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OBSTACLE PROBLEMS FOR DEGENERATE ELLIPTIC EQUATIONS WITH NONHOMOGENEOUS NONLINEAR BOUNDARY CONDITIONS

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In this paper we study the questions of existence and uniqueness of solutions for equations of type $-\operatorname{div}\mathbf{a}(x,Du)+\gamma(u)\ni\phi$, posed in an open bounded subset Ω of $\mathbb{R}^N$, with nonlinear boundary conditions of the form $\mathbf{a}(x,Du)\cdot\eta+\beta(u)\ni\psi$. The nonlinear elliptic operator $\operatorname{div}\mathbf{a}(x,Du)$ modeled on the p-Laplacian operator $\Delta_p(u)=\operatorname{div}(|Du|^{p-2}Du)$, with $p>1$, γ and β maximal monotone graphs in $\mathbb{R}^2$ such that $0\in\gamma(0)\cap\beta(0)$, $\mathbb{R}\neq\overline{D(\gamma)}\subset D(\beta)$ and the data $\phi\in L^1(\Omega)$ and $\psi\in L^1(\partial\Omega)$. Since $D(\gamma)\neq\mathbb{R}$, we are dealing with obstacle problems. For this kind of problems the existence of weak solution, in the usual sense, fails to be true for nonhomogeneous boundary conditions, so a new concept of solution has to be introduced.

Keywords: Obstacle problem; degenerate elliptic equation; p-Laplacian operator; nonlinear boundary conditions.

AMS Subject Classification: 35J60, 35D02

1. Introduction

The purpose of this paper is to establish the existence and uniqueness of solutions for a degenerate elliptic obstacle problem with nonlinear boundary condition of the form

$$(S^{\gamma,\beta}_{\phi,\psi}) \quad \begin{cases} -\operatorname{div} \mathbf{a}(x, Du) + \gamma(u) \ni \phi & \text{in } \Omega, \\ \mathbf{a}(x, Du) \cdot \eta + \beta(u) \ni \psi & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, the function $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray–Lions conditions (see Sec. 3.1), η is the unit outward normal on $\partial\Omega$, $\phi \in L^1(\Omega)$, $\psi \in L^1(\partial\Omega)$ and the nonlinearities γ and β are maximal monotone graphs in $\mathbb{R}^2$ (see, e.g. Ref. 10) such that $0 \in \gamma(0) \cap \beta(0)$ and

$$\mathbb{R} \neq \overline{D(\gamma)} \subset D(\beta).$$

Notice that the general nonlinear diffusion operators of Leray–Lions type, different from the Laplacian, appear when one deals with non-Newtonian fluids (see, e.g. Ref. 4).

Let us remark that since β may be multivalued, this allows to study many nonlinear fluxes on the boundary that occur in some problems in Mechanics and Physics (see, e.g. Ref. 13 or 8). For instance, in the Signorini problem (see, e.g. Refs. 12, 14 and 15) which appears in elasticity and corresponds to the monotone graph

$$\beta(r) = \begin{cases} \emptyset & \text{if } \; r < 0, \\ ]-\infty, 0] & \text{if } \; r = 0, \\ 0 & \text{if } \; r > 0, \end{cases}$$

in problems of optimal control of temperature and in the modelling of semipermeability (see Ref. 13), which corresponds in some cases to the monotone graph

$$\beta(r) = \begin{cases} \emptyset & \text{if } \; r < a, \\ ]-\infty, 0] & \text{if } \; r = a, \\ 0 & \text{if } \; r \in]a, b[, \\ [0, +\infty[& \text{if } \; r = b, \\ \emptyset & \text{if } \; r > b, \end{cases}$$

where $a < 0 < b$.

Observe that if $D(\gamma)$ is not bounded, we are dealing with a one-obstacle problem and with a two-obstacle problem if $D(\gamma)$ is bounded. These problems are also called unilateral problems in the literature. Obstacle problems appear in different physical context, for instance, in deformation of membrane constrained by an obstacle, in bending of elastic isotropic homogeneous plat over an obstacle and in cavitation problems in hydrodynamic lubrication. Notice also that some free boundary problems fall into this scope by using Baiocchi transformation (see Ref. 5), for more details concerning physical applications we refer to Ref. 17 or 13. We want to stress that for this kind of problems the existence of weak solution, in the usual sense, fails to be true for nonhomogeneous boundary conditions.

In the particular case $\mathbf{a}(x,\xi) = \xi$, the problem $(S^{\gamma,\beta}_{\phi,\psi})$ reads

$$(L^{\gamma,\beta}_{\phi,\psi}) \quad \begin{cases} -\Delta u + \gamma(u) \ni \phi & \text{in } \Omega, \\ \partial_\eta u + \beta(u) \ni \psi & \text{on } \partial\Omega, \end{cases}$$

where $\partial_\eta u$ simply denotes the outward normal derivative of u. In the homogeneous case, $\psi \equiv 0$, the pioneering works are the paper by Brezis,[8] in which problem $(L^{\gamma,\beta}_{\phi,0})$ is studied for γ the identity, β a maximal monotone graph and $\phi \in L^2(\Omega)$, and the paper by Brezis and Strauss,[9] in which problem $(L^{\gamma,\beta}_{\phi,0})$ is studied for $\phi \in L^1(\Omega)$ and γ, β continuous nondecreasing functions from $\mathbb{R}$ into $\mathbb{R}$ with $\gamma' \geq \epsilon > 0$. These works were extended by Bénilan, Crandall and Sacks[6] to the case of any γ and β maximal monotone graphs in $\mathbb{R}^2$ such that $0 \in \gamma(0) \cap \beta(0)$. Among other results, in Ref. 6 it is proved that for any $\phi \in L^1(\Omega)$ satisfying some natural range condition (see (2.1)) there exists a unique, up to a constant for u, weak solution of $(L^{\gamma,\beta}_{\phi,0})$, i.e. $[u,z,w] \in W^{1,1}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$, $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, such that

$$\int_\Omega Du \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_\Omega \phi v, \tag{1.1}$$

for all $v \in W^{1,\infty}(\Omega)$. In particular, if $\phi \in L^2(\Omega)$, then $z \in L^2(\Omega)$, $w \in L^2(\partial\Omega)$, $u \in H^1(\Omega)$, and (1.1) is fulfilled for any $v \in H^1(\Omega)$.

In Refs. 2 and 3, we extend the results of Ref. 6 by proving the existence and uniqueness of weak (or entropy) solutions, for the general nonhomogeneous problem $(S^{\gamma,\beta}_{\phi,\psi})$ in the following two cases:

(a) $D(\gamma) = \mathbb{R}$ and, $D(\beta) = \mathbb{R}$ or $\operatorname{div} \mathbf{a}(x, Du) = \Delta_p(u)$,
(b) $\psi \equiv 0$ and, $D(\beta) = \mathbb{R}$ or $\operatorname{div} \mathbf{a}(x, Du) = \Delta_p(u)$.

Recall that in these papers the concept of weak solution, for which existence and uniqueness are proved, is a triple of functions $[u,z,w] \in W^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ such that $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_\Omega \phi v + \int_{\partial\Omega} \psi v,$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

We want to point out that the nonhomogeneous problem, $\psi \not\equiv 0$, is quite different from the homogeneous one, because even if the range condition is satisfied, $(S^{\gamma,\beta}_{\phi,\psi})$ may be ill-posed. For instance, let us consider the obstacle problem

$$(L^{\gamma,0}_{\phi,\psi}) \quad \begin{cases} -\Delta u + \gamma(u) \ni \phi & \text{in } \Omega, \\ \partial_\eta u = \psi & \text{on } \partial\Omega, \end{cases}$$

where γ is a maximal monotone graph with $D(\gamma) = [0,1]$ and $0 \in \gamma(0)$, $\phi \in L^1(\Omega)$, $\phi \leq 0$ a.e. in Ω, and $\psi \in L^1(\partial\Omega)$, $\psi \leq 0$ a.e. in $\partial\Omega$. If we assume there exists a weak solution $[u, z, w]$ of problem $(L^{\gamma,0}_{\phi,\psi})$, then $w = 0$ and $z \in \gamma(u)$. Therefore $0 \leq u \leq 1$ a.e. in Ω, and for any $v \in H^1(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega Du \cdot Dv + \int_\Omega zv = \int_\Omega \phi v + \int_{\partial\Omega} \psi v.$$

Taking $v = u$, since $u \geq 0$, we get

$$0 \leq \int_\Omega |Du|^2 + \int_\Omega zu = \int_\Omega \phi u + \int_{\partial\Omega} \psi u \leq 0.$$

Hence, $\int_\Omega |Du|^2 = 0$, so u is constant and

$$\int_\Omega zv = \int_\Omega \phi v + \int_{\partial\Omega} \psi v,$$

for any $v \in H^1(\Omega) \cap L^\infty(\Omega)$, in particular for any $v \in H^1_0(\Omega) \cap L^\infty(\Omega)$. Consequently, $\phi = z$ a.e. in Ω, and ψ must be 0 a.e. in $\partial\Omega$. Hence in this case, where $D(\gamma) = [0, 1] \subsetneq D(\beta) = \mathbb{R}$, the domain of γ creates some obstruction phenomena for the existence of weak solutions when $\psi \leq 0$, $\psi \not\equiv 0$.

In order to overcome the above problem, the main goal of this paper is to get a new notion of solution for which the existence and uniqueness can be obtained for nonhomogeneous boundary conditions in the case $D(\gamma) \neq \mathbb{R}$. This new notion of solution coincides with the concept of weak solution introduced in Ref. 3 in the cases (a) and (b).

To give an idea of this new concept of solution, let us consider again the problem $(L^{\gamma,0}_{\phi,\psi})$ introduced above with $\phi \in L^{p'}(\Omega)$ and $\psi \in L^{p'}(\partial\Omega)$ and $D(\gamma) = [0, 1]$. In order to get existence it is usual to use the approximate problems

$$(L^{\gamma_r,0}_{\phi,\psi}) \quad \begin{cases} -\Delta u_r + \gamma_r(u_r) = \phi & \text{in } \Omega, \\ \partial_\eta u_r = \psi & \text{on } \partial\Omega, \end{cases}$$

where γ_r is the Yosida approximation of γ. Now, thanks to Ref. 3, the estimates we can obtain for $\gamma_r(u_r)$ are essentially in $L^1(\Omega)$ (see Theorem 3.1) and in $H^1(\Omega)$ for u_r. Therefore we have to pass to the limit weakly-star in the space of measures for $\gamma_r(u_r)$ and weakly in $H^1(\Omega)$ for u_r. So that the standard analysis for this kind of problems allows us to obtain a couple $[u, \mu]$, where $u \in H^1(\Omega)$, μ is a diffuse Radon measure in $\mathbb{R}^N$ concentrated in $\overline{\Omega}$ and

$$\int_\Omega Du \cdot Dv + \int_\Omega v\, d\mu + \int_{\partial\Omega} v\, d\mu = \int_\Omega \phi v + \int_{\partial\Omega} \psi v \quad \forall\, v \in H^1(\Omega) \cap L^\infty(\Omega).$$

In a first step, we prove that the Radon–Nykodym decomposition of μ relatively to the Lebesgue measure, $\mu = \mu_a + \mu_s$, is such that $\mu_a \in \gamma(u)$ a.e. in Ω and μ_s is concentrated on $\{x \in \overline{\Omega};\ u(x) = 0\} \cup \{x \in \overline{\Omega};\ u(x) = 0\}$ with

$$\mu_s \leq 0 \quad \text{on} \quad \{x \in \overline{\Omega};\ u(x) = 0\} \qquad \text{and} \qquad \mu_s \geq 0 \quad \text{on} \quad \{x \in \overline{\Omega};\ u(x) = 1\}.$$

Then, an accurate analysis allows us to prove moreover that μ_s is concentrated on $\partial\Omega$ and it is absolutely continuous with respect to an integrable function on the boundary, which implies that $\mu_s \in L^1(\partial\Omega)$. So, $[u, \mu_a, \mu_s]$ is a weak solution, in the usual sense, of the problem

$$\begin{cases} -\Delta u + \gamma(u) \ni \phi & \text{in } \Omega, \\ Du \cdot \eta + \partial \mathbb{I}_{[0,1]}(u) \ni \psi & \text{on } \partial\Omega, \end{cases}$$

where, for an interval $I \subset \mathbb{R}$, $\partial \mathbb{I}_I$ denotes the subdifferential of the indicator function of I,

$$\mathbb{I}_I(r) = \begin{cases} 0 & \text{if } r \in I, \\ +\infty & \text{if } r \notin I, \end{cases}$$

which is the maximal monotone graph defined by $D(\partial \mathbb{I}_I) = I$ and $\partial \mathbb{I}_I(r) = 0$ for $r \in \mathrm{int}(I)$. For instance, if $\overline{\mathcal{D}(\gamma)} = [a, b]$, then

$$\partial \mathbb{I}_{\overline{\mathcal{D}(\gamma)}}(r) = \begin{cases}]-\infty, 0] & \text{if } \ r = a, \\ 0 & \text{if } \ a < r < b, \\ [0, +\infty[& \text{if } \ r = b. \end{cases}$$

In other words, the boundary condition needs to be fulfilled in the following sense

$$\begin{cases} \partial_\eta u = \psi & \text{on } [0 < u < 1], \\ \partial_\eta u \geq \psi & \text{on } [u = 1], \\ \partial_\eta u \leq \psi & \text{on } [u = 0]. \end{cases} \tag{1.2}$$

The last two conditions of (1.2) disappear whenever the data ϕ and ψ are such that the sets $[u = 1]$ and $[u = 0]$ are negligible. This is the case, for instance, if $\psi \equiv 0$ (see Proposition 2.1 for more general cases).

After a complete analysis of the general problem $(S^{\gamma,\beta}_{\phi,\psi})$ when $\overline{D(\gamma)} \subset D(\beta)$, we get the right notion of solution which coincides with the concept of weak solution for the problem

$$\begin{cases} -\mathrm{div}\, \mathbf{a}(x, Du) + \gamma(u) \ni \phi & \text{in } \Omega, \\ \mathbf{a}(x, Du) \cdot \eta + \beta(u) + \partial \mathbb{I}_{\overline{D(\gamma)}}(u) \ni \psi & \text{on } \partial\Omega. \end{cases}$$

Definition 1.1. Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$. A triple of functions $[u, z, w] \in W^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is a *generalized weak solution* of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x)) + \partial \mathbb{I}_{\overline{D(\gamma)}}(u(x))$ a.e. on $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_\Omega \phi v + \int_{\partial\Omega} \psi v,$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

It is clear that a weak solution is a generalized weak solution. Moreover, thanks to the results in Ref. 3, the two concepts coincide in cases (a) and (b).

The aim of this paper is to prove the existence and uniqueness of solutions in the sense of Definition 1.1 for $(S^{\gamma,\beta}_{\phi,\psi})$ in the case $\mathbb{R} \neq \overline{D(\gamma)} \subset D(\beta)$.

Let us briefly summarize the content of the paper. In Sec. 2, we establish the main results. In Sec. 3.1 we fix the notation and give some preliminaries. Finally, in the last section we give the proofs of the results.

2. Main Results

As in Ref. 3, in order to get the existence of a generalized weak solution, let us introduce the following spaces:

$$V^{1,p}(\Omega) := \left\{\phi \in L^1(\Omega) : \exists\, M > 0 \text{ s.t. } \int_\Omega |\phi v| \leq M\|v\|_{W^{1,p}(\Omega)}\ \forall\, v \in W^{1,p}(\Omega)\right\}$$

and

$$V^{1,p}(\partial\Omega) := \left\{\psi \in L^1(\partial\Omega) : \exists\, M > 0 \text{ s.t. } \int_{\partial\Omega} |\psi v| \leq M\|v\|_{W^{1,p}(\Omega)} \forall\, v \in W^{1,p}(\Omega)\right\}$$

$V^{1,p}(\Omega)$ is a Banach space endowed with the norm

$$\|\phi\|_{V^{1,p}(\Omega)} := \inf\left\{M > 0 : \int_\Omega |\phi v| \leq M\|v\|_{W^{1,p}(\Omega)}\ \forall\, v \in W^{1,p}(\Omega)\right\}$$

and $V^{1,p}(\partial\Omega)$ is a Banach space endowed with the norm

$$\|\psi\|_{V^{1,p}(\partial\Omega)} := \inf\left\{M > 0 : \int_{\partial\Omega} |\psi v| \leq M\|v\|_{W^{1,p}(\Omega)}\ \forall\, v \in W^{1,p}(\Omega)\right\}.$$

Observe that, Sobolev embeddings and Trace theorems imply, for $1 \leq p < N$,

$$L^{p'}(\Omega) \subset L^{(Np/(N-p))'}(\Omega) \subset V^{1,p}(\Omega)$$

and

$$L^{p'}(\partial\Omega) \subset L^{((N-1)p/(N-p))'}(\partial\Omega) \subset V^{1,p}(\partial\Omega).$$

Also,

$$V^{1,p}(\Omega) = L^1(\Omega) \quad \text{and} \quad V^{1,p}(\partial\Omega) = L^1(\partial\Omega) \ \text{ when } p > N,$$

$$L^q(\Omega) \subset V^{1,N}(\Omega) \quad \text{and} \quad L^q(\partial\Omega) \subset V^{1,N}(\partial\Omega) \ \text{ for any } q > 1.$$

Let us state the following notation. For a maximal monotone graph θ in $\mathbb{R} \times \mathbb{R}$ we shall denote, with the agreement that $\inf A = -\infty$ if A is a set unbounded from below and $\sup A = +\infty$ if A is unbounded from above,

$$\theta^{(i)} = \inf D(\theta), \quad \theta^{(s)} = \sup D(\theta),$$

where $D(\theta)$ is the domain of θ, and

$$\theta_- = \inf R(\theta), \quad \theta_+ = \sup R(\theta),$$

$R(\theta)$ being the range of θ.

Moreover, as shown in Ref. 3, in order to obtain the existence of weak solutions of $(S^{\gamma,\beta}_{\phi,\psi})$, ϕ and ψ must necessarily satisfy the following range condition:

$$\mathcal{R}^-_{\gamma,\beta} \leq \int_\Omega \phi + \int_{\partial\Omega} \psi \leq \mathcal{R}^+_{\gamma,\beta}, \tag{2.1}$$

where

$$\mathcal{R}^+_{\gamma,\beta} := \gamma_+|\Omega| + \beta_+|\partial\Omega|, \quad \mathcal{R}^-_{\gamma,\beta} := \gamma_-|\Omega| + \beta_-|\partial\Omega|.$$

In the case $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$, we write $\mathcal{R}_{\gamma,\beta} :=]\,\mathcal{R}^-_{\gamma,\beta}, \mathcal{R}^+_{\gamma,\beta}[$.

Our main result about existence is divided in two statements. Statement (i) corresponds to the existence of generalized weak solutions for one- or two-obstacle problem and regular data. Observe that for the one-obstacle problem, $\mathcal{R}_{\gamma,\beta}$ can be different from $\mathbb{R}$ and for the two-obstacle problem, $\mathcal{R}_{\gamma,\beta} = \mathbb{R}$. Statement (ii) is for the two-obstacle problem and L^1-data.

Theorem 2.1. *Assume* $\mathbb{R} \neq \overline{D(\gamma)} \subset D(\beta)$. *Then,*

(i) *for any* $\phi \in V^{1,p}(\Omega)$ *and* $\psi \in V^{1,p}(\partial\Omega)$ *such that*

$$\int_\Omega \phi + \int_{\partial\Omega} \psi \in \mathcal{R}_{\gamma,\beta}, \tag{2.2}$$

there exists a generalized weak solution $[u, z, w]$ *of problem* $(S^{\gamma,\beta}_{\phi,\psi})$;

(ii) *if* $D(\gamma)$ *is bounded, the existence of a generalized weak solution* $[u, z, w]$ *of problem* $(S^{\gamma,\beta}_{\phi,\psi})$ *holds true for any* $\phi \in L^1(\Omega)$ *and* $\psi \in L^1(\partial\Omega)$.

Remark 2.1. The case in which condition (2.2) is attained at the boundary for the one-obstacle problem with $\mathcal{R}_{\gamma,\beta} \neq \mathbb{R}$ was treated in Ref. 6 in a particular case. For our problem, a similar treatment could be done. Nevertheless the aim of this paper is to deal with the interaction between the imposed constraints in the domain of γ and the boundary condition.

In the case where the obstacle also depends on the space variable, i.e. $\gamma = \gamma(x, .)$, the problem could be treated with the same techniques. However, in this case, we cannot expect z and w to be L^1 functions. They should be diffuse measures such that the singular parts of their Radon–Nikodym decomposition are concentrated on the boundary of the domain of γ. Since the results are of different nature, and in order to keep the presentation simple, we will not deal with that case here, and we shall consider it separately in a forthcoming paper. Some particular cases, like Dirichlet boundary conditions, may be found in Refs. 18 and 1.

With respect to uniqueness, we recall the following result which was obtained in Ref. 3.

Theorem 2.2. (Ref. 3) *Let* $\phi \in L^1(\Omega)$, $\psi \in L^1(\partial\Omega)$. *If* $[u_1, z_1, w_1]$ *and* $[u_2, z_2, w_2]$ *are weak solutions of* $(S^{\gamma,\beta}_{\phi,\psi})$, *then there exists a constant* $c \in \mathbb{R}$

such that

$$u_1 - u_2 = c \quad a.e.\ in\ \Omega,$$

$$z_1 - z_2 = 0 \quad a.e.\ in\ \Omega,$$

and

$$w_1 - w_2 = 0 \quad a.e.\ in\ \partial\Omega.$$

Moreover, if $c \neq 0, z_1 = z_2$ *is constant.*

In Ref. 3, under the assumptions of Theorem 3.1, see below, a contraction principle for weak solutions of problem $(S^{\gamma,\beta}_{\phi,\psi})$ is also given. Now here in order to prove the main result we need a more general contraction principle between sub- and super-weak solutions. As usual, we understand weak-sub and supersolution in the following way. A triple of functions $[u, z, w] \in W^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is a *weak subsolution* (resp. *supersolution*) of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x))$ a.e. on $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv \leq (\text{resp. } \geq) \int_\Omega \phi v + \int_{\partial\Omega} \psi v$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, $v \geq 0$.

Theorem 2.3. *Let* $\phi_1, \phi_2 \in L^1(\Omega), \psi_1, \psi_2 \in L^1(\partial\Omega)$. *If* $[u_1, z_1, w_1]$ *is a weak subsolution of* $(S^{\gamma,\beta}_{\phi_1,\psi_1})$ *and* $[u_2, z_2, w_2]$ *is a weak supersolution of* $(S^{\gamma,\beta}_{\phi_2,\psi_2})$, *then*

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \leq \int_\Omega (\phi_1 - \phi_2)^+ + \int_{\partial\Omega} (\psi_1 - \psi_2)^+.$$

For the uniqueness of a generalized solution, observe that $[u, z, w]$ is a generalized weak solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if and only if $[u, z, w]$ is a weak solution of problem $(S^{\gamma,\beta_\gamma}_{\phi,\psi})$, where $\beta_\gamma := \beta + \partial \mathrm{I\!I}_{\overline{D(\gamma)}}$, that is, β_γ is the maximal monotone graph in $\mathbb{R}^2$ defined by

$$\beta_\gamma(r) = \begin{cases} \left]-\infty, \beta^0(\gamma^{(i)})\right] & \text{if } r = \gamma^{(i)} \text{ (when } \gamma^{(i)} \text{ is finite)}, \\ \beta(r) & \text{if } \ \gamma^{(i)} < r < \gamma^{(s)}, \\ \left[\beta^0(\gamma^{(s)}), +\infty\right[& \text{if } \ r = \gamma^{(s)} \text{ (when } \gamma^{(s)} \text{ is finite)}. \end{cases}$$

Consequently, by Theorems 2.2 and 2.3, we obtain the following result about uniqueness of generalized weak solutions.

Theorem 2.4. *Let* $\phi \in L^1(\Omega)$ *and* $\psi \in L^1(\partial\Omega)$. *Let* $[u_1, z_1, w_1]$ *and* $[u_2, z_2, w_2]$ *be generalized weak solutions of* $(S^{\gamma,\beta}_{\phi,\psi})$. *Then, there exists a constant* $c \in \mathbb{R}$ *such that*

$$u_1 - u_2 = c \quad a.e.\ in\ \Omega,$$

$$z_1 - z_2 = 0 \quad a.e.\ in\ \Omega,$$

and

$$w_1 - w_2 = 0 \quad a.e.\ in\ \partial\Omega.$$

If $c \neq 0, z_1 = z_2$ *is constant.*

Moreover, given $[u_1, z_1, w_1]$ *a generalized weak solution of* $(S^{\gamma,\beta}_{\phi_1,\psi_1})$ *and* $[u_2, z_2, w_2]$ *a generalized weak solution of* $(S^{\gamma,\beta}_{\phi_2,\psi_2})$, *then*

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \leq \int_\Omega (\phi_1 - \phi_2)^+ + \int_{\partial\Omega} (\psi_1 - \psi_2)^+.$$

By the above theorem, we have that $[0, \phi, \psi]$ is the unique generalized weak solution of $(L^{\gamma,0}_{\phi,\psi})$.

Thanks to the example $(L^{\gamma,0}_{\phi,\psi})$ it is clear that, for a generalized weak solution, $w \notin \beta(u)$ in general. In the next result we show that for the homogeneous case $\psi \equiv 0$, $w \in \beta(u)$.

Proposition 2.1. *Assume* $\mathbb{R} \neq \overline{D(\gamma)} \subset D(\beta)$. *Let* $\phi \in L^1(\Omega)$, $\psi \in L^1(\partial\Omega)$ *and* $[u, z, w]$ *be a generalized weak solution of* $(S^{\gamma,\beta}_{\phi,\psi})$.

(i) *If* $\operatorname{esssup}(\psi) \leq \sup \beta(\gamma^{(s)})$, *then* $w \leq \sup \beta(\gamma^{(s)})$ *a.e. on* $\partial\Omega$.
(ii) *If* $\operatorname{essinf}(\psi) \geq \inf \beta(\gamma^{(i)})$, *then* $w \geq \inf \beta(\gamma^{(i)})$ *a.e. on* $\partial\Omega$.

Consequently, if $\inf \beta(\gamma^{(i)}) \leq \operatorname{essinf}(\psi) \leq \operatorname{esssup}(\psi) \leq \sup \beta(\gamma^{(s)})$, *then* $w \in \beta(u)$ *a.e. on* $\partial\Omega$, *and therefore* $[u, z, w]$ *is, in fact, a weak solution of* $(S^{\gamma,\beta}_{\phi,\psi})$.

3. Preliminaries and Proofs

3.1. *Preliminaries*

Throughout the paper, $\Omega \subset \mathbb{R}$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $p > 1$, γ and β are maximal monotone graphs in $\mathbb{R}^2$ such that $0 \in \gamma(0) \cap \beta(0)$ and the Carathéodory function $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies

(H_1) there exists $\Lambda > 0$ such that $\mathbf{a}(x, \xi) \cdot \xi \geq \Lambda |\xi|^p$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$,
(H_2) there exists $\sigma > 0$ and $\varrho \in L^{p'}(\Omega)$ such that $|\mathbf{a}(x, \xi)| \leq \sigma(\varrho(x) + |\xi|^{p-1})$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $p' = \frac{p}{p-1}$,
(H_3) $(\mathbf{a}(x, \xi_1) - \mathbf{a}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0$ for a.e. $x \in \Omega$ and for all $\xi_1, \xi_2 \in \mathbb{R}^N$, $\xi_1 \neq \xi_2$.

The hypotheses $(H_1 - H_3)$ are classical in the study of nonlinear operators in divergence form (cf., Ref. 16). The model example of a function $\mathbf{a}$ satisfying these hypotheses is $\mathbf{a}(x, \xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p-Laplacian operator $\Delta_p(u) = \operatorname{div}(|Du|^{p-2} Du)$.

We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure of $\mathbb{R}^N$ and by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure.

For an open bounded set U of $\mathbb{R}^N$, we define the p-capacity relative to U, $C_p(., U)$, in the following classical way. For any compact subset K of U,

$$C_p(K, U) = \inf\left\{ \int_U |Du|^p \ ; \ u \in \mathcal{C}^\infty_c(U), \ u \geq \chi_K \right\},$$

where χ_K is the characteristic function of K; we will use the convection that $\inf \emptyset = +\infty$. The p-capacity of any open subset $O \subset U$ is defined by

$$C_p(O,U) = \sup\{C_p(K);\ K \subset O \text{ compact}\}.$$

Finally, the p-capacity of any Borel set $A \subset U$ is defined by

$$C_p(A,U) = \inf\{C_p(O);\ O \subset A \text{ open}\}.$$

A function u defined on U is said to be cap_p-quasi-continuous in $A \subset U$ if for every $\varepsilon > 0$, there exists an open set $B_\epsilon \subseteq U$ with $C_p(B_\epsilon, U) < \varepsilon$ such that the restriction of u to $A \setminus B_\epsilon$ is continuous. It is well known that every function in $W^{1,p}(U)$ has a cap_p-quasi-continuous representative, whose values are defined cap_p-quasi everywhere in U, i.e. up to a subset of U of zero p-capacity. When we are dealing with the pointwise values of a function $u \in W^{1,p}(U)$, we always identify u with its cap_p-quasi-continuous representative.

We denote by $\mathcal{M}_b(U)$ the space of all Radon measures in U with bounded total variation. We recall that for a measure $\mu \in \mathcal{M}_b(U)$, and a Borel set $A \subset U$, the measure $\mu \llcorner A$ is defined by $(\mu \llcorner A)(B) = \mu(B \cap A)$ for any Borel set $B \subset U$. If a measure $\mu \in \mathcal{M}_b(U)$ is such that $\mu = \mu \llcorner A$ for a certain Borel set A, the measure μ is said to be concentrated on A. For $\mu \in \mathcal{M}_b(U)$, we denote by μ^+, μ^- and $|\mu|$ the positive part, negative part and the total variation of the measure μ, respectively. By $\mu = \mu_a + \mu_s$ we denote the Radon–Nykodym decomposition of μ relatively to $\mathcal{L}^N$. For simplicity, we also write μ_a for its density respect to $\mathcal{L}^N$, i.e. for the function $f \in L^1(U)$ such that $\mu_a = f\mathcal{L}^N \llcorner U$.

We denote by $\mathcal{M}_b^p(U)$ the space of all diffuse Radon measures in U, i.e. measures which do not charge sets of zero p-capacity. In Ref. 7 it is proved that $\mu \in \mathcal{M}_b(U)$ belongs to $\mathcal{M}_b^p(U)$ if and only if it belongs to $L^1(U) + W^{-1,p'}(U)$, where $W^{-1,p'}(U) = [W_0^{1,p}(U)]^*$. Moreover, if $u \in W^{1,p}(U)$ and $\mu \in \mathcal{M}_b^p(U)$, then u is measurable with respect to μ. If u further belongs to $L^\infty(\Omega)$, then u belongs to $L^\infty(U, d\mu)$, hence to $L^1(U, d\mu)$.

From now on U_Ω will be a fix open bounded subset of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$. If $u \in W^{1,p}(\Omega)$, $1 < p \le \infty$, it is possible to give a pointwise definition of the trace $\tau(u)$ of u on $\partial\Omega$ in the following way (see Ref. 19). Since Ω is an extension domain, there exists $\tilde{u} \in W^{1,p}(U_\Omega)$ an extension of u to all of U_Ω. Consequently, every point of U_Ω except possibly a set of zero p-capacity is a Lebesgue point of $\tilde{u}$. Since $p > 1$, the sets of zero p-capacity are of $\mathcal{H}^{N-1}$-measure zero and therefore $\tilde{u}$ is defined $\mathcal{H}^{N-1}$-almost everywhere on $\partial\Omega$, and we have $\tau(u) = \tilde{u}$ on $\partial\Omega$. We denote $\tau(u)$ by u in the rest of the paper. This definition is independent of the open set U_Ω and also of the extension $\tilde{u}$.

We define

$$\mathcal{M}_b^p(\overline{\Omega}) := \{\mu \in \mathcal{M}_b(U_\Omega) \cap W^{-1,p'}(U_\Omega) :\ \mu \text{ is concentrated on } \overline{\Omega}\}.$$

For $u \in W^{1,p}(\Omega)$ and $\mu \in \mathcal{M}_b^p(\overline{\Omega})$, we have

$$\langle \mu, \tilde{u} \rangle = \int_\Omega u\, d\mu + \int_{\partial\Omega} u\, d\mu.$$

This definition is independent of the open set U_Ω. Observe also that if $\mu \in \mathcal{M}_b^p(\overline{\Omega})$ then $\mu \in \mathcal{M}_b^p(U_\Omega)$.

We denote

$$\mathrm{sign}_0(r) := \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0, \end{cases} \quad \text{and} \quad \mathrm{sign}_0^+(r) := \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r \leq 0. \end{cases}$$

We need to use the truncation functions

$$T_k(r) := [k - (k - |r|)^+]\mathrm{sign}_0(r) \quad \text{and}$$
$$T_k^+(r) := [k - (k - |r|)^+]\mathrm{sign}_0^+(r),\ k > 0,\ r \in \mathbb{R}.$$

Let θ be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. For $r \in \mathbb{N}$, the *Yosida approximation* θ_r of θ is given by $\theta_r = r(I - (I + \frac{1}{r}\theta)^{-1})$. The function θ_r is maximal monotone and Lipschitz. We recall the definition of the *main section* θ^0 of θ

$$\theta^0(s) := \begin{cases} \text{the element of minimal absolute value of}\, \theta(s) & \text{if } \theta(s) \neq \emptyset \\ +\infty \quad \text{if } [s, +\infty[\cap D(\theta) = \emptyset \\ -\infty \quad \text{if }]-\infty, s] \cap D(\theta) = \emptyset. \end{cases}$$

We have that, $|\theta_r|$ is increasing in r, if $s \in D(\theta)$, $\theta_r(s) \to \theta^0(s)$ as $r \to +\infty$, and if $s \notin D(\theta)$, $|\theta_r(s)| \to +\infty$ as $r \to +\infty$.

If $0 \in D(\theta)$, $j_\theta(r) = \int_0^r \theta^0(s)ds$ defines a convex lower semi-continuous function such that $\theta = \partial j_\theta$. If j_θ^* is the Legendre transformation of j_θ then $\theta^{-1} = \partial j_\theta^*$.

Finally, let us recall the existence result given in Ref. 3 that will be useful for the proof of the main results of this paper.

Theorem 3.1. (Ref. 3) *Assume $D(\gamma) = \mathbb{R} = D(\beta)$. For any $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$ with*

$$\int_\Omega \phi + \int_{\partial\Omega} \psi \in \mathcal{R}_{\gamma,\beta},$$

there exists a weak solution $[u, z, w] \in W^{1,p}(\Omega) \times V^{1,p}(\Omega) \times V^{1,p}(\partial\Omega)$ of $(S_{\phi,\psi}^{\gamma,\beta})$ such that

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_\Omega \phi v + \int_{\partial\Omega} \psi v, \quad \forall\, v \in W^{1,p}(\Omega),$$
$$\|z^\pm\|_{L^1(\Omega)} + \|w^\pm\|_{L^1(\partial\Omega)} \leq \|\phi^\pm\|_{L^1(\Omega)} + \|\psi^\pm\|_{L^1(\partial\Omega)},$$
$$\int_\Omega |zv| + \int_{\partial\Omega} |wv| \leq \int_\Omega |\phi v| + \int_{\partial\Omega} |\psi v| + \sigma \left(\|\varrho\|_{L^{p'}(\Omega)} + \|Du\|_{L^p(\Omega)}^{p-1} \right) \|Dv\|_{L^p(\Omega)},$$
$$\forall\, v \in W^{1,p}(\Omega),$$
$$\|Du\|_{L^p(\Omega)}^{p-1} \leq \frac{c(\Omega, N, p)}{\Lambda} \left(\|\phi\|_{V^{1,p}(\Omega)} + \|\psi\|_{V^{1,p}(\partial\Omega)} \right),$$

and

$$\left\| u - \frac{1}{|\Omega|}\int_\Omega u \right\|_{W^{1,p}(\Omega)} \leq c(\Omega, N, p)\left(\|\phi\|_{V^{1,p}(\Omega)} + \|\phi\|_{V^{1,p}(\partial\Omega)}\right)$$

for some $c(\Omega, N, p) > 0$.

3.2. *Proofs*

In this section we give the proof of Theorems 2.1 and 2.3. In order to prove Theorem 2.1, we first obtain an existence result for which we need the following definition.

Definition 3.1. Let $u \in W^{1,p}(\Omega)$ and $\mu \in \mathcal{M}_b^p(\overline{\Omega})$. We say that

$$\mu \in \gamma(u)$$

if the following conditions are satisfied,

(i) $\mu_a \in \gamma(u)\,\mathcal{L}^N$-a.e. in Ω,
(ii) μ_s is concentrated on the set $\{x \in \overline{\Omega} : u = \gamma^{(i)}\} \cup \{x \in \overline{\Omega} : u = \gamma^{(s)}\}$,
(iii) $\mu_s \leq 0$ on $\{x \in \overline{\Omega} : u = \gamma^{(i)}\}$ and $\mu_s \geq 0$ on $\{x \in \overline{\Omega} : u = \gamma^{(s)}\}$.

Let us point out the similitude of the above concept with the definition of $\mu \in \gamma(u)$ for $\mu \in \mathcal{M}_b^p(\Omega)$ given in Ref. 11, from which we have taken some ideas for the proof of Proposition 3.1.

Remark 3.1. Let $u \in W^{1,p}(\Omega)$ and $\mu \in \mathcal{M}_b^p(\overline{\Omega})$. Recalling that the set $\{x \in \overline{\Omega} : u = \pm\infty\}$ has zero p-capacity relative to U_Ω and that $\mu \in \mathcal{M}_b^p(U_\Omega)$, we have that

$$\mu_s = 0 \quad \text{on } \{x \in \overline{\Omega} : u = \pm\infty\}.$$

In particular, if $D(\gamma) = \mathbb{R}$, then

$$\mu \in \gamma(u) \quad \text{if and only if} \quad \mu \in L^1(\Omega) \quad \text{and} \quad \mu \in \gamma(u) \text{ a.e. in } \Omega.$$

Proposition 3.1. *Assume* $\mathbb{R} \neq \overline{D(\gamma)} \subset D(\beta)$. *Then, for any* $\phi \in V^{1,p}(\Omega)$ *and* $\psi \in V^{1,p}(\partial\Omega)$ *such that*

$$\int_\Omega \phi + \int_{\partial\Omega} \psi \in \mathcal{R}_{\gamma,\beta}, \tag{3.1}$$

there exists $[u, \mu, w] \in W^{1,p}(\Omega) \times M_b^p(\overline{\Omega}) \times V^{1,p}(\partial\Omega)$ *such that* $w \in \beta(u)$ *a.e. in* $\partial\Omega, \mu \in \gamma(u)$ *and*

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \langle \mu, \tilde{v} \rangle + \int_{\partial\Omega} wv = \int_\Omega \phi v + \int_{\partial\Omega} \psi v \quad \forall\, v \in W^{1,p}(\Omega). \tag{3.2}$$

Proof. We divide the proof into three steps.

Step 1. *Approximation and uniform estimates.* In order to apply Theorem 3.1, we have to approximate the nonlinearities γ and β by maximal monotone graphs everywhere defined.

Let $\tilde{\beta}$ be the maximal monotone graph with defined by

$$\tilde{\beta}(s) = \begin{cases} \beta(s) & \text{if } s \in \,]\,\gamma^{(i)}, \gamma^{(s)}[, \\ \beta^0(\gamma^{(i)}) & \text{if } s < \gamma^{(i)} \text{ (when } \gamma^{(i)} \text{ is finite)}, \\ \beta^0(\gamma^{(s)}) & \text{if } s > \gamma^{(s)} \text{ (when } \gamma^{(s)} \text{ is finite)}. \end{cases}$$

On the other hand, the approximation γ^r of γ depends on the domain of γ. In the case domain of γ is bounded, i.e. $\gamma^{(i)}$ and $\gamma^{(s)}$ are both finite, for every $r \in \mathbb{N}$, we take $\gamma^r = \gamma_r$ to be the Yosida approximation of γ. And in the case $D(\gamma)$ is not bounded, we consider that $\gamma^{(i)} = -\infty$ and $\gamma^{(s)}$ is finite (the other case, $\gamma^{(i)}$ finite and $\gamma^{(s)} = +\infty$, being similar), for every $r \in \mathbb{N}$, we take γ^r the maximal monotone graph defined by

$$\gamma^r(s) = \begin{cases} \gamma(s) & \text{if } s < 0, \\ \gamma_r(s) & \text{if } s > 0. \end{cases}$$

It is clear that in the last case we are regularizing just the positive part, the regularization of the negative part is not necessary since it is everywhere defined. Now, since $D(\gamma^r) = \mathbb{R} = D(\tilde{\beta})$, by Theorem 3.1, problem $(S_{\phi,\psi}^{\gamma_r,\tilde{\beta}})$ has a weak solution, i.e. there exists a triple of functions $[u_r, z_r, w_r] \in W^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ such that $z_r(x) \in \gamma^r(u_r(x))$ a.e. in Ω, $w_r \in \tilde{\beta}(u_r)$ a.e. in $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x, Du_r) \cdot Dv + \int_\Omega z_r v + \int_{\partial\Omega} w_r v = \int_\Omega \phi v + \int_{\partial\Omega} \psi v, \tag{3.3}$$

for all $v \in W^{1,p}(\Omega)$. Moreover, for any r, we have

$$\|w_r^\pm\|_{L^1(\partial\Omega)} + \|z_r^\pm\|_{L^1(\Omega)} \le \|\phi^\pm\|_{L^1(\Omega)} + \|\psi^\pm\|_{L^1(\partial\Omega)}, \tag{3.4}$$

$$\begin{aligned} &\int_{\partial\Omega} |w_r v| + \int_\Omega |z_r v| \\ &\quad \le \int_\Omega |\phi v| + \int_{\partial\Omega} |\psi v| + \sigma\left(\|g\|_{L^{p'}(\Omega)} + \|Du_r\|_{L^p(\Omega)}^{p-1}\right) \|Dv\|_{L^p(\Omega)}, \quad \forall\, v \in W^{1,p}(\Omega), \end{aligned} \tag{3.5}$$

$$\|Du_r\|_{L^p(\Omega)}^{p-1} \le \frac{c(\Omega, N, p)}{\Lambda} \left(\|\phi\|_{V^{1,p}(\Omega)} + \|\psi\|_{V^{1,p}(\partial\Omega)}\right) \tag{3.6}$$

and

$$\left\| u_r - \frac{1}{|\Omega|} \int_\Omega u_r \right\|_{W^{1,p}(\Omega)} \le c(\Omega, N, p) \left(\|\phi\|_{V^{1,p}(\Omega)} + \|\phi\|_{V^{1,p}(\partial\Omega)}\right). \tag{3.7}$$

Let us prove that

$$\left\{ \int_\Omega u_r \right\}_r \quad \text{is bounded.} \tag{3.8}$$

Indeed, if there exists a subsequence, denoted equal, such that $\int_\Omega u_r \to +\infty$, by (3.7), there exists another subsequence, still denoted equal, such that $u_r \to +\infty$

a.e. in Ω and $u_r \to +\infty$ a.e. in $\partial\Omega$. Now, since $z_r \in \gamma(u_r)$ a.e. in Ω and $w_r \in \tilde{\beta}(u_r)$ a.e. in $\partial\Omega$, in the case $\gamma^{(s)}$ is finite, $\lim_r z_r^+ = +\infty$ a.e. in Ω, which contradicts that $\int_{\partial\Omega} w_r^+ + \int_\Omega z_r^+ \leq \int_\Omega \phi^+ + \int_{\partial\Omega} \psi^+$, and in the case $\gamma^{(s)} = +\infty$, $\lim_r z_r^+ = \gamma_+$ a.e. in Ω and $\lim_r w_r^+ = \beta_+$ a.e. in Ω, which contradicts that $\int_{\partial\Omega} w_r^+ + \int_\Omega z_r^+ \leq \int_\Omega \phi^+ + \int_{\partial\Omega} \psi^+ < \mathcal{R}^+_{\gamma,\beta}$. A similar argument shows that there is not a subsequence such that$\int_\Omega u_r \to -\infty$.

Step 2. *Convergences.* By (3.7) and (3.8), we obtain that $\{u_r\}$ is bounded in $W^{1,p}(\Omega)$. Thus, we can assume that, as r goes to $+\infty$

$$\begin{aligned} u_r \to u \qquad & \text{in } W^{1,p}(\Omega)\text{-weak}, \\ & \text{in } L^p(\Omega), \\ & \text{in } L^p(\partial\Omega). \end{aligned} \tag{3.9}$$

Let us prove now that $\{w_r\}_r$ is convergent. Observe that, for the two-obstacle problem, for any r,

$$\beta^0(\gamma^{(i)}) \leq w_r \leq \beta^0(\gamma^{(s)}). \tag{3.10}$$

On the other hand, for the one-obstacle problem, the choice of γ^r implies that, for any r,

$$w_r \geq w_{r+1}. \tag{3.11}$$

Indeed, letting

$$\overline{z}_r(x) = \begin{cases} z_r(x) & \text{if } u_r(x) < 0, \\ \gamma^{r+1}(u_r) & \text{if } u_r(x) \geq 0, \end{cases}$$

we have that, since γ^{r+1} is nondecreasing in r,

$$\int_\Omega \mathbf{a}(x, Du_r) \cdot Dv + \int_\Omega \overline{z}_r v + \int_{\partial\Omega} w_r v \geq \int_\Omega \phi v + \int_{\partial\Omega} \psi v, \quad \forall\, v \in W^{1,p}(\Omega)^+.$$

From here we obtain, by Theorem 2.3, that

$$w_r \geq w_{r+1}. \tag{3.12}$$

In the two-obstacle problem, thanks to (3.10), we can assume that there exists $w \in L^\infty(\partial\Omega)$ such that

$$w_r \rightharpoonup w \quad L^\infty(\partial\Omega)\text{-weak}^* \text{ as } r \text{ goes to } +\infty. \tag{3.13}$$

In the one-obstacle problem, thanks to (3.11) and (3.4), we can assume that there exists $w \in L^1(\partial\Omega)$ such that

$$w_r \to w \quad \text{in} \quad L^1(\partial\Omega) \quad \text{as } r \text{ goes to } +\infty. \tag{3.14}$$

Observe that, by (3.5), $w \in V^{1,p}(\partial\Omega)$.

As a consequence of (3.9) and (3.13) or (3.14), we have

$$w \in \tilde{\beta}(u) \quad \text{a.e. in } \partial\Omega. \tag{3.15}$$

Taking now

$$\hat{z}_r(x) = \begin{cases} z_r(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in U_\Omega \setminus \Omega, \end{cases}$$

thanks to (3.4), we can assume there exists $\mu \in \mathcal{M}_b(U_\Omega)$, such that

$$\hat{z}_r \rightharpoonup \mu \quad \mathcal{M}_b(U_\Omega)\text{-weak}^* \quad \text{ as } r \text{ goes to } +\infty. \tag{3.16}$$

Moreover, μ is concentrated on $\overline{\Omega}$. Now, by (3.5) and (3.6), if we define Ψ_r as

$$\langle \Psi_r, v \rangle := \int_{U_\Omega} \hat{z}_r v \, dx, \quad \forall\, v \in W_0^{1,p}(U_\Omega),$$

$\{\Psi_r \,:\, r > 0\}$ is bounded in $[W_0^{1,p}(U_\Omega)]^*$. Then, we can assume that $\mu \in [W_0^{1,p}(U_\Omega)]^*$ and

$$\langle \mu, v \rangle = \int_{U_\Omega} v \, d\mu \quad \forall\, v \in W_0^{1,p}(U_\Omega).$$

Consequently, $\mu \in \mathcal{M}_b^p(\overline{\Omega})$, and

$$\lim_{r \to +\infty} \int_\Omega z_r v \, dx = \langle \mu, \tilde{v} \rangle \quad \forall\, v \in W^{1,p}(\Omega). \tag{3.17}$$

Having in mind (H_2), since $\{Du_r\}$ is bounded in $L^p(\Omega)$, we can assume that there exists $\chi \in L^{p'}(\Omega)$ such that

$$\mathbf{a}(x, Du_r) \rightharpoonup \chi \quad \text{ in } L^{p'}(\Omega)\text{-weak, as } r \to +\infty. \tag{3.18}$$

By (3.9), (3.17), (3.18) and (3.13) in the two-obstacle problem or (3.12) and (3.14) in the one-obstacle problem, passing to the limit in (3.3) we get,

$$\int_\Omega \chi \cdot Dv + \langle \mu, \tilde{v} \rangle + \int_{\partial\Omega} wv = \int_\Omega \phi v + \int_{\partial\Omega} \psi v, \quad \forall\, v \in W^{1,p}(\Omega). \tag{3.19}$$

Step 3. *Identification of the nonlinearities.* Let $\xi \in \mathcal{D}(\mathbb{R}^N)$, $\xi \geq 0$. Using $(u - u_r)\xi$ as test function in (3.3), by (H_3), we obtain

$$\begin{aligned} \int_{\partial\Omega} w_r (u - u_r)\xi + \int_\Omega z_r (u - u_r)\xi &\geq \int_\Omega \phi\,(u - u_r)\xi + \int_{\partial\Omega} \psi\,(u - u_r)\xi \\ &\quad - \int_\Omega \xi \mathbf{a}(x, Du) \cdot D(u - u_r) - \int_\Omega (u - u_r)\mathbf{a}(x, Du_r) \cdot D\xi, \end{aligned}$$

which implies, on account of the above convergence and using Fatou's Lemma in the first integral, that

$$\liminf_{r \to +\infty} \int_\Omega z_r (u - u_r)\xi \geq 0. \tag{3.20}$$

For simplicity, we set $j = j_\gamma$ and $j_r = j_{\gamma_r}$. Since $z_r \in \partial j_r(u_r)$, we have, for all $v \in W^{1,p}(\Omega)$,

$$j_r(v) \geq j_r(u_r) + z_r(v - u_r) \quad \mathcal{L}^N\text{-a.e. in } \Omega.$$

Now, using the fact that j_r is increasing, for any $r > s > 0$, we have

$$\begin{aligned}\int_\Omega j(v)\xi\,dx &\geq \int_\Omega j_r(v)\xi\,dx \\ &\geq \int_\Omega j_r(u_r)\xi\,dx + \int_\Omega z_r(v - u_r)\xi\,dx \\ &\geq \int_\Omega j_s(u_r)\xi\,dx + \int_\Omega z_r(v - u_r)\xi\,dx \\ &\geq \int_\Omega j_s(u_r)\xi\,dx + \int_\Omega z_r(v - u)\xi\,dx + \int_\Omega z_r(u - u_r)\xi\,dx.\end{aligned}$$

Letting $r \to +\infty$ and using Fatou's Lemma, we deduce that

$$\int_\Omega j(v)\xi\,dx \geq \int_\Omega j_s(u)\xi\,dx + \langle \mu, (\tilde{v} - \tilde{u})\xi\rangle + \liminf_{r\to+\infty} \int_\Omega z_r(u - u_r)\xi\,dx.$$

Passing to the limit as $s \to +\infty$, we get

$$\int_\Omega j(v)\xi\,dx \geq \int_\Omega j(u)\xi\,dx + \langle \mu, (\tilde{v} - \tilde{u})\xi\rangle + \liminf_{r\to+\infty} \int_\Omega z_r(u - u_r)\xi\,dx, \tag{3.21}$$

consequently, $u \in \overline{D(\gamma)}$ a.e. and $j(u) \in L^1(\Omega)$. Moreover, by (3.15),

$$w \in \beta(u) \quad \text{ a.e. in } \partial\Omega. \tag{3.22}$$

By (3.20), (3.21) implies that

$$\int_\Omega j(v)\xi\,dx \geq \int_\Omega j(u)\xi\,dx + \langle \mu, (\tilde{v} - \tilde{u})\xi\rangle. \tag{3.23}$$

Taking $v = u$ in (3.21) we obtain that

$$\liminf_{r\to+\infty} \int_\Omega z_r(u - u_r)\xi \leq 0,$$

and by (3.20),

$$\liminf_{r\to+\infty} \int_\Omega z_r(u - u_r)\xi = 0.$$

Therefore, passing to a subsequence if necessary,

$$\lim_{r\to+\infty} \int_\Omega z_r(u - u_r) = 0.$$

Let us see that $\chi = \mathbf{a}(x, Du)$. To do that we apply the Minty–Browder's method. In order to do this, let us see first that

$$\limsup_{r\to+\infty} \int_\Omega \mathbf{a}(x, Du_r) \cdot Du_r \le \int_\Omega \chi \cdot Du. \tag{3.24}$$

Using u_r as test function in (3.3), by (3.19) we get

$$\begin{aligned}\int_\Omega \mathbf{a}(x, Du_r) \cdot Du_r &= \int_\Omega \phi u_r + \int_{\partial\Omega} \psi u_r - \int_\Omega z_r u_r - \int_{\partial\Omega} w_r u_r \\ &= \int_\Omega \chi \cdot Du_r \ + \langle \mu - z_r, \tilde{u}_r\rangle - \int_{\partial\Omega} (w_r - w) u_r \\ &= \int_\Omega \chi \cdot Du_r \ + \langle \mu - z_r, \tilde{u}\rangle + \langle \mu, \tilde{u}_r - \tilde{u}\rangle \\ &\quad - \int_\Omega z_r(u_r - u) - \int_{\partial\Omega} (w_r - w) u_r.\end{aligned}$$

From here, having in mind the above convergence and using Fatou's Lemma in the last integral, it follows (3.24).

Now, by (H_3), for any $\rho \in (L^p(\Omega))^N$ we have

$$\int_\Omega \mathbf{a}(x, \rho) \cdot (Du_r - \rho) \le \int_\Omega \mathbf{a}(x, Du_r) \cdot (Du_r - \rho).$$

Passing to the limit and using (3.24), we get

$$\int_\Omega \mathbf{a}(x, \rho) \cdot (Du - \rho) \le \int_\Omega \chi \cdot (Du - \rho).$$

Then taking $\rho = Du - \lambda\xi$, for $\lambda > 0$ and $\xi \in (L^p(\Omega))^N$, we get

$$\int_\Omega \mathbf{a}(x, Du - \lambda\xi) \cdot \xi \le \int_\Omega \chi \cdot \xi.$$

From here, letting $\lambda \to 0$, we obtain

$$\int_\Omega \mathbf{a}(x, Du) \cdot \xi \le \int_\Omega \chi \cdot \xi, \qquad \text{for any } \xi \in (L^p(\Omega))^N,$$

which implies that

$$\mathbf{a}(x, Du) = \chi \qquad \text{a.e. in } \Omega.$$

Therefore, (3.19) can be rewritten as

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \langle \mu, \tilde{v}\rangle + \int_{\partial\Omega} wv = \int_\Omega \phi v + \int_{\partial\Omega} \psi v, \quad \forall\, v \in W^{1,p}(\Omega). \tag{3.25}$$

From (3.23), by density, we have that for all $\xi \in C(\overline{\Omega})$, $\xi \ge 0$,

$$\int_\Omega j(v)\xi\, dx \ge \int_\Omega j(u)\xi\, dx + \langle \mu, (\tilde{v} - \tilde{u})\xi\rangle \quad \forall\, v \in W^{1,p}(\Omega). \tag{3.26}$$

Therefore, if we take in (3.26) $v(x) = t$ for all $x \in \Omega$, $t \in]\gamma^{(i)}, \gamma^{(s)}[$, we obtain

$$(j(t) - j(u))\mathcal{L}^N \geq (t-u)\mu \quad \text{as measures.} \tag{3.27}$$

Consequently, the absolutely continuous part with respect to the Lebesgue measure $\mathcal{L}^N$ of (3.27) satisfies

$$(j(t) - j(u)) \geq (t-u)\mu_a,$$

i.e. $\mu_a \in \gamma(u)$ a.e. in Ω. On the other hand, taking the singular part in (3.27), it follows that

$$(t-u)\mu_s \leq 0 \quad \forall\, t \in]\gamma^{(i)}, \gamma^{(s)}[,$$

which implies

$$\mu_s \leq 0 \quad \text{ on } \{x \in \overline{\Omega} \,:\, u(x) < t\} \quad \text{ and } \quad \mu_s \geq 0 \quad \text{ on } \{x \in \overline{\Omega} \,:\, u(x) > t\}.$$

Since $t \in]\gamma^{(i)}, \gamma^{(s)}[$ is arbitrary, we deduce that

$$\mu_s \quad \text{is concentrated on the set} \quad \{x \in \overline{\Omega} \,:\, u(x) = \gamma^{(i)}\} \cup \{x \in \overline{\Omega} \,:\, u(x) = \gamma^{(s)}\},$$

$$\mu_s \leq 0 \quad \text{on } \{x \in \overline{\Omega} \,:\, u(x) = \gamma^{(i)}\} \quad \text{ and } \quad \mu_s \geq 0 \quad \text{on } \{x \in \overline{\Omega} \,:\, u(x) = \gamma^{(s)}\},$$

so $\mu \in \gamma(u)$ and the proof is complete. □

In order to be more precise about the singular part μ_s obtained in the above proposition, we establish the following technical result.

Lemma 3.1. *Let $\eta \in W^{1,p}(\Omega)$, $\nu \in M_b^p(\overline{\Omega})$ and $\lambda \in \mathbb{R}$ be such that $\eta \leq \lambda$ (resp. $\eta \geq \lambda$) a.e. Ω. If*

$$-\mathrm{div}\, \mathbf{a}(x, D\eta) = \nu$$

in the sense that $\int_\Omega \mathbf{a}(x, D\eta) \cdot D\xi = \int_{\overline{\Omega}} \xi \, d\nu$, for any $\xi \in W^{1,p}(\Omega)$, then

$$\int_{\{x \in \overline{\Omega} : \eta(x) = \lambda\}} \xi \, d\nu \geq 0 \tag{3.28}$$

(*resp.*

$$\int_{\{x \in \overline{\Omega} : \eta(x) = \lambda\}} \xi \, d\nu \leq 0, \tag{3.29}$$

for any $\xi \in W^{1,p}(\Omega)$, $\xi \geq 0$).

Proof. For $n \geq 1$, let $\varphi_n(r) = \inf(1, (nr + 1 - n\,\lambda)^+)$. Since $\varphi_n(\eta)$ converges to $\chi_{\{x \in \overline{\Omega} : \eta(x) = \lambda\}}$, ν-a.e. in Ω (indeed, $\varphi_n(r)$ converges to $\chi_{[\lambda, \infty)}(r)$ for every $r \in \mathbb{R}$, so $\varphi_n(\eta(x))$ converges to $\chi_{[\lambda,\infty)}(\eta(x))$ at every x where $\eta(x)$ is defined. As η is defined quasi everywhere and $\chi_{[\lambda,\infty)} \circ \eta = \chi_{\{x\in\overline{\Omega}:\eta(x)=\lambda\}}$, then the convergence of $\varphi_n(\eta)$ to $\chi_{\{x\in\overline{\Omega}:\eta(x)=\lambda\}}$ is quasi everywhere; finally, since ν is diffuse then the convergence is also ν-a.e. in Ω) then

$$\begin{aligned}\int_{\{x\in\overline{\Omega}:\eta(x)=\lambda\}} \xi\, d\nu &= \lim_{n\to\infty} \int_{\overline{\Omega}} \xi\, \varphi_n(\eta)\, d\nu \\ &= \lim_{n\to\infty} \int_{\Omega} \mathbf{a}(x, D\eta) D(\xi\, \varphi_n(\eta)) \\ &\geq \lim_{n\to\infty} \int_{\Omega} \mathbf{a}(x, D\eta) D\xi\ \ \varphi_n(\eta) \\ &\geq -\|D\xi\|_\infty \lim_{n\to\infty} \int_{\{x\in\overline{\Omega}:\lambda-1/n\leq\eta(x)\leq\lambda]} |\mathbf{a}(x, D\eta)| = 0.\end{aligned}$$

Finally, if $\eta \geq \lambda$, then it is enough to do the same with $\hat{\eta} = -\eta$, $\hat{\lambda} = -\lambda$, $\hat{\mu} = -\mu$ and $\hat{\mathbf{a}}(x, z) = -\mathbf{a}(x, -z)$. $\square$

Proof of Theorem 2.1. (i) By Proposition 3.1, given $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$ satisfying the range condition (2.2), there exists $[u, \mu, \sigma] \in W^{1,p}(\Omega) \times M_b^p(\overline{\Omega}) \times V^{1,p}(\partial\Omega)$ such that $\sigma \in \beta(u)$ a.e. in $\partial\Omega$, $\mu \in \gamma(u)$ and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \langle \mu, \tilde{v} \rangle + \int_{\partial\Omega} \sigma v = \int_\Omega \phi v + \int_{\partial\Omega} \psi v \quad \forall\, v \in W^{1,p}(\Omega). \tag{3.30}$$

Now, applying (3.28) of Lemma 3.1 with $\eta = u$, $\nu := -\mu + \phi\, \mathcal{L}^N \llcorner \Omega + (\psi - \sigma)$ $\mathcal{H}^{N-1} \llcorner \partial\Omega$, we get that for any $\xi \in \mathcal{C}_c^1(\Omega)$, $\xi \geq 0$,

$$\int_{\{x\in\Omega:u(x)=\gamma^{(s)}\}} \xi\, d\nu \geq 0,$$

which implies that

$$\int_\Omega \xi\, d\mu_s^+ \leq \int_{\{x\in\Omega:u(x)=\gamma^{(s)}\}} \xi\, (\phi - \mu_a)\, dx.$$

Consequently $\mu_s^+ \llcorner \Omega = 0$. Similarly, applying (3.29) of Lemma 3.1, it holds that

$$\int_\Omega \xi\, d\mu_s^- \leq \int_{\{x\in\Omega:u(x)=\gamma^{(i)}\}} \xi\, (\mu_a - \phi)\, dx,$$

and therefore $\mu_s^- \llcorner \Omega = 0$.

Let now Γ be an open subset of $\partial\Omega$. For $n \in \mathbb{N}$ large enough, consider $\Omega_n = \{x \in \mathbb{R}^N : d(x, \overline{\Gamma}) < 1/n\}$, and let $\xi \in \mathcal{C}_c^1(\Omega_n)$, $\xi \geq 0$ and $\xi = 1$ in Γ. Applying

Lemma 3.1, we obtain

$$\int_\Gamma d\mu_s^+ \leq \int_{\Omega_n} \xi\,(\phi - \mu_a)\chi_{\{x\in\Omega : u(x)=\gamma^{(s)}\}} + \int_{\Omega_n} \xi\,(\psi - \sigma)\chi_{\{x\in\partial\Omega : u(x)=\gamma^{(s)}\}}.$$

Letting n go to $+\infty$, we obtain that

$$\int_\Gamma d\mu_s^+ \leq \int_\Gamma (\psi - \sigma)\chi_{\{x\in\partial\Omega : u(x)=\gamma^{(s)}\}}.$$

Consequently,

$$\mu_s^+ \llcorner \partial\Omega \leq (\psi - \sigma)\chi_{\{x\in\partial\Omega : u(x)=\gamma^{(s)}\}}\mathcal{H}^{N-1}\llcorner\partial\Omega. \tag{3.31}$$

Similarly, we obtain that

$$\int_\Gamma d\mu_s^- \leq \int_\Gamma (\sigma - \psi)\chi_{\{x\in\partial\Omega : u(x)=\gamma^{(i)}\}},$$

and therefore

$$\mu_s^- \llcorner \partial\Omega \leq (\sigma - \psi)\chi_{\{x\in\partial\Omega : u(x)=\gamma^{(i)}\}}\mathcal{H}^{N-1}\llcorner\partial\Omega.$$

Therefore $\mu_s = g\mathcal{H}^{N-1}\llcorner\partial\Omega$, $g \in L^1(\partial\Omega)$ and $g(x) \in \partial\mathrm{I\!I}_{\overline{D(\gamma)}}(u(x))$ a.e. on $\partial\Omega$. Hence, if we set $z = \mu_a$ and $w = \sigma + g$, we have that $[u, z, w]$ is a generalized weak solution of problem $(S_{\phi,\psi}^{\gamma,\beta})$.

(ii) Given L^1-data $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$, we take $\phi_m := T_m(\phi)$ and $\psi_m := T_m(\psi)$, respectively. Then, by the first part, there exists $[u_m, z_m, w_m]$ a generalized weak solution of $(S_{\phi_m,\psi_m}^{\gamma,\beta})$, that is, $z_m(x) \in \gamma(u_m(x))$ a.e. in Ω, $w_m(x) \in \beta(u_m(x)) + \partial\mathrm{I\!I}_{\overline{D(\gamma)}}(u_m(x))$ a.e. on $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x, Du_m)\cdot Dv + \int_\Omega z_m v + \int_{\partial\Omega} w_m v = \int_\Omega \phi_m v + \int_{\partial\Omega} \psi_m v, \tag{3.32}$$

for all $v \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$. Now, since $D(\gamma)$ is bounded, we can take $v = u_m$ in (3.32), to obtain

$$\Lambda \int_\Omega |Du_m|^p \leq M(\|\phi\|_1 + \|\psi\|_1).$$

Hence, $\{u_m\}$ is bounded in $W^{1,p}(\Omega)$. On the other hand, by Theorem 2.4, we have

$$\int_\Omega |z_m - z_n| + \int_{\partial\Omega} |w_m - w_n| \leq \int_\Omega |\phi_n - \phi_m| + \int_{\partial\Omega} |\psi_n - \psi_m|.$$

Therefore, there exists a subsequence, denoted equal, $[u, z, w] \in W^{1,p}(\Omega)\times L^1(\Omega)\times L^1(\partial\Omega)$ and $\chi \in L^p(\Omega)$ such that

$$u_m \text{ converges to } u \text{ weakly in } W^{1,p}(\Omega),$$
$$u_m \text{ converges to } u \text{ in } L^p(\Omega),$$
$$u_m \text{ converges to } u \text{ in } L^p(\partial\Omega),$$
$$\mathbf{a}(., Du_m) \text{ converges to } \chi \text{ weakly in } L^p(\Omega),$$
$$z_m \to z \quad \text{ in } L^1(\Omega)$$

and

$$w_m \to w \qquad \text{in } L^1(\partial\Omega).$$

Since $z_m \in \gamma(u_m)$ a.e. in Ω and $w_m \in \beta_\gamma(u_m)$ a.e. on $\partial\Omega$, by monotonicity, we have $z(x) \in \gamma(u(x))$ a.e. in Ω and $w(x) \in \beta_\gamma(u(x))$ a.e. on $\partial\Omega$. Finally, it is not difficult to see that $\chi = \mathbf{a}(x, Du)$ using again Minty–Browder's method, therefore passing to the limit in (3.32), we obtain that $[u, z, w]$ is a generalized weak solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$. □

Let us now prove Proposition 2.1.

Proof of Proposition 2.1. Let us see the proof of (i), is the proof of (ii) similar. Take $\phi_n \in L^\infty(\Omega)$ and $\psi_n \in L^\infty(\partial\Omega)$ such that esssup $(\psi_n) \leq$ esssup (ψ), ϕ_n and ψ_n satisfy (2.2) and converge in L^1 to ϕ and ψ, respectively. Then, by Theorem 2.1, there exists a generalized weak solution $[u_n, z_n, w_n]$ of problem $(S^{\gamma,\beta}_{\phi_n,\psi_n})$ and moreover, by (3.31), $w_n\chi_{\{x\in\partial\Omega:u_n(x)=\gamma^{(s)}\}} \leq \psi_n\chi_{\{x\in\partial\Omega:u_n(x)=\gamma^{(s)}\}}$. Therefore, since esssup $(\psi_n) \leq$ esssup $(\psi) \leq \sup\beta(\gamma^{(s)})$, $w_n \leq \sup\beta(\gamma^{(s)})$ a.e. on $\partial\Omega$. Finally, by Theorem 2.4, $w_n \to w$ in $L^1(\partial\Omega)$, and we deduce that $w \leq \sup\beta(\gamma^{(s)})$ a.e. on $\partial\Omega$. □

Finally we prove the contraction principle given in Theorem 2.3.

Proof of Theorem 2.3. Consider $\varphi \in \mathcal{D}(\Omega)$, $0 \leq \varphi \leq 1$, and $\rho \in W^{1,p}(\Omega)$, $0 \leq \rho \leq 1$. Then,

$$\begin{aligned}&\int_\Omega \mathbf{a}(x, Du_1) \cdot D\left(\frac{T_k^+}{k}(u_1 - u_2 + k\rho)\varphi\right) + \int_\Omega z_1 \frac{T_k^+}{k}(u_1 - u_2 + k\rho)\varphi \\ &\quad \leq \int_\Omega \phi_1 \frac{T_k^+}{k}(u_1 - u_2 + k\rho)\varphi\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \mathbf{a}(x, Du_2) \cdot D\left(\frac{T_k^+}{k}(u_1 - u_2 + k\rho)\varphi\right) + \int_\Omega z_2 \frac{T_k^+}{k}(u_1 - u_2 + k\rho)\varphi \\ &\quad \geq \int_\Omega \phi_2 \frac{T_k^+}{k}(u_1 - u_2 + k\rho)\varphi.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_\Omega (z_1 - z_2)\frac{T_k^+}{k}(u_1 - u_2 + k\rho)\varphi \\ &\qquad + \int_\Omega (\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot \frac{T_k^+}{k}(u_1 - u_2 + k\rho)D\varphi \\ &\qquad + \int_\Omega (\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot \frac{(T_k^+)'}{k}(u_1 - u_2 + k\rho)(Du_1 - Du_2)\varphi\end{aligned}$$

$$+\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot (T_k^+)'(u_1-u_2+k\rho)D\rho\varphi$$
$$\le \int_\Omega (\phi_1-\phi_2)\frac{T_k^+}{k}(u_1-u_2+k\rho)\varphi.$$

Now, since the third term is positive,

$$\int_\Omega (z_1-z_2)\frac{T_k^+}{k}(u_1-u_2+k\rho)\varphi$$
$$+\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot \frac{T_k^+}{k}(u_1-u_2+k\rho)D\varphi$$
$$+\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot (T_k^+)'(u_1-u_2+k\rho)D\rho\varphi$$
$$\le \int_\Omega (\phi_1-\phi_2)\frac{T_k^+}{k}(u_1-u_2+k\rho)\varphi.$$

Taking limit when k goes to 0 in the above expression, having in mind that $Du_1 = Du_2$ where $u_1 = u_2$, we obtain that

$$\int_\Omega (z_1-z_2)\mathrm{sign}_0^+(u_1-u_2)\varphi+\int_\Omega (z_1-z_2)\rho\chi_{\{u_1=u_2\}}\varphi$$
$$+\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot \mathrm{sign}_0^+(u_1-u_2)D\varphi$$
$$\le \int_\Omega (\phi_1-\phi_2)\mathrm{sign}_0^+(u_1-u_2)\varphi+\int_\Omega (\phi_1-\phi_2)\rho\chi_{\{u_1=u_2\}}\varphi.$$

Since

$$\mathrm{sign}_0^+(u_1-u_2)=\mathrm{sign}_0^+(z_1-z_2)\chi_{\{u_1\neq u_2\}}+\mathrm{sign}_0^+(u_1-u_2)\chi_{\{z_1=z_2\}}, \tag{3.33}$$

$$\int_\Omega (z_1-z_2)^+\varphi+\int_\Omega (z_1-z_2)(\rho-\mathrm{sign}_0^+(z_1-z_2))\chi_{\{u_1=u_2\}}\varphi$$
$$+\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot \mathrm{sign}_0^+(u_1-u_2)D\varphi \le \int_\Omega (\phi_1-\phi_2)^+\varphi.$$

By approximation, we can take $\rho = \mathrm{sign}_0^+(z_1-z_2)$ in the above expression, then

$$\int_\Omega (z_1-z_2)^+\varphi+\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot \mathrm{sign}_0^+(u_1-u_2)D\varphi$$
$$\le \int_\Omega (\phi_1-\phi_2)^+\varphi. \tag{3.34}$$

Now,

$$\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot \operatorname{sign}_0^+(u_1-u_2)D\varphi$$
$$= \int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot \operatorname{sign}_0^+(u_1-u_2)D(\varphi-1).$$

Then, since

$$\operatorname{sign}_0^+(u_1-u_2) = \lim_{k\to 0}\frac{T_k^+}{k}(u_1-u_2+k\hat\rho) - \hat\rho\chi_{\{u_1=u_2\}},$$

$\hat\rho \in W^{1,p}(\Omega)$, $0 \le \hat\rho \le 1$, we have that

$$\begin{aligned}
&\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot \operatorname{sign}_0^+(u_1-u_2)D\varphi \\
&\quad = \lim_{k\to 0}\Bigg(\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot D\left(\frac{T_k^+}{k}(u_1-u_2+k\hat\rho)(\varphi-1)\right) \\
&\qquad - \int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot \frac{(T_k^+)'}{k}(u_1-u_2+k\hat\rho) \\
&\qquad \times (Du_1-Du_2+kD\hat\rho)(\varphi-1)\Bigg) \\
&\quad \ge \lim_{k\to 0}\Bigg(\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot D\left(\frac{T_k^+}{k}(u_1-u_2+k\hat\rho)(\varphi-1)\right) \\
&\qquad - \int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot (T_k^+)'(u_1-u_2+k\hat\rho)D\hat\rho(\varphi-1)\Bigg) \\
&\quad = \lim_{k\to 0}\left(\int_\Omega (\mathbf{a}(x,Du_1)-\mathbf{a}(x,Du_2))\cdot D\left(\frac{T_k^+}{k}(u_1-u_2+k\hat\rho)(\varphi-1)\right)\right) \\
&\quad \ge \lim_{k}\Bigg(-\int_\Omega (z_1-z_2)\frac{T_k^+}{k}(u_1-u_2+k\hat\rho)(\varphi-1) \\
&\qquad + \int_{\partial\Omega}(w_1-w_2)\frac{T_k^+}{k}(u_1-u_2+k\hat\rho) \\
&\qquad + \int_\Omega (\phi_1-\phi_2)\frac{T_k^+}{k}(u_1-u_2+k\hat\rho)(\varphi-1) - \int_{\partial\Omega}(\psi_1-\psi_2)\frac{T_k^+}{k}(u_1-u_2+k\hat\rho)\Bigg).
\end{aligned}$$

Therefore, by (3.34),

$$\int_\Omega (z_1-z_2)^+\varphi - \int_\Omega \left(z_1-z_2\right)(\operatorname{sign}_0^+(u_1-u_2)+\hat\rho\chi_{\{u_1=u_2\}})\,(\varphi-1)$$
$$+\int_{\partial\Omega}(w_1-w_2)\left(\operatorname{sign}_0^+(u_1-u_2)+\hat\rho\chi_{\{u_1=u_2\}}\right)$$

$$
\begin{aligned}
&+\int_\Omega (\phi_1-\phi_2)\left(\mathrm{sign}_0^+(u_1-u_2)+\hat\rho\chi_{\{u_1=u_2\}}\right)(\varphi-1)\\
&\le \int_\Omega (\phi_1-\phi_2)^+\varphi+\int_{\partial\Omega}(\psi_1-\psi_2)\left(\mathrm{sign}_0^+(u_1-u_2)+\hat\rho\chi_{\{u_1=u_2\}}\right)\\
&\le \int_\Omega (\phi_1-\phi_2)^+ + \int_{\partial\Omega}(\psi_1-\psi_2)^+. \qquad (3.35)
\end{aligned}
$$

Now, it is easy to see that

$$\mathrm{sign}_0^+(u_1-u_2)=\mathrm{sign}_0^+(w_1-w_2)\chi_{\{u_1\neq u_2\}}+\mathrm{sign}_0^+(u_1-u_2)\chi_{\{w_1=w_2\}},$$

and consequently,

$$
\begin{aligned}
&\int_{\partial\Omega}(w_1-w_2)\left(\mathrm{sign}_0^+(u_1-u_2)+\hat\rho\chi_{\{u_1=u_2\}}\right)\\
&\quad=\int_{\partial\Omega}(w_1-w_2)\mathrm{sign}_0^+(w_1-w_2)+\int_{\partial\Omega}(w_1-w_2)\mathrm{sign}_0^+(u_1-u_2)\chi_{\{w_1=w_2\}}\\
&\qquad+\int_{\partial\Omega}(w_1-w_2)(\hat\rho-\mathrm{sign}_0^+(w_1-w_2))\chi_{\{u_1=u_2\}}\\
&\quad=\int_{\partial\Omega}(w_1-w_2)^+ + \int_{\partial\Omega}(w_1-w_2)(\hat\rho-\mathrm{sign}_0^+(w_1-w_2))\chi_{\{u_1=u_2\}}.
\end{aligned}
$$

Therefore, using the above expression in (3.35), and letting $\varphi\to 1$, we obtain

$$
\begin{aligned}
&\int_\Omega (z_1-z_2)^+ + \int_{\partial\Omega}(w_1-w_2)^+ + \int_{\partial\Omega}(w_1-w_2)(\hat\rho-\mathrm{sign}_0^+(w_1-w_2))\chi_{\{u_1=u_2\}}\\
&\quad\le \int_\Omega(\phi_1-\phi_2)^+ + \int_{\partial\Omega}(\psi_1-\psi_2)^+. \qquad (3.36)
\end{aligned}
$$

Finally, taking, by approximation, $\hat\rho=\mathrm{sign}_0^+(w_1-w_2)$ in (3.36) we obtain the contraction principle. □

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