega$  and the unique viscosity solution of the boundary value problem

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Our Theorem 2.3 gives this characterization in the case of the discrete distance. We will see in the next section that we can also obtain the  $\text{AMLE}_\varepsilon$  extension by taking the limit as  $p \rightarrow \infty$  in a nonlocal  $p$ -Laplacian problem, which represents the nonlocal version of the approximation of the local problem with the  $p$ -Laplacian.

### 3 Existence of $\text{AMLE}_\varepsilon$ by a nonlocal $L^p$ -variational approach

First, let us introduce some notation. Given  $f : \Omega_\varepsilon \setminus \Omega \rightarrow \mathbb{R}$  and  $u : \Omega \rightarrow \mathbb{R}$ , we will denote

$$u_f(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ f(x) & \text{if } x \in \Omega_\varepsilon \setminus \Omega. \end{cases}$$

Given a convex set  $K \subset L^2(\Omega)$ , we denote by  $\mathbb{I}_K$  to the indicator function of  $K$ , that is, the function defined as

$$\mathbb{I}_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Let  $J : \mathbb{R}^N \rightarrow \mathbb{R}$  be a nonnegative, radial, continuous function, strictly positive in  $B_1(0)$ , vanishing in  $\mathbb{R}^N \setminus B_1(0)$  and such that  $\int_{\mathbb{R}^N} J(z) dz = 1$ . For  $1 < p <$

$+\infty$  and  $f : \Omega_1 \setminus \overline{\Omega} \rightarrow \mathbb{R}$  such that  $|f|^{p-1} \in L^1(\Omega_1 \setminus \overline{\Omega})$ , we define in  $L^1(\Omega)$  the operator  $B_{p,f}^J$  by

$$\begin{aligned} B_{p,f}^J(u)(x) &:= - \int_{\Omega} J(x-y) |u(y) - u(x)|^{p-2} (u(y) - u(x)) dy \\ &\quad - \int_{\Omega_1 \setminus \overline{\Omega}} J(x-y) |f(y) - u(x)|^{p-2} (f(y) - u(x)) dy, \quad x \in \Omega, \end{aligned}$$

that is,

$$B_{p,f}^J(u)(x) = - \int_{\Omega_1} J(x-y) |u_f(y) - u(x)|^{p-2} (u_f(y) - u(x)) dy, \quad x \in \Omega.$$

In [1] (see also [3]) we have seen that the nonlocal version of the Dirichlet problem (1.1), with boundary value  $f$ , can be written as

$$B_{p,f}^J(u) = 0. \quad (3.1)$$

We have also established the following Poincaré's type inequality for such kind of integral operators.

**Proposition 3.1** ([1]) *Given  $J : \mathbb{R}^N \rightarrow \mathbb{R}$  as above,  $p \geq 1$  and  $f \in L^p(\Omega_1 \setminus \overline{\Omega})$ , there exists  $\lambda = \lambda(J, \Omega, p) > 0$  such that*

$$\lambda \int_{\Omega} |u(x)|^p dx \leq \int_{\Omega} \int_{\Omega_1} J(x-y) |u_f(y) - u(x)|^p dy dx + \int_{\Omega_1 \setminus \overline{\Omega}} |f(y)|^p dy \quad (3.2)$$

for all  $u \in L^p(\Omega)$ .

We say that  $u$  is a supersolution (resp. subsolution) of the nonlocal Dirichlet problem (3.1) with boundary value  $f$  if  $B_{p,f}^J(u) \geq 0$  (resp.  $B_{p,f}^J(u) \leq 0$ ). We have the following comparison principle.

**Lemma 3.2** *Let  $J : \mathbb{R}^N \rightarrow \mathbb{R}$  as above,  $p \geq 2$  and  $\overline{f}, \underline{f} \in L^p(\Omega_1 \setminus \overline{\Omega})$ , with  $\overline{f} \geq \underline{f}$ . If  $\overline{u}$  is a supersolution of the Dirichlet problem (3.1) with boundary value  $\overline{f}$  and  $\underline{u}$  is a subsolution of the Dirichlet problem (3.1) with boundary value  $\underline{f}$  then  $\overline{u} \geq \underline{u}$ .*

**Proof.** By assumption we have

$$0 \geq - \int_{\Omega_1} J(x-y) |\underline{u}_f(y) - \underline{u}(x)|^{p-2} (\underline{u}_f(y) - \underline{u}(x)) dy \quad \forall x \in \Omega$$

and

$$0 \leq - \int_{\Omega_1} J(x-y) |\overline{u}_f(y) - \overline{u}(x)|^{p-2} (\overline{u}_f(y) - \overline{u}(x)) dy \quad \forall x \in \Omega.$$

Then, multiplying by  $(\underline{u} - \bar{u})^+(x)$ , integrating and having in mind that

$$(\underline{u}_f(x) - \bar{u}_{\bar{f}}(x))^+ = (\underline{f}(x) - \bar{f}(x))^+ = 0$$

if  $x \in \Omega_1 \setminus \Omega$ , we get

$$\begin{aligned} 0 &\geq - \int_{\Omega_1 \times \Omega_1} J(x-y) |\underline{u}_f(y) - \underline{u}_f(x)|^{p-2} (\underline{u}_f(y) - \underline{u}_f(x)) (\underline{u}_f(x) - \bar{u}_{\bar{f}}(x))^+ dy dx \\ &= \frac{1}{2} \int_{\Omega_1 \times \Omega_1} J(x-y) |\underline{u}_f(y) - \underline{u}_f(x)|^{p-2} (\underline{u}_f(y) - \underline{u}_f(x)) \\ &\quad \times \left[ (\underline{u}_f(y) - \bar{u}_{\bar{f}}(y))^+ - (\underline{u}_f(x) - \bar{u}_{\bar{f}}(x))^+ \right] dy dx \end{aligned}$$

and

$$\begin{aligned} 0 &\geq \int_{\Omega_1 \times \Omega_1} J(x-y) |\bar{u}_{\bar{f}}(y) - \bar{u}_{\bar{f}}(x)|^{p-2} (\bar{u}_{\bar{f}}(y) - \bar{u}_{\bar{f}}(x)) (\underline{u}_f(x) - \bar{u}_{\bar{f}}(x))^+ dy dx \\ &= - \int_{\Omega_1 \times \Omega_1} \frac{J(x-y)}{2} |\bar{u}_{\bar{f}}(y) - \bar{u}_{\bar{f}}(x)|^{p-2} (\bar{u}_{\bar{f}}(y) - \bar{u}_{\bar{f}}(x)) \\ &\quad \times \left[ (\underline{u}_f(y) - \bar{u}_{\bar{f}}(y))^+ - (\underline{u}_f(x) - \bar{u}_{\bar{f}}(x))^+ \right] dy dx. \end{aligned}$$

Then, adding the last two inequalities we obtain that

$$\begin{aligned} &\int_{\Omega_1 \times \Omega_1} J(x-y) \left( |\underline{u}_f(y) - \underline{u}_f(x)|^{p-2} (\underline{u}_f(y) - \underline{u}_f(x)) \right. \\ &\quad \left. - |\bar{u}_{\bar{f}}(y) - \bar{u}_{\bar{f}}(x)|^{p-2} (\bar{u}_{\bar{f}}(y) - \bar{u}_{\bar{f}}(x)) \right) \\ &\quad \times \left( (\underline{u}_f(y) - \bar{u}_{\bar{f}}(y))^+ - (\underline{u}_f(x) - \bar{u}_{\bar{f}}(x))^+ \right) dy dx \leq 0. \end{aligned}$$

Therefore, since  $(|r|^{p-2}r - |s|^{p-2}s)(r^+ - s^+) \geq 0$ , we get that

$$\begin{aligned} &J(x-y) \left( |\underline{u}_f(y) - \underline{u}_f(x)|^{p-2} (\underline{u}_f(y) - \underline{u}_f(x)) \right. \\ &\quad \left. - |\bar{u}_{\bar{f}}(y) - \bar{u}_{\bar{f}}(x)|^{p-2} (\bar{u}_{\bar{f}}(y) - \bar{u}_{\bar{f}}(x)) \right) \\ &\quad \times \left( (\underline{u}_f(y) - \bar{u}_{\bar{f}}(y))^+ - (\underline{u}_f(x) - \bar{u}_{\bar{f}}(x))^+ \right) = 0 \end{aligned} \tag{3.3}$$

for a.e.  $(x, y) \in \Omega_1 \times \Omega_1$ .

Let  $\tilde{\Omega} := \{x \in \Omega : \underline{u}(x) > \bar{u}(x)\}$ . Since  $(|r|^{p-2}r - |s|^{p-2}s)(r - s) \geq C|r - s|^p$ , from (3.3) we obtain that, for a.e.  $(x, y) \in \tilde{\Omega} \times \tilde{\Omega}$ ,

$$\begin{aligned} 0 &= J(x-y) (|\underline{u}(y) - \underline{u}(x)|^{p-2} (\underline{u}(y) - \underline{u}(x)) - |\bar{u}(y) - \bar{u}(x)|^{p-2} (\bar{u}(y) - \bar{u}(x))) \\ &\quad \times (\underline{u}(y) - \underline{u}(x) - (\bar{u}(y) - \bar{u}(x))) \\ &\geq C J(x-y) |\underline{u}(y) - \underline{u}(x) - (\bar{u}(y) - \bar{u}(x))| \end{aligned}$$

that is,

$$J(x-y)|\underline{u}(y) - \bar{u}(y) - (\underline{u}(x) - \bar{u}(x))|^p = 0 \quad \text{for a.e. } (x, y) \in \tilde{\Omega} \times \tilde{\Omega}. \quad (3.4)$$

Therefore, if

$$|\tilde{\Omega}| > 0, \quad (3.5)$$

from (3.4), we get there exists  $\Lambda \subset \tilde{\Omega}$  with  $|\Lambda| > 0$  such that

$$\underline{u}(x) - \bar{u}(x) = \lambda > 0 \quad \text{a.e. in } \Lambda.$$

Then, taking the above conclusion in (3.3) we have that, for a.e.  $(x, y) \in (\Omega_1 \setminus \tilde{\Omega}) \times \Lambda$ ,

$$\begin{aligned} 0 &= J(x-y) \left( |\underline{u}(y) - \underline{u}_f(x)|^{p-2} (\underline{u}(y) - \underline{u}_f(x)) \right. \\ &\quad \left. - |\bar{u}(y) - \bar{u}_f(x)|^{p-2} (\bar{u}(y) - \bar{u}_f(x)) \right) (\underline{u}(y) - \bar{u}(y)) \\ &= J(x-y) \left( |\bar{u}(y) - (\underline{u}_f(x) - \lambda)|^{p-2} (\bar{u}(y) - (\underline{u}_f(x) - \lambda)) \right. \\ &\quad \left. - |\bar{u}(y) - \bar{u}_f(x)|^{p-2} (\bar{u}(y) - \bar{u}_f(x)) \right) \lambda. \end{aligned}$$

Now, since  $|r|^{p-2}r - |s|^{p-2}s = 0$  if and only if  $r = s$  we conclude that

$$\underline{u}_f(x) - \lambda = \bar{u}_f(x) \quad \text{a.e. in } \Omega_1 \setminus \tilde{\Omega},$$

which contradicts that  $\Omega_1 \setminus \tilde{\Omega}$  contains the non-null set  $\Omega_1 \setminus \Omega$  (since  $\bar{f} \geq \underline{f}$ ). Therefore (3.5) is false, and then  $\underline{u} \leq \bar{u}$  a.e. in  $\Omega$ .  $\square$

For the energy functional

$$\begin{aligned} G_{p,f}^J(u) &= \frac{1}{2p} \int_{\Omega} \int_{\Omega} J(x-y) |u(y) - u(x)|^p dy dx \\ &\quad + \frac{1}{p} \int_{\Omega} \int_{\Omega_1 \setminus \Omega} J(x-y) |f(y) - u(x)|^p dy dx, \end{aligned}$$

we have the following result:

**Theorem 3.3** *Assume that  $p \geq 2$ . Then, there exists a unique  $u_p \in L^p(\Omega)$  such that*

$$G_{p,f}^J(u_p) = \min\{G_{p,f}^J(u) : u \in L^p(\Omega)\}. \quad (3.6)$$

*Moreover,  $u_p$  is the unique solution of the nonlocal Euler–Lagrange equation  $B_{p,f}^J(u_p) = 0$ , and it has a continuous representative in  $\Omega$ .*

**Proof.** Let  $v_n \in L^p(\Omega)$  a minimizing sequence, that is,

$$m := \inf\{G_{p,f}^J(u) : u \in L^p(\Omega)\} = \lim_{n \rightarrow +\infty} G_{p,f}^J(v_n).$$

Then, by the Poincaré inequality (3.2), we have

$$\|v_n\|_p \leq \left( \frac{1}{\lambda} \left( m + 1 + \int_{\Omega_1 \setminus \bar{\Omega}} |f(y)|^p dy \right) \right)^{\frac{1}{p}}.$$

Therefore, we can assume that  $v_n \rightharpoonup u_p$  weakly in  $L^2(\Omega)$ . Hence, since the functional  $G_{p,f}^J$  is weakly lower semi-continuous in  $L^2(\Omega)$ , we get

$$G_{p,f}^J(u_p) \leq \liminf_{n \rightarrow \infty} G_{p,f}^J(v_n) = m,$$

consequently,  $m = G_{p,f}^J(u_p)$  and (3.6) holds.

By results in [1], we know that the operator  $B_{p,f}^J$  is completely accretive and verifies the range condition  $L^p(\Omega) \subset \text{Ran}(I + B_{p,f}^J)$ . Let us see that

$$\partial G_{p,f}^J = \overline{B_{p,f}^J}^{L^2(\Omega)}. \quad (3.7)$$

Since  $G_{p,f}^J$  is convex and lower semi-continuous in  $L^2(\Omega)$ , to prove (3.7) it is enough to show that

$$B_{p,f}^J \subset \partial G_{p,f}^J. \quad (3.8)$$

Let us see that (3.8) holds. Set  $v = B_{p,f}^J(u)$  and let  $w \in D(G_{p,f}^J)$ . Then

$$\begin{aligned} & \int_{\Omega} v(x)(w(x) - u(x)) dx \\ &= - \int_{\Omega} \int_{\Omega_1} J(x-y) |u_f(y) - u(x)|^{p-2} (u_f(y) - u(x)) dy (w(x) - u(x)) dx \\ &= - \int_{\Omega_1} \int_{\Omega_1} J(x-y) |u_f(y) - u_f(x)|^{p-2} (u_f(y) - u_f(x)) dy (w_f(x) - u_f(x)) dx \\ &= \frac{1}{2} \int_{\Omega_1} \int_{\Omega_1} J(x-y) |u_f(y) - u_f(x)|^{p-2} (u_f(y) - u_f(x)) \\ & \quad \times (w_f(y) - w_f(x) - (u_f(x) - u_f(y))) dy dx. \end{aligned}$$

From here, using the numerical inequality

$$\frac{1}{2} |r|^{p-2} r (s - r) \leq \frac{1}{2p} (|s|^p - |r|^p),$$

we obtain that

$$\begin{aligned} G_{p,f}^J(w) - G_{p,f}^J(u) &= \frac{1}{2p} \int_{\Omega_1} \int_{\Omega_1} J(x-y) |w_f(y) - w_f(x)|^p dy dx \\ & \quad - \frac{1}{2p} \int_{\Omega_1} \int_{\Omega_1} J(x-y) |u_f(y) - u_f(x)|^p dy dx \\ & \geq \int_{\Omega} v(x)(w(x) - u(x)) dx, \end{aligned}$$

from where it follows (3.8). The second part of the theorem is a consequence of (3.7).

Now,  $B_{p,f}^J(u_p) = 0$  can be written as

$$\int_{\Omega_1} J(x-y) \varphi_p((u_p)_f(y) - u_p(x)) dy = 0 \quad \forall x \in \Omega \setminus N, \text{ meas}(N) = 0,$$

for  $\varphi_p(r) := |r|^{p-2}r$ . Then, the continuity of  $u_p$  in  $\Omega$  follows by the above conclusion and the Implicit Function Theorem ([12]): since  $J$  is continuous and  $\varphi_p$  is continuous and increasing,

$$F(x, \alpha) := \int_{\Omega_1} J(x-y) \varphi_p((u_p)_f(y) - \alpha) dy$$

is continuous in  $\Omega \times \mathbb{R}$  and for fixed  $x \in \Omega$  it is decreasing in  $\alpha$ . Therefore, by [12, Theorem 1.1]  $F(x, \alpha) = 0$  has a unique solution  $\alpha(x)$  continuous in  $\Omega$ . In fact, this can be proved in a direct way as follows. Since  $\lim_{\alpha \rightarrow -\infty} F(x, \alpha) = +\infty$ ,  $\lim_{\alpha \rightarrow +\infty} F(x, \alpha) = -\infty$  and  $F(x, \cdot)$  is continuous and decreasing, there exists a unique  $\alpha(x)$  such that  $F(x, \alpha(x)) = 0$ . Now  $\alpha(x)$  is l.s.c. at  $x_0 \in \Omega$ ; indeed, take  $\alpha < \alpha(x_0)$ , therefore

$$\int_{\Omega_1} J(x-y) \varphi_p((u_p)_f(y) - \alpha) dy > 0.$$

Since  $J$  is continuous, there exists  $r > 0$  such that

$$\int_{\Omega_1} J(x-y) \varphi_p((u_p)_f(y) - \alpha) dy > 0 \quad \forall x \in B_r(x_0).$$

Therefore

$$\alpha < \alpha(x) \quad \forall x \in B_r(x_0).$$

Similarly  $\alpha(x)$  is u.s.c. Finally, since  $\alpha(x) = u_p(x)$  in  $\Omega \setminus N$  we conclude that  $u_p$  has a continuous representative.  $\square$

From now on, we will suppose that minimizers  $u_p$  of  $G_{p,f}^J$  are continuous and satisfy the Euler–Lagrange equation  $B_{p,f}^J(u_p) = 0$  everywhere.

At this step we also rescale the kernel  $J$  in order to deal with  $d_\varepsilon$  instead of with  $d_1$ . So, let

$$J_\varepsilon(z) = \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right).$$

We want to study the limit as  $p \rightarrow \infty$  of the minimizers  $u_p^\varepsilon$  of  $G_{p,f}^{J_\varepsilon}$ . From now on, we assume that  $f \in L^\infty(\Omega_\varepsilon \setminus \bar{\Omega})$ .

In [1], we have proved that

$$\lim_{p \rightarrow +\infty} G_{p,f}^{J_\varepsilon} = G_{\infty,f}^\varepsilon \quad \text{in the sense of Mosco,} \quad (3.9)$$



where

$$G_{\infty,f}^\varepsilon(u) = \begin{cases} 0 & \text{if } |u(x) - u(y)| \leq \varepsilon, \text{ for } x, y \in \Omega, |x - y| \leq \varepsilon, \text{ and} \\ & |f(y) - u(x)| \leq \varepsilon, \text{ for } x \in \Omega, y \in \Omega_\varepsilon \setminus \Omega, |x - y| \leq \varepsilon, \\ +\infty & \text{in other case.} \end{cases}$$

Now, by Hölder's and Poincaré's inequality (3.2), we have  $\|u_p^\varepsilon\|_2 \leq C\|f\|_\infty$  for every  $p \geq 2$ . Therefore, we can assume that

$$u_p^\varepsilon \rightharpoonup v_\infty \quad \text{weakly in } L^2(\Omega) \text{ as } p \rightarrow +\infty. \quad (3.10)$$

Then, by (3.9), we have

$$G_{\infty,f}^\varepsilon(v_\infty) \leq \liminf_{p \rightarrow \infty} G_{p,f}^{J_\varepsilon}(u_p^\varepsilon). \quad (3.11)$$

Given  $v \in D(G_{\infty,f}^\varepsilon)$ , by the definition of Mosco convergence, there exists  $v_p \in D(G_{p,f}^J)$ , such that  $v_p \rightarrow v$  in  $L^2(\Omega)$ , and such that

$$G_{\infty,f}^\varepsilon(v) \geq \limsup_{p \rightarrow \infty} G_{p,f}^{J_\varepsilon}(v_p). \quad (3.12)$$

Now, by (3.6),  $G_{p,f}^{J_\varepsilon}(u_p^\varepsilon) \leq G_{p,f}^{J_\varepsilon}(v_p)$  for all  $p \geq 2$ , and therefore, by (3.11) and (3.12), we obtain that  $G_{\infty,f}^\varepsilon(v_\infty) \leq G_{\infty,f}^\varepsilon(v)$ , and consequently

$$G_{\infty,f}^\varepsilon(v_\infty) = \min\{G_{\infty,f}^\varepsilon(u) : u \in D(G_{\infty,f}^\varepsilon)\}.$$

Therefore,

$$0 \in \partial G_{\infty,f}^\varepsilon(v_\infty). \quad (3.13)$$

If we define

$$K_{\infty,f}^\varepsilon := \left\{ u \in L^2(\Omega) : \begin{array}{l} |u(x) - u(y)| \leq \varepsilon \text{ for } x, y \in \Omega, |x - y| \leq \varepsilon, \text{ and} \\ |f(y) - u(x)| \leq \varepsilon \text{ for } x \in \Omega, y \in \Omega_\varepsilon \setminus \overline{\Omega}, |x - y| \leq \varepsilon \end{array} \right\},$$

we have that the functional  $G_{\infty,f}^\varepsilon$  is given by the indicator function of  $K_{\infty,f}^\varepsilon$ , that is,  $G_{\infty,f}^\varepsilon = \mathbb{I}_{K_{\infty,f}^\varepsilon}$ . Therefore, the Euler-Lagrange equation (3.13) can be written as

$$0 \in \partial \mathbb{I}_{K_{\infty,f}^\varepsilon}(v_\infty). \quad (3.14)$$

Observe that  $K_{\infty,f}^\varepsilon$  is not empty if we assume that  $L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega) \leq 1$ . In this case it is not difficult to see that

$$\Psi(f), \Lambda(f) \in \left\{ u \in L^2(\Omega) : |u_f(x) - u_f(y)| \leq d_\varepsilon(x, y), x, y \in \Omega_\varepsilon \right\} \subset K_{\infty,f}^\varepsilon, \quad (3.15)$$

being  $\Psi(f)$  and  $\Lambda(f)$  the McShane-Whitney extensions.

Now, to study the Lipschitz extension problem we can always assume that  $f$  satisfies

$$L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega) = 1. \quad (3.16)$$

In fact: given  $f : \Omega_\varepsilon \setminus \Omega \rightarrow \mathbb{R}$  Lipschitz continuous respect to the distance  $d_\varepsilon$ , consider  $\tilde{f}(x) := \frac{f(x)}{k}$ ,  $k := L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega)$ . Now, if  $v : \Omega_\varepsilon \rightarrow \mathbb{R}$  is  $\text{AMLE}_\varepsilon(\tilde{f}, \Omega)$ , then,  $u(x) := kv(x)$  is  $\text{AMLE}_\varepsilon(f, \Omega)$ .

Consequently, if  $f$  satisfies (3.16), on account of (3.14) and (3.15),  $(v_\infty)_f \in K_{\infty, f}^\varepsilon$ .

Remark that for  $\Omega$  convex, if  $f$  satisfies (3.16) then  $\text{MLE}(f, \Omega_\varepsilon) = \{u_f : u \in K_{\infty, f}^\varepsilon\}$ , and we conclude, directly, that  $(v_\infty)_f \in \text{MLE}(f, \Omega_\varepsilon)$ .

Our aim is to see that  $(v_\infty)_f$  is  $\text{AMLE}_\varepsilon(f, \Omega)$ . To this aim we need the following result.

**Lemma 3.4** *Let  $\delta > 0$ . There exists a unique solution  $u_{\infty, \delta}$  of*

$$\begin{cases} -\Delta_\infty^\varepsilon u = \delta & \text{in } \Omega, \\ u = f + \delta & \text{in } \Omega_\varepsilon \setminus \Omega, \end{cases} \quad (3.17)$$

where

$$\Delta_\infty^\varepsilon u(x) := \sup_{y \in \overline{B}_\varepsilon(x)} u(y) + \inf_{y \in \overline{B}_\varepsilon(x)} u(y) - 2u(x), \quad (3.18)$$

and we have a bound of the form

$$u_{\infty, \delta}(x) - C(\Omega, \varepsilon)\delta \leq u_\infty(x) \leq u_{\infty, \delta}(x),$$

where  $u_\infty$  is the solution of Problem (2.3).

Analogously, there is a unique solution  $u_{\infty, -\delta}$  of

$$\begin{cases} -\Delta_\infty^\varepsilon u = -\delta & \text{in } \Omega, \\ u = f - \delta & \text{in } \Omega_\varepsilon \setminus \Omega, \end{cases} \quad (3.19)$$

and we have a bound of the form

$$u_{\infty, -\delta}(x) + C(\Omega, \varepsilon)\delta \geq u_\infty(x) \geq u_{\infty, -\delta}(x).$$

**Proof.** We use probabilistic arguments. The existence and uniqueness of  $u_{\infty, \delta}$  comes from the fact that it can be obtained as the value of the tug-of-war game with running payoff  $\delta$  and final payoff  $f(x) + \delta$ , see [20], [16]. In fact,

the equation verified by  $u_{\infty,\delta}$  is just the dynamic programming principle that holds for the value function of this game, see [15].

Hence we are left with the proof of the bounds. The fact that  $u_{\infty}(x) \leq u_{\infty,\delta}(x)$  is almost immediate since both functions can be seen as values of the same tug-of-war game in which the running payoff and the final payoff for  $u_{\infty}$  are strictly below than those for  $u_{\infty,\delta}$ . In fact the rules of the Tug-of-War game are the same; to obtain  $u_{\infty,\delta}$  we use running payoff  $\delta$  and final payoff  $f(x) + \delta$ , while for  $u_{\infty}$  we use zero running payoff and final payoff  $f(x)$ . See [16] for a detailed proof of a comparison principle. To see the other bound,

$$u_{\infty,\delta}(x) - C(\Omega, \varepsilon)\delta \leq u_{\infty}(x),$$

we argue as follows. Fix  $\eta > 0$  and using the value function  $u_{\infty,\delta}$  let us choose a strategy for Player I in the game without running payoff and with final payoff  $f$  (this will provide a lower bound for  $u$ ). Player II follows any strategy and Player I follows a strategy  $S_I^0$  such that at  $x_{k-1} \in \Omega$  he chooses to step to a point that almost maximizes  $u_{\infty,\delta}$ , that is, to a point  $x_k \in \overline{B}_{\varepsilon}(x_{k-1})$  such that

$$u_{\infty,\delta}(x_k) \geq \sup_{\overline{B}_{\varepsilon}(x_{k-1})} u_{\infty,\delta} - \eta 2^{-k}.$$

We start from the point  $x_0$ . The following inequality for the expectation holds:

$$\begin{aligned} & \mathbb{E}_{S_I^0, S_{II}}^{x_0} [u_{\infty,\delta}(x_k) + k\delta - \eta 2^{-k} \mid x_0, \dots, x_{k-1}] \\ & \geq \frac{1}{2} \left\{ \inf_{\overline{B}_{\varepsilon}(x_{k-1})} u_{\infty,\delta} + k\delta - \eta 2^{-k} + \sup_{\overline{B}_{\varepsilon}(x_{k-1})} u_{\infty,\delta} - \eta 2^{-k} + k\delta - \eta 2^{-k} \right\} \\ & \geq u_{\infty,\delta}(x_{k-1}) + (k-1)\delta - \eta 2^{-(k-1)}, \end{aligned}$$

where we have estimated the strategy of Player II by inf and used the fact that  $u_{\infty,\delta}$  verifies (3.17). Thus  $M_k = u_{\infty,\delta}(x_k) + k\delta - \eta 2^{-k}$  is a submartingale and consequently, if  $\tau$  is the stopping time of the game, and  $S_{II}^0$  is a quasioptimal strategy for Player II in the game without running payoff and with final payoff  $f$ , that is a strategy such that

$$\inf_{S_{II}} \mathbb{E}_{S_I^0, S_{II}}^{x_0} [f(x_{\tau})] \geq \mathbb{E}_{S_I^0, S_{II}^0}^{x_0} [f(x_{\tau})] - \eta,$$

we deduce that

$$\begin{aligned}
u_\infty(x_0) &= \inf_{S_{II}} \sup_{S_I} \mathbb{E}_{S_I, S_{II}}^{x_0} [f(x_\tau)] \\
&\geq \mathbb{E}_{S_I^0, S_{II}^0}^{x_0} [f(x_\tau)] - \eta \\
&\geq \mathbb{E}_{S_I^0, S_{II}^0}^{x_0} [f(x_\tau) + \delta + \delta\tau - \eta 2^{-\tau} - \delta(\tau + 1)] - \eta \\
&= \mathbb{E}_{S_I^0, S_{II}^0}^{x_0} [M_\tau - \delta(\tau + 1)] - \eta \\
&\geq \limsup_{k \rightarrow \infty} \mathbb{E}_{S_I^0, S_{II}^0}^{x_0} [M_{\tau \wedge k}] - \mathbb{E}_{S_I^0, S_{II}^0}^{x_0} [\delta(\tau + 1)] - \eta \\
&\geq \mathbb{E}_{S_I^0, S_{II}^0}^{x_0} [M_0] - \delta \mathbb{E}_{S_I^0, S_{II}^0}^{x_0} [\tau + 1] - \eta \\
&= u_{\infty, \delta}(x_0) - 2\eta - \delta \mathbb{E}_{S_I^0, S_{II}^0}^{x_0} [\tau + 1],
\end{aligned}$$

where we have used Fatou's Lemma and the Optional Stopping Theorem for the submartingale  $M_k$ . Now, we just observe that, under strategies  $S_I^0, S_{II}^0$ , the game finishes if, in some moment, Player II obtains  $n = n(\Omega, \varepsilon)$  consecutive victories. Now, the expected number of tosses to get  $n$  consecutive victories of Player II is a finite number  $N = N(n)$ . Therefore

$$\mathbb{E}_{S_I^0, S_{II}^0}^{x_0} [\tau] \leq N = c(\Omega, \varepsilon).$$

Consequently, we have

$$u_\infty(x_0) \geq u_{\infty, \delta}(x_0) - 2\eta - \delta C(\Omega, \varepsilon),$$

and, since  $\eta$  was arbitrary, this implies the desired estimate.  $\square$

**Remark 3.5** From [19, Proposition 7.1] and [14, Theorem 1.9] (see also [5]), the expected value for the stopping time for a standard  $\varepsilon$ -tug-of-war game, is  $O(\varepsilon^{-2})$  (see also [16]). Since we are looking at this problem with a fixed  $\varepsilon > 0$  we don't need this more precise estimate.

**Lemma 3.6** *Let  $u \in L^\infty(\Omega)$ . Then,*

$$\lim_{p \rightarrow +\infty} \left( G_{p, f}^{J_\varepsilon}(u) \right)^{\frac{1}{p}} = L_\varepsilon(u_f, \Omega). \quad (3.20)$$

**Proof.** We have

$$\begin{aligned}
\left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} &= \left(\frac{1}{2p} \int_{\Omega} \int_{\Omega} J_\varepsilon(x-y) |u(y) - u(x)|^p dy dx \right. \\
&\quad \left. + \frac{1}{p} \int_{\Omega} \int_{\Omega_\varepsilon \setminus \Omega} J_\varepsilon(x-y) |u_f(y) - u(x)|^p dy dx \right)^{\frac{1}{p}} \\
&\leq \left( (L_\varepsilon(u_f, \Omega))^p \frac{1}{2p} \int_{\Omega} \int_{\Omega} J_\varepsilon(x-y) dy dx \right. \\
&\quad \left. + (L_\varepsilon(u_f, \Omega))^p \frac{1}{p} \int_{\Omega} \int_{\Omega_\varepsilon \setminus \Omega} J_\varepsilon(x-y) dy dx \right)^{\frac{1}{p}} \\
&= L_\varepsilon(u_f, \Omega) \left( \frac{1}{2p} \int_{\Omega} \int_{\Omega} J_\varepsilon(x-y) dy dx + \frac{1}{p} \int_{\Omega} \int_{\Omega_\varepsilon \setminus \Omega} J_\varepsilon(x-y) dy dx \right)^{\frac{1}{p}}.
\end{aligned}$$

Hence,

$$\limsup_{p \rightarrow +\infty} \left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} \leq L_\varepsilon(u_f, \Omega).$$

On the other hand, suppose that

$$\alpha := \liminf_{p \rightarrow +\infty} \left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} < L_\varepsilon(u_f, \Omega).$$

Let  $\tilde{\alpha}$  be such that  $\alpha < \tilde{\alpha} < L_\varepsilon(u_f, \Omega)$ . Then, there exists a set  $A \subset \{(x, y) \in \Omega \times \Omega_\varepsilon : |x - y| \leq \varepsilon\}$ , with positive measure, such that  $|u_f(x) - u_f(y)| > \tilde{\alpha}$  if  $(x, y) \in A$ . Consequently,

$$\begin{aligned}
\left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} &\geq \left(\frac{1}{2p} \int_A J_\varepsilon(x-y) |u_f(y) - u_f(x)|^p dy dx\right)^{\frac{1}{p}} \\
&> \tilde{\alpha} \left(\frac{1}{2p} \int_A J_\varepsilon(x-y) dy dx\right)^{\frac{1}{p}},
\end{aligned}$$

from where it follows the contradiction

$$\alpha = \liminf_{p \rightarrow +\infty} \left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} \geq \tilde{\alpha} > \alpha.$$

Therefore,

$$L_\varepsilon(u_f, \Omega) \leq \liminf_{p \rightarrow +\infty} \left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}},$$

and we have concluded the proof.  $\square$

In the next result we denote  $M_p^\varepsilon := G_{p,f}^{J_\varepsilon}(u_p^\varepsilon) = \min\{G_{p,f}^{J_\varepsilon}(u) : u \in L^p(\Omega)\}$ .

**Theorem 3.7** *Let  $f \in L^\infty(\Omega_\varepsilon \setminus \overline{\Omega})$ ,  $L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega) = 1$ , and let  $u_p^\varepsilon$  a minimizer of  $G_{p,f}^{J_\varepsilon}$ ,  $p \geq 2$ . Then, there exists a sequence  $p_i \rightarrow +\infty$ , as  $i \rightarrow +\infty$ , such that*

$$u_{p_i}^\varepsilon \rightharpoonup v_\infty \in L^\infty(\Omega) \quad \text{in } L^q(\Omega) \quad \text{as } i \rightarrow +\infty, \quad (3.21)$$

$$(M_p^\varepsilon)^{1/p} \rightarrow \inf_{u \in L^\infty(\Omega)} L_\varepsilon(u_f, \Omega) \quad \text{as } p \rightarrow +\infty, \quad (3.22)$$

$$\inf_{u \in L^\infty(\Omega)} L_\varepsilon(u_f, \Omega) = L_\varepsilon((v_\infty)_f, \Omega), \quad (3.23)$$

and  $(v_\infty)_f$  is  $\text{AMLE}_\varepsilon(f, \Omega)$ . Moreover,  $u_p^\varepsilon \rightarrow v_\infty$  pointwise and hence strongly in any  $L^q(\Omega)$ .

**Proof.** Set  $M_\infty^\varepsilon := \inf_{u \in L^\infty(\Omega)} L_\varepsilon(u_f, \Omega)$ . Let  $v \in L^\infty(\Omega)$  such that  $L_\varepsilon(v_f, \Omega) \leq M_\infty^\varepsilon + \delta$ . Then, for  $p$  large enough,

$$\begin{aligned} (M_p^\varepsilon)^{1/p} &\leq G_{p,f}^{J_\varepsilon}(v)^{1/p} \\ &\leq L_\varepsilon(v_f, \Omega) \left( \frac{1}{2p} \int_\Omega \int_\Omega J_\varepsilon(x-y) dy dx + \frac{1}{p} \int_\Omega \int_{\Omega_\varepsilon \setminus \Omega} J_\varepsilon(x-y) dy dx \right)^{\frac{1}{p}} \\ &\leq (M_\infty^\varepsilon + \delta) \left( \frac{1}{p} \int_{\Omega_\varepsilon \times \Omega_\varepsilon} J(x-y) dx dy \right)^{1/p} \leq M_\infty^\varepsilon + 2\delta, \end{aligned}$$

and consequently

$$\limsup_p M_p^{1/p} \leq M_\infty.$$

Fix now  $q \geq 2$ . For  $p > q$ , by Holder inequality,

$$q^{1/q} G_{q,f}^{J_\varepsilon}(u_p^\varepsilon)^{1/q} \leq (2p)^{1/p} (M_p^\varepsilon)^{1/p} \left( \int_{\Omega_\varepsilon \times \Omega_\varepsilon} J_\varepsilon(x-y) dx dy \right)^{1/q-1/p}.$$

Therefore, by Poincaré's inequality (3.2), there exist a subsequence  $p_i$  such that, for any  $q \geq 2$ ,

$$u_{p_i}^\varepsilon \rightharpoonup v_\infty \quad \text{in } L^q(\Omega) \quad \text{as } i \rightarrow +\infty, \quad (3.24)$$

$v_\infty \in L^\infty(\Omega)$ . Moreover, by the lower semicontinuity of  $G_{q,f}^{J_\varepsilon}$ ,

$$G_{q,f}^{J_\varepsilon}(v_\infty)^{1/q} \leq \limsup_p (M_p^\varepsilon)^{1/p} \left( \frac{1}{q} \int_{\Omega_\varepsilon \times \Omega_\varepsilon} J_\varepsilon(x-y) dx dy \right)^{1/q}.$$

Letting now  $q$  to  $+\infty$ , and having in mind Lemma 3.6, we get

$$L_\varepsilon((v_\infty)_f, \Omega) \leq \limsup_p (M_p^\varepsilon)^{1/p} \leq M_\infty^\varepsilon,$$

and we have proved (3.21), (3.22) and (3.23).

Let us prove now that  $(v_\infty)_f$  is  $\text{AMLE}_\varepsilon(f, \Omega)$ . By Theorem 2.3 we need to prove that  $v_\infty$  coincide with the unique solution  $u_\infty$  of Problem (2.3). To this end we want to use comparison arguments. Take  $u_{\infty,\delta}$  as in Lemma 3.4 and regularize it as follows:

$$u_{\infty,\delta}^\theta(x) = u_{\infty,\delta} * \rho_\theta(x),$$

where  $\rho_\theta$  is a usual mollifier. Here the convolution is taken in the whole  $\Omega_\varepsilon$ . As  $u_{\infty,\delta}$  is a solution of (3.17), we get that  $u_{\infty,\delta}^\theta$  is a continuous function that, for  $\theta$  small, verifies pointwise

$$\begin{cases} -\Delta_\infty^\varepsilon u \geq \frac{\delta}{2} & \text{in } \bar{\Omega}, \\ u \geq f + \frac{\delta}{2} & \text{in } \Omega_\varepsilon \setminus \Omega. \end{cases} \quad (3.25)$$

Now, we claim that there exists  $p_{\delta,\theta}$ ,  $p_{\delta,\theta} \rightarrow +\infty$  as  $\delta, \theta \rightarrow 0$ , such that for every  $p \geq p_{\delta,\theta}$  the following inequality holds:

$$\int_{\Omega_\varepsilon} J_\varepsilon(x-y) \varphi_p((u_{\infty,\delta}^\theta)_f(y) - u_{\infty,\delta}^\theta(x)) dy \leq 0 \quad \forall x \in \Omega,$$

being  $\varphi_p(r) := |r|^{p-2}r$ . To see this fact we argue by contradiction. Assume that there exists  $p_n \rightarrow \infty$  and  $x_n \in \Omega$  such that

$$\int_{\Omega_\varepsilon} J_\varepsilon(x_n - y) \varphi_{p_n}((u_{\infty,\delta}^\theta)_f(y) - u_{\infty,\delta}^\theta(x_n)) dy > 0.$$

We rewrite this as

$$\begin{aligned} & \int_{\Omega_\varepsilon \cap \{(u_{\infty,\delta}^\theta)_f(y) > (u_{\infty,\delta}^\theta)_f(x_n)\}} J_\varepsilon(x_n - y) \varphi_{p_n}((u_{\infty,\delta}^\theta)_f(y) - u_{\infty,\delta}^\theta(x_n)) dy \\ & > \int_{\Omega_\varepsilon \cap \{(u_{\infty,\delta}^\theta)_f(y) < (u_{\infty,\delta}^\theta)_f(x_n)\}} J_\varepsilon(x_n - y) \varphi_{p_n}((u_{\infty,\delta}^\theta)_f(x_n) - u_{\infty,\delta}^\theta(y)) dy. \end{aligned}$$

Thus,

$$\begin{aligned} & \left( \int_{\Omega_\varepsilon \cap \{(u_{\infty,\delta}^\theta)_f(y) > (u_{\infty,\delta}^\theta)_f(x_n)\}} J_\varepsilon(x_n - y) \varphi_{p_n}((u_{\infty,\delta}^\theta)_f(y) - (u_{\infty,\delta}^\theta)_f(x_n)) dy \right)^{\frac{1}{p_n-1}} \\ & > \left( \int_{\Omega_\varepsilon \cap \{(u_{\infty,\delta}^\theta)_f(y) < (u_{\infty,\delta}^\theta)_f(x_n)\}} J_\varepsilon(x_n - y) \varphi_{p_n}((u_{\infty,\delta}^\theta)_f(x_n) - (u_{\infty,\delta}^\theta)_f(y)) dy \right)^{\frac{1}{p_n-1}}. \end{aligned}$$

Then, passing to the limit, using that  $\bar{\Omega}$  is compact (hence we can assume that  $x_n \rightarrow x_0$ ) and that  $u_{\infty,\delta}^\theta$  is a uniformly continuous function that does not depend on  $n$ , we obtain

$$\sup_{y \in \bar{B}_\varepsilon(x_0)} u_{\infty,\delta}^\theta(y) + \inf_{y \in \bar{B}_\varepsilon(x_0)} u_{\infty,\delta}^\theta(y) - 2u_{\infty,\delta}^\theta(x_0) \geq 0,$$

a contradiction with the fact that  $u_{\infty,\delta}^\theta$  verifies (3.25).

Therefore  $u_{\infty,\delta}^\theta$  is a supersolution of the problem for every  $p \geq p_{\delta,\theta}$  and, using the comparison principle given in Lemma 3.2, we have  $u_p^\varepsilon \leq u_{\infty,\delta}^\theta$  for every

$p \geq p_{\delta,\theta}$ . Therefore, letting  $p \rightarrow \infty$ , we get  $v_\infty \leq u_{\infty,\delta}^\theta$ . Now, we let  $\theta \rightarrow 0$  and use the bounds in Lemma 3.4 to obtain

$$v_\infty \leq u_\infty(x) + C\delta.$$

Finally, we take  $\delta \rightarrow 0$  and conclude that

$$v_\infty \leq u_\infty.$$

A symmetric argument using a regularization of  $u_{\infty,-\delta}$  as subsolution proves the reverse inequality. Hence we have that

$$v_\infty = u_\infty.$$

In addition, since

$$u_{\infty,-\delta}^\theta \leq u_p^\varepsilon \leq u_{\infty,\delta}^\theta \quad \forall p \geq p_{\delta,\theta},$$

and  $u_{\infty,-\delta}^\theta, u_{\infty,\delta}^\theta \rightarrow u_\infty$  pointwise as  $\theta, \delta \rightarrow 0$ , we have

$$u_p^\varepsilon \rightarrow v_\infty \quad \text{pointwise as } p \rightarrow +\infty.$$

□

## 4 Viscosity solutions

The solutions of problem (1.2) are usually understood in the viscosity sense, nevertheless, in Theorem 2.3, we have understood the solution of Problem (2.3) in the pointwise sense:  $u \in L^\infty(\Omega_\varepsilon)$  is a solution of (2.3) if

$$\begin{cases} \sup_{y \in \overline{B}_\varepsilon(x)} u(y) + \inf_{y \in \overline{B}_\varepsilon(x)} u(y) - 2u(x) = 0 & \text{for all } x \in \Omega, \\ u(x) = f(x) & \text{for all } x \in \Omega_\varepsilon \setminus \Omega. \end{cases}$$

In this section we will see that this concept implies also the viscosity one.

Since the solutions of Problem (2.3) are discontinuous in general (see the Appendix), to work with viscosity solutions we need to use the generalized definition of discontinuous viscosity solutions. Let us consider the upper and lower semi-continuous envelopes of  $u$  in  $\Omega_\varepsilon$  defined as

$$u^*(x) := \limsup_{y \in \Omega_\varepsilon, y \rightarrow x} u(y) \quad \text{and} \quad u_*(x) := \liminf_{y \in \Omega_\varepsilon, y \rightarrow x} u(y),$$



respectively. Then, we say that  $u \in L^\infty(\Omega_\varepsilon)$  is a *viscosity subsolution* of problem (2.3) if  $u(x) = f(x)$  for almost all  $x \in \Omega_\varepsilon \setminus \Omega$  and  $-\Delta_\infty^\varepsilon \phi(x_0) \leq 0$  when  $\phi \in C(\Omega_\varepsilon)$ ,  $\phi(x_0) = u^*(x_0)$  and  $u^* - \phi$  achieves a maximum at  $x_0 \in \Omega$ . Likewise,  $u \in L^\infty(\Omega_\varepsilon)$  is a *viscosity supersolution* of problem (2.3) if  $u(x) = f(x)$  for almost all  $x \in \Omega_\varepsilon \setminus \Omega$  and  $-\Delta_\infty^\varepsilon \phi(x_0) \geq 0$  when  $\phi \in C(\Omega_\varepsilon)$ ,  $\phi(x_0) = u_*(x_0)$  and  $u_* - \phi$  achieves a minimum at  $x_0 \in \Omega$ . We say that  $u$  is a *viscosity solution* of problem (2.3) if  $u$  is both a viscosity subsolution and a viscosity supersolution.

**Proposition 4.1** *Let  $u \in L^\infty(\Omega_\varepsilon)$ . We have*

- (i) *If  $-\Delta_\infty^\varepsilon u(x) \leq 0$  for all  $x \in \Omega$ , then  $-\Delta_\infty^\varepsilon u^*(x) \leq 0$  for all  $x \in \Omega$  and consequently  $u$  is a viscosity subsolution of Problem (2.3).*
- (ii) *If  $-\Delta_\infty^\varepsilon u(x) \geq 0$  for all  $x \in \Omega$ , then  $-\Delta_\infty^\varepsilon u_*(x) \geq 0$  for all  $x \in \Omega$  and consequently  $u$  is a viscosity supersolution of Problem (2.3).*
- (iii) *If  $-\Delta_\infty^\varepsilon u(x) = 0$  for all  $x \in \Omega$ , then  $u$  is a viscosity solution of Problem (2.3).*

**Proof.** We are going to prove (i), the proof of (ii) is similar, and (iii) is a consequence of (i) and (ii).

Fix  $x_0 \in \Omega$ , and let  $x_k \in \Omega$  such that  $x_k \rightarrow x_0$  and  $u^*(x_0) = \lim_{k \rightarrow \infty} u(x_k)$ . Fix  $0 < \delta < \frac{\varepsilon}{2}$ , and select for each  $k \in \mathbb{N}$ , point  $y_k, z_k \in B_\varepsilon(x_k)$  such that

$$\sup_{y \in B_\varepsilon(x_k)} u(y) \leq u(y_k) + \delta, \quad \inf_{z \in B_\varepsilon(x_k)} u(z) \geq u(z_k) - \delta. \quad (4.1)$$

By taking subsequences, we may assume that  $y_k \rightarrow y \in \overline{B}_\varepsilon(x_0)$  and  $z_k \rightarrow z \in \overline{B}_\varepsilon(x_0)$ . Then,

$$\begin{aligned} \sup_{x \in \overline{B}_\varepsilon(x_0)} u^*(x) - u^*(x_0) &\geq u^*(y) - u^*(x_0) \geq \limsup_{k \rightarrow +\infty} (u(y_k) - u(x_k)) \\ &\geq \limsup_{k \rightarrow +\infty} \left( \sup_{y \in \overline{B}_\varepsilon(x_k)} u(y) - \delta - u(x_k) \right). \end{aligned}$$

Sending  $\delta \rightarrow 0^+$ , we get

$$\sup_{x \in \overline{B}_\varepsilon(x_0)} u^*(x) - u^*(x_0) \geq \limsup_{k \rightarrow +\infty} \left( \sup_{y \in \overline{B}_\varepsilon(x_k)} u(y) - u(x_k) \right). \quad (4.2)$$

On the other hand,

$$\begin{aligned} u^*(x_0) - \inf_{x \in \overline{B}_\varepsilon(x_0)} u^*(x) &\leq u^*(x_0) - u^*(z) \leq \liminf_{k \rightarrow +\infty} (u(x_k) - u(z_k)) \\ &\leq \liminf_{k \rightarrow +\infty} \left( u(x_k) - \inf_{z \in \overline{B}_\varepsilon(x_k)} u(z) + \delta \right). \end{aligned}$$

Sending  $\delta \rightarrow 0^+$ , we get

$$u^*(x_0) - \inf_{x \in \overline{B}_\varepsilon(x_0)} u^*(x) \leq \liminf_{k \rightarrow +\infty} \left( u(x_k) - \inf_{z \in \overline{B}_\varepsilon(x_k)} u(z) \right). \quad (4.3)$$

From (4.2) and (4.3), and having in mind that by hypothesis we have  $-\Delta_\infty^\varepsilon u \leq 0$ , we obtain that

$$\begin{aligned} -\Delta_\infty^\varepsilon u_*(x_0) &= - \left( \sup_{x \in \overline{B}_\varepsilon(x_0)} u^*(x) - u^*(x_0) \right) - \left( \inf_{x \in \overline{B}_\varepsilon(x_0)} u^*(x) - u^*(x_0) \right) \\ &\leq \liminf_{k \rightarrow +\infty} \left( u(x_k) - \sup_{y \in \overline{B}_\varepsilon(x_k)} u(y) + u(x_k) - \inf_{z \in \overline{B}_\varepsilon(x_k)} u(z) \right) \\ &= \liminf_{k \rightarrow +\infty} (-\Delta_\infty^\varepsilon u(x_k)) \leq 0. \end{aligned}$$

This ends the proof.  $\square$

Problem (2.3) has not continuous solutions even for continuous boundary data, however assuming the continuity of the data and the continuity of the solution at the boundary, adapting an argument due to Le Gruyer and J. C. Archer [13] (see also [4]), we obtain the following result.

**Proposition 4.2** *Let  $f : \Omega_\varepsilon \setminus \Omega \rightarrow \mathbb{R}$  be a continuous function. If  $u : \Omega_\varepsilon \rightarrow \mathbb{R}$  is a solution of*

$$\begin{cases} -\Delta_\infty^\varepsilon u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Omega_\varepsilon \setminus \Omega, \end{cases}$$

*and we assume that  $u^*(x) = u_*(x) = f(x)$  for all  $x \in \Omega_\varepsilon \setminus \Omega$ , then  $u$  is continuous in  $\Omega_\varepsilon$ .*

**Proof.** By Lemma 4.1, we have

$$-\Delta_\infty^\varepsilon u^*(x) \leq 0 \leq -\Delta_\infty^\varepsilon u_*(x) \quad \text{for all } x \in \Omega. \quad (4.4)$$

Set  $\alpha := \sup\{u^*(x) - u_*(x) : x \in \overline{\Omega}\}$ . To prove the result it is enough to show that  $\alpha = 0$ . Arguing by contradiction, we suppose  $\alpha > 0$ . By the upper semi-continuity of the function  $u^* - u_*$  and having in mind that  $u^*(x) = u_*(x) = f(x)$  for all  $x \in \Omega_\varepsilon \setminus \Omega$ , we have the set

$$A := \{x \in \Omega_\varepsilon : (u^* - u_*)(x) = \alpha\}$$

is nonempty, closed and contained in  $\Omega$ . Define  $B := \{x \in A : u^*(x) = \max_A u^*\}$ . By the upper semi-continuity of the function  $u^*$ ,  $B$  is nonempty.

Then, take  $x_0 \in \partial B$ . Since  $x_0 \in A$ , we have

$$(u^* - u_*)(x_0) \geq \sup_{x \in \overline{B_\varepsilon(x_0)}} (u^* - u_*)(x) \geq \inf_{x \in \overline{B_\varepsilon(x_0)}} u^*(x) - \inf_{x \in \overline{B_\varepsilon(x_0)}} u_*(x). \quad (4.5)$$

First suppose that

$$u^*(x_0) = \sup_{x \in \overline{B_\varepsilon(x_0)}} u^*(x).$$

Then, since  $-\Delta_\infty^\varepsilon u^*(x_0) \leq 0$ , we have

$$u^*(x_0) = \inf_{x \in \overline{B_\varepsilon(x_0)}} u^*(x),$$

and by (4.5) we deduce that

$$u_*(x_0) = \inf_{x \in \overline{B_\varepsilon(x_0)}} u_*(x).$$

Then, since  $-\Delta_\infty^\varepsilon u_*(x_0) \geq 0$ , we obtain that

$$u_*(x_0) = \sup_{x \in \overline{B_\varepsilon(x_0)}} u_*(x).$$

Therefore,  $u^*$  and  $u_*$  are constant in  $\overline{B_\varepsilon(x_0)}$ , contradicting our assumption that  $x_0 \in \partial B$ .

It remains to arrive to a contradiction in the case

$$u^*(x_0) < \sup_{x \in \overline{B_\varepsilon(x_0)}} u^*(x).$$

By the upper semi-continuity of the function  $u^*$  there is  $y_0 \in \overline{B_\varepsilon(x_0)}$  such that

$$u^*(y_0) = \sup_{x \in \overline{B_\varepsilon(x_0)}} u^*(x).$$

Since  $u^*(y_0) > u^*(x_0)$  and  $x_0 \in B$ , we see that  $y_0 \notin A$ . Then,

$$u^*(y_0) - u_*(y_0) < \alpha = u^*(x_0) - u_*(x_0).$$

Hence,

$$\begin{aligned} \sup_{x \in \overline{B_\varepsilon(x_0)}} u_*(x) - u_*(x_0) &\geq u_*(y_0) - u_*(x_0) \\ &> u^*(y_0) - u^*(x_0) = \sup_{x \in \overline{B_\varepsilon(x_0)}} u^*(x) - u^*(x_0). \end{aligned} \quad (4.6)$$

Combining (4.5) and (4.6), we obtain  $-\Delta_\infty^\varepsilon u_*(x_0) < -\Delta_\infty^\varepsilon u^*(x_0)$ , which contradicts (4.4), and the proposition follows.  $\square$

## 5 Appendix: Examples

In this appendix we collect some concrete examples that are illustrative of the difficulties of the problem. In the first example we see that there exists  $f$  for which the  $\text{AMLE}_1$  of  $f$  is not AMLE of  $f$  in the sense of Definition 1.1 (in fact, there is no AMLE of  $f$  in that sense).

**Example 5.1** For  $\varepsilon = 1$ ,  $\Omega = (0, \frac{1}{2})$  and  $f = 0\chi_{(-1,0]} + 1\chi_{[\frac{1}{2}, \frac{3}{2})}$ , for any  $z$  defined in  $(0, \frac{1}{2})$  such that  $z(x) \in [0, 1]$ ,  $f + z\chi_{(0, \frac{1}{2})} \in \text{MLE}(f, \Omega_1)$ . Between all of them,  $u = f + \frac{1}{2}\chi_{(0, \frac{1}{2})}$  is the unique  $\text{AMLE}_1(f, \Omega)$  (it is very easy to prove that it is solution of (2.3)). On the other hand, there is not AMLE of  $f$  in the sense of Definition 1.1. In fact, if  $u$  is AMLE of  $f$ , then if  $B = (-\frac{1}{2}, \frac{1}{2})$ , the function  $g = 0\chi_{(-1, \frac{1}{2})} + 1\chi_{(\frac{1}{2}, \frac{3}{2})} \in \text{MLE}(f, \Omega_1)$  and  $g = u$  in  $\Omega_1 \setminus B$ , therefore  $L_{d_1}(u, B) \leq L_{d_1}(g, B) = 0$ , and, hence,  $u$  is constant in  $B$ , that is,  $u = 0$  in  $(0, \frac{1}{2})$ . Similarly, we can prove that  $u = 1$  in  $(0, \frac{1}{2})$  by taking  $B = (0, 1)$  and  $g = 0\chi_{(-1,0)} + 1\chi_{(0, \frac{3}{2})}$ , which gives a contradiction.

**Example 5.2** For  $\varepsilon = 1$ ,  $\Omega = (0, 2)$  and  $f = x\chi_{(-1,0]} + 2\chi_{[2,3)}$ , the unique solution  $u$  of (2.3) can be explicitly found as follows. First, we observe that  $u$  is increasing in  $x$ . Indeed, Since  $L_{d_1}(f, \Omega_1 \setminus \Omega) = 1$ , it is easy to see that the McShane-Whitney extensions are given in  $\Omega$  by

$$\Psi(f)(x) = x, \quad \Lambda(f)(x) = 0\chi_{(0,1)}(x) + 1\chi_{[1,2)}(x).$$

Then, if  $u$  is the solution of (2.3), since  $\Omega$  is convex, by Theorem 2.3,  $u \in \text{MLE}(f, \Omega_1)$  and therefore

$$0\chi_{(0,1)}(x) + 1\chi_{[1,2)}(x) \leq u(x) \leq x \quad \forall x \in (0, 2). \quad (5.1)$$

By (5.1), for any  $x \in (0, 1)$  we have

$$u(x) = \frac{1}{2}x + \frac{1}{2} \sup_{y \in [1, x+1]} u(y),$$

so it is nondecreasing in this interval. For any  $x \in (1, 2)$  we have

$$u(x) = \frac{1}{2} \inf_{y \in [x-1, 1]} u(y) + 1,$$

so it also is nondecreasing in this interval. So, taking into account again (5.1),  $u$  is nondecreasing in all  $\Omega = (0, 2)$ . Therefore, for any  $x \in (0, 1)$  we have

$$u(x+1) = 2u(x) + 1 - x$$

and for any  $z \in (1, 2)$  we have

$$2 = u(z + 1) = 2u(z) - u(z - 1)$$

but taking  $z - 1 = x$  we get,

$$2 = 2u(x + 1) - u(x) = 3u(x) + 2 - 2x$$

and we conclude that

$$u(x) = \frac{2}{3}x, \quad x \in (0, 1).$$

This implies

$$u(x) = 1 + \frac{1}{3}(x - 1), \quad x \in (1, 2).$$

Finally,  $u(1) = 1$ .

Note that  $u_*(2) = \frac{4}{3} < 2 = u^*(2) = f(2)$  and  $u$  is discontinuous at  $x = 1$ , therefore, the assumption  $u^*(x) = u_*(x) = f(x)$  for all  $x \in \Omega_\varepsilon \setminus \Omega$  in Proposition 4.2 is necessary for the continuity of  $u$  on  $\Omega$ .

**Example 5.3** For  $\varepsilon = 3/2$ ,  $\Omega = (0, 2)$  and  $f = x\chi_{(-\frac{3}{2}, 0]} + 2\chi_{[2, \frac{7}{2}]}$ , the unique solution  $u$  of (2.3) can also be explicitly found as follows. Since  $L_{d_{\frac{3}{2}}}(f, \Omega_{\frac{3}{2}} \setminus \Omega) = 1$ , it is easy to see that the McShane-Whitney extensions are given in  $\Omega$  by

$$\Psi(f)(x) = x, \quad \Lambda(f)(x) = -1\chi_{(0, \frac{1}{2})}(x) + \frac{1}{2}\chi_{[\frac{1}{2}, 2)}(x).$$

Then, if  $u$  is the solution of (2.3), since  $\Omega$  is convex, by Theorem 2.3,  $u \in \text{MLE}(f, \Omega_{\frac{3}{2}})$  and therefore

$$-1\chi_{(0, \frac{1}{2})}(x) + \frac{1}{2}\chi_{[\frac{1}{2}, 2)}(x) \leq u(x) \leq x \quad \forall x \in (0, 2). \quad (5.2)$$

By (5.2), for any  $x \in (0, 1/2)$  we have

$$u(x) = \frac{1}{2} \left( x - \frac{3}{2} \right) + \frac{1}{2} \sup_{y \in [\frac{1}{2}, x + \frac{3}{2}]} u(y),$$

so it is nondecreasing in this interval. For any  $x \in (1/2, 2)$  we have

$$u(x) = \frac{1}{2} \inf_{y \in [x - \frac{3}{2}, \frac{1}{2}]} u(y) + 1,$$

so it also is nondecreasing in this interval. So, taking into account again (5.2),  $u$  is nondecreasing in all  $\Omega = (0, 2)$ . Consequently, we have

$$u(x) = \frac{1}{2} \left( x - \frac{3}{2} \right) + \frac{1}{2} u \left( x + \frac{3}{2} \right) \quad \text{if } x \in (0, 1/2) \quad (5.3)$$

and

$$u(x) = \frac{1}{2}u\left(x - \frac{3}{2}\right) + 1 \quad \text{if } x \in (1/2, 2). \quad (5.4)$$

Now, if  $x \in (3/2, 2)$ , since  $x - \frac{3}{2} \in (0, 1/2)$ , by (5.3) and (5.4), we have

$$u(x) = \frac{1}{2}\left(\frac{1}{2}(x - 3) + \frac{1}{2}u(x)\right) + 1,$$

from where it follows that

$$u(x) = \frac{1}{3} + \frac{1}{3}x, \quad x \in (3/2, 2).$$

Similarly, if  $x \in (0, 1/2)$ , since  $x + \frac{3}{2} \in (3/2, 2)$ , by (5.3) and (5.4), we have

$$u(x) = \frac{1}{2}\left(x - \frac{3}{2}\right) + \frac{1}{2}\left(\frac{1}{2}u(x) + 1\right),$$

from where it follows that

$$u(x) = \frac{2}{3}x - \frac{1}{3}, \quad x \in (0, 1/2).$$

Let us see now how  $u$  is in  $[\frac{1}{2}, \frac{3}{2}]$ . If  $x \in [\frac{1}{2}, \frac{7}{6}]$ , since  $\inf_{\overline{B}_{\frac{3}{2}}(x)} u(y)$  is taken in  $x - \frac{3}{2}$ , we have

$$u(x) = \frac{1}{2}\left(x - \frac{3}{2}\right) + \frac{1}{2}2 = \frac{1}{2}x + \frac{1}{4}.$$

And, if  $x \in [\frac{7}{6}, \frac{3}{2}]$  since  $\inf_{\overline{B}_{\frac{3}{2}}(x)} u(y) = -\frac{1}{3}$ , we have

$$u(x) = \frac{5}{6}, \quad x \in [7/6, 3/2].$$

Finally  $u(1/2) = 1/2$ . And we have arrived at

$$u(x) = \begin{cases} \frac{2}{3}x - \frac{1}{3}, & x \in (0, 1/2), \\ \frac{1}{2}x + \frac{1}{4}, & x \in [1/2, 7/6], \\ \frac{5}{6}, & x \in [7/6, 3/2], \\ \frac{1}{3} + \frac{1}{3}x, & x \in [3/2, 2). \end{cases}$$

Observe that  $u(x) < 0$  for  $0 < x < \frac{1}{2}$ ;  $u$  is increasing in  $\Omega$  but not in the whole  $\Omega_\varepsilon$ .

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