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Global existence for a degenerate nonlinear diffusion problem with nonlinear gradient term and source

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1. Introduction

We consider the following degenerate nonlinear diffusion problem with a nonlinear gradient term and source

$$(I) \qquad \begin{cases} u_t = \Delta u^m - \|\nabla u^\alpha\|^q + u^p & \text{in } Q = \Omega \times (0,\infty) \\ u = 0 & \text{on } S = \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) \geq 0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$, $m \geq 1$, $\alpha > 0$, $p \geq 1$ and $q \geq 1$.

Equation (I) without the gradient term has been extensively studied (see for instance [Sa] and the references therein). It is known that if $p < m$ there exists a global mild solution for initial data $u_0 \in L^1(\Omega)$ and if $p > m$ solutions may blow up in finite time.

For the case $m = \alpha = 1$, equation (I) was introduced by M. Chipot and F. B. Weissler [ChW] in order to investigate the effect of a damping term on existence or nonexistence of solutions. On the other hand, Ph. Souplet in [So$_2$] proposes a model in population dynamics, where this type of equations

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describes the evolution of the population density of a biological species under the effect of certain natural mechanism.

In the nondegenerate semilinear case, several authors have studied the existence of nonglobal positive solutions, giving conditions for blow-up under certain assumptions on p, q, N and Ω (see for instance [ChW], [KP], [F], [Q_1], [Q_2], [So_1], [So_2], [SW_1], [SW_2]). Global existence for nonnegative initial data has been proved in the case $q \geq p > 1$ (see [F], [Q_2], [SW_2]). Now, in the degenerate case we only know the blow-up results of M. Wiegner ([W_1], [W_2]) and Ph. Souplet and F. B. Weissler ([SW_2]) for classical solutions. In particular, in [SW_2] and [W_2] it is remarked that problem (I) does not admit global classical solutions in the following cases:

(i) $p > m \geq 2,\ \alpha q < p$,

(ii) $\alpha = \frac{m}{2}, q = 2,\ p > m$,

(iii) $\alpha = \frac{m}{2}, q = 2,\ p = m,\ m > 2$.

On the other hand, the existence of solutions of the Cauchy problem for the equation

$$u_t = \Delta u^m + \|\nabla u^\alpha\|^q$$

has been studied by D. Andreucci in [An] under optimal assumptions on initial data.

Concerning equation (I), we want to stress that the diffusion term Δu^m degenerates for $u = 0$, so that one can not expect in general an existence result for classical solutions.

The aim of this paper is to prove the existence of global weak solutions for nonnegative initial data in $L^{m+1}(\Omega)$ under the assumptions:

$$(H) \qquad m \geq 1,\ \ \alpha > \frac{m}{2},\ \ 1 \leq q < 2 \ \text{ and } \ 1 \leq p < \alpha q.$$

Remark that a more natural assumption would be $1 \leq p < \max\{m, \alpha q\}$. We assume (H) for the rest of the paper since in the case $1 \leq p < m$ the existence of solutions of problem (I) follows by comparison with solutions of $u_t = \Delta u^m + u^p$.

Finally, let us notice that, to our knowledge, the question of uniqueness of solutions of this kind of equations is still an open problem.

To finish this introduction we give some of the notation and definitions used later. Concerning the vector-valued functions we follow the notation and definitions of [Br]. We denote by $\mathcal{D}'(]0,T[;X)$ the space of the X-valued distributions on $]0,T[$, i.e., the space of all continuous linear functions from $\mathcal{D}(]0,T[)$ into X. Given a distribution $u \in \mathcal{D}'(]0,T[;X)$, its derivative is denoted by $\partial_t u$ and it is defined as the distribution

$$\langle \partial_t u, \varphi \rangle = -\langle u, \varphi' \rangle, \qquad \forall\, \varphi \in \mathcal{D}(]0,T[).$$

If $u \in W^{1,1}(0,T;X)$, there exists almost every where the derivative of u, defined by

$$\frac{du}{dt}(t) = \lim_{h\to 0} \frac{u(t+h)-u(t)}{h},$$

moreover, $\frac{du}{dt} \in L^1(0,T;X)$. We will use later the following result given in [Br, Proposition A.6].

Lemma 1.1. *Given* $u \in L^1(0,T;X)$*, if* $\partial_t u \in L^1(0,T;X)$*, then there exists* $\tilde{u} \in W^{1,1}(0,T;X)$*, such that* $u = \tilde{u}$ *and* $\partial_t u = \frac{d\tilde{u}}{dt}$ *a.e. in* $]0,T[$.

This paper is organized as follows. Some a priori estimates for smooth solutions are obtained in Section 2. In the third section we establish the existence of global weak solutions for initial data in $L^{m+1}(\Omega)$. In Section 4 we present a model in population dynamics involving problem (I). In appendix A, we prove the existence of weak solutions for initial data in $L^1(\Omega)$ under the restriction $m < \alpha q$. Without this assumption the existence of mild solutions is obtained. Finally, in appendix B, using a modification of the Bernstein technique due to Ph. Bénilan, it is shown more regularity for the solution in the one dimensional case.

2. A priori estimates for smooth solutions

In this section we shall establish a priori estimates for the smooth solutions which will be fundamental for the rest of the paper. From now on we assume Ω to be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ of class C^1.

For $k \in \mathbb{R}$, $0 < k \le 1$, let $f \in C^1(\mathbb{R})$, satisfying $f(k) = 0$ and $f(r) \le (r-k)^p$ for all $r \ge 0$ and $F \in C^1(\mathbb{R}^N, \mathbb{R})$, satisfying $\|\xi\|^q - 1 \le F(\xi) \le \|\xi\|^q$ for all $\xi \in \mathbb{R}^N$. Consider the problem

$$(P) \qquad \begin{cases} u_t = \Delta u^m - F(\nabla u^\alpha) + f(u) & \text{in } Q_T = \Omega \times (0,T) \\ u = k & \text{on } S_T = \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \ge k & \text{in } \Omega. \end{cases}$$

Proposition 2.1. *Let u be a smooth solution of problem (P). Under the assumptions:*

$$(H_1) \qquad m \geq 1, \;\; \alpha > 0, \;\; q \geq 1 \;\; \textit{and} \;\; 1 \leq p < \alpha q,$$

for any $1 \leq s < \infty$ and $\tau > 0$, there exists a constant $C(\tau, \, s)$ such that

$$\|u(t)\|_{L^s(\Omega)} \leq C(\tau, \, s) \quad \textit{for every} \quad t \geq \tau > 0. \tag{2.1}$$

Moreover, if $u_0 \in L^{m+1}(\Omega)$, then

$$\|u(t)\|_{L^{m+1}(\Omega)} \leq C(\|u_0\|_{L^{m+1}(\Omega)}) \quad \textit{for every} \quad t \geq 0. \tag{2.2}$$

Proof. By the maximum principle we have $k \leq u$. Given $s > 0$, multiplying the equation of (P) by $(u^\alpha - k^\alpha)^s$ and performing obvious manipulations it yields

$$\frac{d}{dt}\int_\Omega \Phi_s(u(t)) + \left(\frac{q}{s+q}\right)^q \int_\Omega \|\nabla((u(t)^\alpha - k^\alpha))^{\frac{s+q}{q}}\|^q \leq$$
$$\leq C_1 \int_\Omega (u(t)^\alpha - k^\alpha)^{s+\frac{p}{\alpha}} + C_1 \int_\Omega (u(t)^\alpha - k^\alpha)^s,$$

where

$$\Phi_s(r) = \int_k^r (\tau^\alpha - k^\alpha)^s \, d\tau.$$

Now, by the Poincaré inequality, we have

$$\int_\Omega (u(t)^\alpha - k^\alpha)^{s+q} \leq C_2 \int_\Omega \|\nabla(u(t)^\alpha - k^\alpha)^{\frac{s+q}{q}}\|^q.$$

Thus, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \Phi_s(u(t)) + C_3(s)\int_\Omega (u(t)^\alpha - k^\alpha)^{s+q} \leq \\ &\leq C_1 \int_\Omega (u(t)^\alpha - k^\alpha)^{s+\frac{p}{\alpha}} + C_1 \int_\Omega (u(t)^\alpha - k^\alpha)^s. \end{aligned} \tag{2.3}$$

On the other hand, since $p < \alpha q$, using Young inequality, we have for every $\epsilon > 0$,

$$\begin{aligned} &\int_\Omega (u(t)^\alpha - k^\alpha)^{s+\frac{p}{\alpha}} + \int_\Omega (u(t)^\alpha - k^\alpha)^s \leq \\ &\leq \; \epsilon \int_\Omega (u(t)^\alpha - k^\alpha)^{q+s} + C(\epsilon, s) \end{aligned} \tag{2.4}$$

Then, taking ϵ small enough, by (2.3) and (2.4), we get

$$\frac{d}{dt}\int_{\Omega} \Phi_s(u(t)) + C_4(s)\int_{\Omega} (u(t)^{\alpha} - k^{\alpha})^{s+q} \leq C_5(s). \tag{2.5}$$

Now, since $\alpha q > 1$, by Hölder inequality, it is not difficult to see that

$$C_6(s)\int_{\Omega} (u(t)^{\alpha} - k^{\alpha})^{s+q} + C_7(s) \geq \left(\int_{\Omega} \Phi_s(u(t))\right)^{\frac{\alpha(s+q)}{\alpha s+1}}.$$

Hence, from (2.5) we get the following differential inequality,

$$\frac{d}{dt}\int_{\Omega} \Phi_s(u(t)) + d_1(s)\left(\int_{\Omega} \Phi_s(u(t))\right)^{\frac{\alpha(q+s)}{\alpha s+1}} \leq d_2(s),$$

which implies, from a lemma of Ghidaglia [T, Lemma 5.1], that

$$\begin{aligned}\int_{\Omega}\int_k^u (\tau^{\alpha} - k^{\alpha})^s d\tau \leq \left(\frac{d_2(s)}{d_1(s)}\right)^{\frac{\alpha s+1}{\alpha(q+s)}} + \\ +\left(d_3(s, \|u_0\|_{L^{\alpha s+1}(\Omega)}) + d_1(s)\left(\frac{\alpha(q+s)}{\alpha s+1} - 1\right)t\right)^{-\frac{\alpha s+1}{\alpha q-1}}.\end{aligned}$$

From here, by convexity, we finish the proof of (2.1).

On the other hand, in the case $u_0 \in L^{m+1}(\Omega)$, we obtain

$$\int_{\Omega}\int_k^u (\tau^{\alpha} - k^{\alpha})^{\frac{m}{\alpha}} d\tau \leq d_4(m) + \left(d_5(\|u_0\|_{L^{m+1}(\Omega)})\right)^{-\frac{m+1}{\alpha q-1}},$$

and consequently (2.2) holds.

Our main goal now is to get uniform estimates for smooth solutions independent on time when the initial data are in $L^{\infty}(\Omega)$, and also an $L^{m+1}-L^{\infty}$ regularizing effect.

Proposition 2.2. *Assume that (H$_1$) holds. Let u be a global smooth solution of problem (P). Then*

(i) If $u_0 \in L^{\infty}(\Omega)$, there exists a constant C, depending only on $\|u_0\|_{L^{\infty}(\Omega)}$, such that

$$\|u\|_{L^{\infty}(Q)} \leq C. \tag{2.6}$$

(ii) If $u_0 \in L^{m+1}(\Omega)$ *and* $\tau > 0$*, there exists a constant* $C = C(\tau, \|u_0\|_{L^{m+1}(\Omega)})$*, such that*

$$\|u(t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t \ge \tau. \tag{2.7}$$

Proof. (i) The case $p < m$ is a consequence of [Sa, Theorem 1.3] and the maximum principle.

If $m \le p$, we consider the function $w(x) = Ce^{a \cdot x}$, where C is a constant and $a \in \mathbb{R}^N$ is a fixed vector. Let L be the differential operator

$$L(v) = v_t - \Delta\varphi(v) + F(\nabla v^\alpha) - f(v)$$

with F, f choosen as above and $\varphi \in C^\infty(\mathbb{R}), \varphi(r) = r^m$ for $r \ge k$ and $\varphi' \ge c > 0$. It is easy to see that

$$L(w) \ge -m^2 C^m \|a\|^2 e^{mR\|a\|} - C^p e^{pR\|a\|} + \alpha^q C^{\alpha q} \|a\|^q e^{-\alpha q R\|a\|},$$

where $R > 0$ is such that $\Omega \subset B(0, R)$. Consequently, since $p < \alpha q$, choosing C large enough it follows that $\|u_0\|_{L^\infty(\Omega)} \le w$ and $L(u) \le 0 \le L(w)$. Then, by the maximum principle, $u \le w$ and the conclusion holds.

(ii) Let $\tau > 0$ fixed, applying the above proposition, [K, Theorem 1] and the maximum principle,

$$\|u(\tau)\|_{L^\infty(\Omega)} \le C(\tau, \|u_0\|_{L^{m+1}(\Omega)}).$$

Now, if we consider problem (P) with initial datum $u(\tau)$, by (i) the proof is concluded.

In the next result we obtain some estimates for the gradients.

Proposition 2.3. *Assume (H$_1$) holds. Let* u *be a smooth solution of problem (P) in* Q_T *with initial datum* $u_0 \in L^{m+1}(\Omega)$*. Then, for any* $0 < \epsilon \le m$ *there exists a constant* K*, depending on* $\|u_0\|_{L^{m+1}(\Omega)}, T$ *and* ϵ*, such that*

$$\int_0^T \int_\Omega \|\nabla u^{\frac{m+\epsilon}{2}}\|^2 \le K. \tag{2.8}$$

In particular, if $u_0 \in L^\infty(\Omega)$ *and* $\alpha > \frac{m}{2}$*, then*

$$\int_0^T \int_\Omega \|\nabla u^\alpha\|^2 \le K. \tag{2.9}$$

Proof. By the maximun principle we have $u(t)^\epsilon - k^\epsilon \geq 0$. Hence, multiplying the equation of (P) by $u(t)^\epsilon - k^\epsilon$ and performing obvious manipulations it yields

$$\begin{aligned}(2.10)\qquad &\int_0^T \frac{d}{dt}\int_\Omega \Psi(u(t)) + \frac{4m\epsilon}{(m+\epsilon)^2}\int_{Q_T} \|\nabla(u(t))^{\frac{m+\epsilon}{2}}\|^2 \leq \\ &\leq \int_{Q_T} f(u(t))(u(t)^\epsilon - k^\epsilon) + \int_{Q_T} (u(t)^\epsilon - k^\epsilon),\end{aligned}$$

where

$$\Psi(r) = \int_k^r (\tau^\epsilon - k^\epsilon)\, d\tau.$$

On the other hand, multiplying the equation of (P) by $(u(t)^\alpha - k^\alpha)^{\frac{m}{\alpha}}$ and working as in the proof of Proposition 2.1, it is not difficult to obtain

$$(2.11)\qquad \int_{Q_T} u^{\alpha q+m} \leq C(T, \|u_0\|_{L^{m+1}(\Omega)}).$$

Finally, from (2.10) and (2.11), (2.8) is obtained. (2.9) is a consequence of (2.8) and (i) of Proposition 2.2.

In the particular case $q = 1$, we can obtain the following energy estimates.

Proposition 2.4. *Let u be a smooth solution of problem (P) in Q_T with initial datum $u_0 \in L^\infty(\Omega)$. Under the assumptions:*

$$(H_2)\qquad m \geq 1,\ \ \alpha > \frac{m}{2},\ \ q = 1 \ \text{ and }\ 1 \leq p < \alpha,$$

given $\tau > 0$, there exist constants C and K such that

$$(2.12)\qquad \|\nabla u(t)^m\|_{L^2(\Omega)} \leq C(\tau, \|u_0\|_{L^\infty(\Omega)}), \quad \forall\, \tau \leq t \leq T.$$

$$(2.13)\qquad \int_\tau^T \int_\Omega u_t (u^m)_t \leq K(\tau, T, \|u_0\|_{L^\infty(\Omega)}).$$

Proof. Multiplying the equation of (P) by $(u^m)_t$ we get

$$\begin{aligned}(2.14)\qquad &0 \leq \int_\Omega u_t(u^m)_t = \\ &= -\int_\Omega \nabla u^m \cdot \nabla (u^m)_t - \int_\Omega F(\nabla u^\alpha)(u^m)_t + \int_\Omega f(u)(u^m)_t.\end{aligned}$$

If we set

$$G(r) = \int_0^r f(\tau) m\tau^{m-1}\, d\tau,$$

from (2.14) it follows that

$$0 \le \int_\Omega u_t (u^m)_t = -\frac{d}{dt} E[u(t)] - \int_\Omega F(\nabla u^\alpha)(u^m)_t, \tag{2.15}$$

where the energy $E[u(t)]$ is defined by

$$E[u(t)] = \frac{1}{2}\int_\Omega \|\nabla u(t)^m\|^2 - \int_\Omega G(u(t)).$$

On the other hand, by Young inequality, we have for any $\epsilon > 0$,

$$\left| \int_\Omega F(\nabla u^\alpha)(u^m)_t \right| \le \frac{1}{2\epsilon^2}\int_\Omega F(\nabla u^\alpha)^2 + \frac{\epsilon^2}{2}\int_\Omega ((u^m)_t)^2. \tag{2.16}$$

Now, by Proposition 2.2, there is a constant $M_1 > 0$ such that

$$\|u\|_{L^\infty(Q_T)} \le M_1. \tag{2.17}$$

Hence,

$$((u^m)_t)^2 = mu^{m-1} u_t (u^m)_t \le M_2 u_t (u^m)_t.$$

Thus, taking ϵ such that $\frac{\epsilon^2}{2} M_2 < \frac{1}{2}$, from (2.15) and (2.16) it follows that

$$0 \le \frac{1}{2}\int_\Omega u_t (u^m)_t \le -\frac{d}{dt} E[u(t)] + M_3 \int_\Omega F(\nabla u^\alpha)^2. \tag{2.18}$$

Consequently

$$\frac{d}{dt} E[u(t)] \le M_3 \int_\Omega F(\nabla u^\alpha)^2. \tag{2.19}$$

Now, as a consequence of (2.17), there exists a constant $N > 0$, such that $E[u(t)] + N \ge 0$, and from (2.19), we can write

$$\frac{d}{dt}\Big(E[u(t)] + N \Big) \le M_3 \int_\Omega F(\nabla u^\alpha)^2. \tag{2.20}$$

On the other hand, by Proposition 2.3, it is easy to see that

$$\int_t^{t+h} \Big(E[u(s)] + N\Big)\, ds \le C_1 h + C_2 \tag{2.21}$$

and

$$\int_t^{t+h} \int_\Omega F(\nabla u^\alpha)^2 \le C_3 h + C_4. \tag{2.22}$$

Then, from (2.20), (2.21), (2.22) and the uniform Gronwall's Lemma [T, Lemma 1.1], we obtain (2.12). Finally, integrating (2.18) over $]\tau, T[$, we get

$$\int_\tau^T \int_\Omega u_t (u^m)_t \le 2\Big(E[u(\tau)] - E[u(T)]\Big) + 2M_3 \int_\tau^T \int_\Omega F(\nabla u^\alpha)^2.$$

From here, using (2.12) and (2.22), we obtain (2.13).

3. Existence of global weak solutions

In this section we prove the existence of a global weak solution of problem (I) when the initial datum is in $L^{m+1}(\Omega)$ and is nonnegative. Since we only get the second energy estimate in the particular case $q = 1$ (Proposition 2.4), we can not apply the classical compactness methods. To overcome this difficulty, we use a modification of the method introduced by H. W. Alt and S. Luckhaus in [AL].

Definition. *Given* $0 \le u_0 \in L^{m+1}(\Omega)$*, by a weak solution of problem (I) on* Q_T *we mean a function* $u \in L^\infty(]\tau, T[\times\Omega)$ *for every* $0 < \tau < T$*, such that* $u^m \in L^2(0, T; H_0^1(\Omega))$*,* $u^\alpha \in L^q(0, T; W_0^{1,q}(\Omega))$ *and satisfies the identity*

$$\int_{Q_T} (u_0 - u)\xi_t + \nabla u^m \cdot \nabla \xi + \|\nabla u^\alpha\|^q \xi - u^p \xi = 0$$

for any function $\xi \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q_T) \cap W^{1,\infty}(0, T; L^\infty(\Omega))$ *with* $\xi(T) = 0$*.*

We shall say that u *is a global weak solution of problem (I) if* u *is a weak solution on* Q_T *for all positive* T*.*

Let us now state our existence result.

Theorem 3.1 (Global Existence). *Assume that (H) holds. For every non-negative initial datum $u_0 \in L^{m+1}(\Omega)$ there exists a global weak solution of problem (I) such that, for every $\tau > 0$*

$$\|u\|_{L^\infty(]\tau,T[\times\Omega)} \le C(\|u_0\|_{L^{m+1}(\Omega)},\tau). \tag{3.1}$$

Moreover, if $u_0 \in L^\infty(\Omega)$ then

$$\|u\|_{L^\infty(Q)} \le C(\|u_0\|_{L^\infty(\Omega)}). \tag{3.2}$$

Proof. To prove the existence of solution we will consider a sequence of approximated nondegenerate problems which can be solved in a classical sense. For simplicity we consider the case $q > 1$ for which it is not necessary to approximate the norm. To avoid the degeneracy we consider a sequence of functions $u_{0,n} \in \mathcal{D}(\Omega)$, satisfying:

$$u_{0,n} \to u_0 \quad \text{in } L^1(\Omega) \text{ and } \|u_{0,n}\|_{m+1} \le \|u_0\|_{m+1} \quad \forall\, n \in \mathbb{N}.$$

Consider also sequences of functions (φ_n), (f_n) and (g_n) satisfying:

$$\begin{cases} \varphi_n \in C^\infty(\mathbb{R}),\ \ \varphi_n'(r) \ge c_n > 0\ \ \forall\, r > 0,\ \ \varphi_n(0) = 0, \\ \qquad\qquad \text{and } \varphi_n(r) = r^m \quad \forall\, r \ge \frac{1}{n} \\ f_n \in C^\infty(\mathbb{R}),\ \ f_n(r) = (r - \frac{1}{n})^p\ \ \forall\, r \in [\frac{1}{n}, M_n],\ \ f_n(r) \le (r - \frac{1}{n})^p \\ \qquad\qquad \text{and } f_n(r) \text{ constant for } r > M_n + \frac{1}{n} \\ g_n \in C^\infty(\mathbb{R}),\ \ g_n(r) = r^\alpha \quad \text{for all } r \in [\frac{1}{n}, N_n] \\ \qquad\qquad \text{and } g_n(r) \text{ constant for } r > N_n + \frac{1}{n}. \end{cases}$$

By a classical result [LSU, Theorem 6.1], the problem

$$\begin{cases} v_t = \Delta\varphi_n(v) - \|\nabla g_n(v)\|^q + f_n(v) \quad \text{in } Q_T = \Omega \times (0,T) \\ v = \frac{1}{n} \qquad \text{on } S_T = \partial\Omega \times (0,T) \\ v(x,0) = u_{0,n}(x) + \frac{1}{n} \qquad \text{in } \Omega \end{cases}$$

has a unique smooth solution u_n in Q_T satisfying

$$\frac{1}{n} \le u_n(x,t) \le N_n(T) \quad \text{for all} \quad (x,t) \in \overline{Q_T},$$

with $N_n(T)$ independent of N_n. Hence, taking $N_n = N_n(T)$, u_n is a smooth solution of the problem

$$\begin{cases} w_t = \Delta w^m - \|\nabla w^\alpha\|^q + f_n(w) & \text{in } Q_T = \Omega \times (0,T) \\ w = \frac{1}{n} & \text{on } S_T = \partial\Omega \times (0,T) \\ w(x,0) = u_{0,n}(x) + \frac{1}{n} & \text{in } \Omega. \end{cases}$$

Then, by Proposition 2.2, for $\tau > 0$ we have

$$u_n(x,t) \le C(\|u_0\|_{m+1}, \tau) \quad \text{for all } n \in \mathbb{N},\ x \in \Omega \text{ and } t \ge \tau\ .$$

Consequently we get that u_n is a smooth solution of the problem

$$(P_n)\begin{cases} (u_n)_t = \Delta u_n^m - \|\nabla u_n^\alpha\|^q + (u_n - \frac{1}{n})^p & \text{in } Q_T = \Omega \times (0,T) \\ u_n = \frac{1}{n} & \text{on } S_T = \partial\Omega \times (0,T) \\ u_n(x,0) = u_{0,n}(x) + \frac{1}{n} & \text{in } \Omega. \end{cases}$$

Moreover, as a consequence of Propositions 2.2 and 2.3, the following estimates hold:

(3.3)
$$\|u_n(t)\|_{L^\infty(\Omega)} \le C(\|u_0\|_{L^{m+1}(\Omega)}, \tau), \quad \text{for all } t \ge \tau > 0 \text{ and } n \in \mathbb{N},$$

(3.4)
$$\int_0^T \int_\Omega \|\nabla u_n^m\|^2 \le K_1(\|u_0\|_{L^{m+1}(\Omega)}, T) \quad \text{for all } n \in \mathbb{N},$$

(3.5)
$$\int_\tau^T \int_\Omega \|\nabla u_n^\alpha\|^2 \le K_2(\|u_0\|_{L^{m+1}(\Omega)}, \tau, T) \text{ for all } T \ge \tau > 0 \text{ and } n \in \mathbb{N}.$$

In the next step we are going to see that

(3.6) $\{u_n : n \in \mathbb{N}\}$ is relatively compact in $L^1(Q_T)$.

Given $\tau \in]0,T[$ fixed, we consider the Banach space

$$W = L^2(\tau,T;H_0^1(\Omega)) \cap L^{(\frac{2}{q})'}(]\tau,T[\times\Omega),$$

with dual

$$W' = L^2(\tau,T;H^{-1}(\Omega)) + L^{\frac{2}{q}}(]\tau,T[\times\Omega).$$

Let us see first that

$$(3.7) \qquad \|(u_n)_t\|_{W'} \leq C_2(\tau, T, \|u_0\|_{L^{m+1}(\Omega)}) \qquad \forall n \in \mathbb{N}.$$

In fact: let $\xi \in W$ with $\|\xi\|_W \leq 1$. Then, since $(u_n)_t \in L^2(\tau, T; L^2(\Omega))$, using (3.4) and (3.5), we have

$$\begin{aligned} |\langle (u_n)_t, \xi \rangle_{W',W}| &= \left| \int_\tau^T \int_\Omega (u_n)_t \xi \right| = \\ &= \left| \int_\tau^T \int_\Omega -\nabla u_n^m \cdot \nabla \xi - \|\nabla u_n^\alpha\|^q \xi + (u_n - \frac{1}{n})^p \xi \right| \leq \\ &\leq \left(\int_\tau^T \int_\Omega \|\nabla u_n^m\|^2 \right)^{\frac{1}{2}} \left(\int_\tau^T \int_\Omega \|\nabla \xi\|^2 \right)^{\frac{1}{2}} + \\ &+ \left(\int_\tau^T \int_\Omega \|\nabla u_n^\alpha\|^2 \right)^{\frac{q}{2}} \|\xi\|_{L^{(\frac{2}{q})'}(]\tau,T[\times\Omega)} + \\ &+ C\|\xi\|_{L^2(]\tau,T[\times\Omega)} \leq K_1 + K_2 + C. \end{aligned}$$

The first step in proving (3.6) is to show that

$$(3.8) \quad \lim_{h\to 0^+} \int_\tau^{T-h} \int_\Omega |u_n(r+h) - u_n(r)| \, dxdr = 0 \text{ uniformly in } n \in \mathbb{N}.$$

First, we claim that

(3.9)

$$\int_\tau^{T-h} \int_\Omega (u_n(r+h) - u_n(r))(u_n^m(r+h) - u_n^m(r)) \, dxdr \leq C_3 h^{\frac{2-q}{2}} \quad \forall n \in \mathbb{N}.$$

In fact: taking $0 < h \leq 1$ and having in mind (3.4) and (3.7), it follows that

$$\begin{aligned} &\int_\tau^{T-h} \int_\Omega (u_n(r+h) - u_n(r))(u_n^m(r+h) - u_n^m(r)) \, dxdr = \\ &= \int_\tau^{T-h} \int_\Omega \left(\int_\tau^T (u_n)_s(s) \mathbf{1}_{[r,r+h]}(s) \, ds \right) (u_n^m(r+h) - u_n^m(r)) \, dxdr = \\ &= \int_\tau^{T-h} \langle (u_n)_s, \mathbf{1}_{[r,r+h]}(s)(u_n^m(r+h) - u_n^m(r)) \rangle_{W',W} \, dr \leq \\ &\leq \int_\tau^{T-h} \|(u_n)_s\|_{W'} \|\mathbf{1}_{[r,r+h]}(s)(u_n^m(r+h) - u_n^m(r))\|_W \, dr \leq \\ &\leq C_2 \int_\tau^{T-h} \|\mathbf{1}_{[r,r+h]}(s)(u_n^m(r+h) - u_n^m(r))\|_W \, dr = \end{aligned}$$

$$= C_2 \int_\tau^{T-h} \Bigg[\bigg(\int_\tau^T \int_\Omega \|\mathbf{1}_{[r,r+h]}(s) \nabla (u_n^m(r+h) - u_n^m(r))\|^2 \, dx \, ds \bigg)^{\frac{1}{2}} +$$

$$+ \bigg(\int_\tau^T \int_\Omega |\mathbf{1}_{[r,r+h]}(s)(u_n^m(r+h) - u_n^m(r))|^{(\frac{2}{q})'} \, dx \, ds \bigg)^{\frac{2-q}{2}} \Bigg] \, dr =$$

$$= C_2 \int_\tau^{T-h} \Bigg[h^{\frac{1}{2}} \bigg(\int_\Omega \|\nabla (u_n^m(r+h) - u_n^m(r))\|^2 \, dx \bigg)^{\frac{1}{2}} +$$

$$+ h^{\frac{2-q}{2}} \bigg(\int_\Omega |u_n^m(r+h) - u_n^m(r)|^{(\frac{2}{q})'} \, dx \bigg)^{\frac{2-q}{2}} \Bigg] \, dr \leq$$

$$\leq C_2 h^{\frac{2-q}{2}} \Bigg[T^{\frac{1}{2}} \bigg(\int_\tau^{T-h} \int_\Omega \|\nabla (u_n^m(r+h) - u_n^m(r))\|^2 \, dx \, dr \bigg)^{\frac{1}{2}} + CT \Bigg]$$

$$\leq C_3 h^{\frac{2-q}{2}}.$$

Therefore, (3.9) holds. Now, to prove (3.8), we consider for large M the set

$$E := \{ r \in]\tau, T-h[: \ \|u_n^m(r+h)\|_{H_0^1(\Omega)} + \|u_n^m(r)\|_{H_0^1(\Omega)} +$$

$$+ \frac{1}{h^{\frac{2-q}{2}}} \int_\Omega (u_n(r+h) - u_n(r))(u_n^m(r+h) - u_n^m(r)) \, dx > M \}.$$

By (3.9) we have

$$\lambda_1(E) \leq \frac{C_4}{M},$$

where $\lambda_1(E)$ is the measure of E, and consequently, by (3.3)

$$\int_E \int_\Omega |u_n(r+h) - u_n(r)| \, dx \, dr \leq \frac{C_5}{M}.$$

On the other hand, by [AL, Lemma 1.8] we have for $r \in]\tau, T-h[\setminus E$

$$\int_\Omega |u_n(r+h) - u_n(r)| \, dx \leq \omega_M(h^{\frac{2-q}{2}}),$$

with continuous functions ω_M satisfying $\omega_M(0) = 0$. Therefore,

$$\int_\tau^{T-h} \int_\Omega |u_n(r+h) - u_n(r)| \, dx \, dr \leq T \omega_M(h^{\frac{2-q}{2}}) + \frac{C_5}{M},$$

which yields (3.8) by appropriate choice of M and h.

Now, by (3.3), (3.4) and [Si, Theorem 3] we can suppose (up to extraction of a subsequence, if necessary) that

$$u_n \to u \qquad a.e. \text{ in } Q_T.$$

Then, by the Vitali Convergence Theorem, we have that

$$u_n \to u \qquad \text{in } L^1(Q_T). \tag{3.10}$$

So, by (3.4) we can suppose (up to extraction of a subsequence, if necessary) that

$$\nabla u_n^m \to \nabla u^m \quad \text{weakly in } L^2(Q_T). \tag{3.11}$$

Consequently, $u^m \in L^2(0,T;H_0^1(\Omega))$.

For $\tau \in]0,T[$ fixed, we consider the Banach space W defined as above. By (3.7) we can suppose (up to extraction of a subsequence, if necessary) that

$$(u_n)_t \to w \quad \text{with respect to } \sigma(W',W). \tag{3.12}$$

Consider the Banach space

$$X := H_0^1(\Omega) \cap L^{(\frac{2}{q})'}(\Omega),$$

with dual

$$X' := H^{-1}(\Omega) + L^{\frac{2}{q}}(\Omega).$$

It is easy to see that

$$W' \subset L^1(\tau,T;X'). \tag{3.13}$$

Since $u \in L^{m+1}(Q_T)$, we have $u \in \mathcal{D}'(]\tau,T[;X')$. Thus, there exists $\partial_t u \in \mathcal{D}'(]\tau,T[;X')$. Let us see that $w = \partial_t u$. Take $\varphi \in \mathcal{D}(]\tau,T[)$ and $\psi \in X$. Then, since $\varphi\psi \in W$, we have

$$\langle w, \varphi\psi\rangle_{W',W} = \lim_{n\to\infty} \langle (u_n)_t, \varphi\psi\rangle_{W',W} =$$

$$= -\lim_{n\to\infty} \int_\Omega \int_\tau^T u_n(x,t)\varphi'(t)\psi(x)\, dtdx =$$

$$= -\int_\Omega \int_\tau^T u(x,t)\varphi'(t)\psi(x)\, dtdx = -\int_\tau^T \varphi'(t)\langle u(t),\psi\rangle_{X',X}\, dt =$$

$$= -\langle \int_\tau^T \varphi'(t)u(t)\, dt, \psi\rangle_{X',X}.$$

Now, by (3.13) the map $t \to \varphi(t)w(t)$ belongs to $L^1(\tau, T; X')$, hence

$$\langle w, \varphi\psi \rangle_{W',W} = \int_\tau^T \langle w(t), \varphi(t)\psi \rangle_{X',X}\, dt = \langle \int_\tau^T \varphi w(t)\, dt, \psi \rangle_{X',X}.$$

Consequently, $w = \partial_t u$. Therefore, $u \in L^1(\tau, T; X')$ and $\partial_t u \in L^1(\tau, T; X')$. So, as a consequence of Lemma 1.1, there exists $\tilde{u} \in W^{1,1}(\tau, T; X')$ such that

$$u = \tilde{u} \quad \text{and} \quad \partial_t u = \frac{d\tilde{u}}{dt} \quad \text{a.e. in }]\tau, T[.$$

Therefore

$$\tilde{u}(t) - \tilde{u}(\tau) = \int_\tau^t \partial_s u(s)\, ds \quad \forall\, t \in [\tau, T]$$

and

$$\lim_{h \to 0^+} \int_\tau^T \| \frac{u(t+h) - u(t)}{h} - \partial_t u(t) \|_{X'}\, dt = 0. \tag{3.14}$$

In the next step we are going to see that for almost all $\tau, t \in]0, T]$, $\tau < t$, the following formula holds

$$\frac{1}{m+1} \int_\Omega u(t)^{m+1} - u(\tau)^{m+1} = \int_\tau^t \langle \partial_s u(s), u(s)^m \rangle_{X',X}\, ds. \tag{3.15}$$

We have for almost all $s \in]\tau, T[$ pointwise in Ω

$$\frac{1}{m+1}\Big(u(s)^{m+1} - u(s-h)^{m+1} \Big) \le \Big(u(s) - u(s-h) \Big) u(s)^m, \tag{3.16}$$

and for $s > h$

$$\frac{1}{m+1}\Big(u(s)^{m+1} - u(s-h)^{m+1} \Big) \ge \Big(u(s) - u(s-h) \Big) u(s-h)^m. \tag{3.17}$$

Integrating in (3.16), we obtain

$$\begin{aligned} &\int_\tau^t \int_\Omega \frac{1}{m+1}\Big(u(s)^{m+1} - u(s-h)^{m+1} \Big) \le \\ &\le \int_\tau^t \int_\Omega \Big(u(s) - u(s-h) \Big) u(s)^m. \end{aligned} \tag{3.18}$$

Now,

$$\lim_{h \to 0^+} \frac{1}{h} \int_\tau^t \int_\Omega \frac{1}{m+1}\Big(u(s)^{m+1} - u(s-h)^{m+1} \Big) =$$

$$= \lim_{h\to 0^+} \frac{1}{h}\left(\int_{t-h}^{t} \int_\Omega \frac{1}{m+1} u(s)^{m+1} - \int_{\tau-h}^{\tau} \int_\Omega \frac{1}{m+1} u(s)^{m+1} \right) =$$

$$= \frac{1}{m+1}\left(\int_\Omega u(t)^{m+1} - u(\tau)^{m+1} \right). \tag{3.19}$$

On the other hand, by (3.14) we have that

$$\lim_{h\to 0^+} \frac{1}{h} \int_\tau^t \int_\Omega \Big(u(s) - u(s-h) \Big) u(s)^m = \int_\tau^t \langle \partial_s u(s), u(s)^m \rangle_{X',X} \, ds. \tag{3.20}$$

Consequently, from (3.18) (3.19) and (3.20), we obtain

$$\frac{1}{m+1}\left(\int_\Omega u(t)^{m+1} - u(\tau)^{m+1} \right) \le \int_\tau^t \langle \partial_s u(s), u(s)^m \rangle_{X',X} \, ds. \tag{3.21}$$

Similarly, integrating in (3.17) and taking limit as $h \to 0^+$, we obtain

$$\frac{1}{m+1}\left(\int_\Omega u(t)^{m+1} - u(\tau)^{m+1} \right) \ge \int_\tau^t \langle \partial_s u(s), u(s)^m \rangle_{X',X} \, ds.$$

and (3.15) holds.

In the next step we show that

$$\nabla u_n^m \to \nabla u^m \quad \text{in } L^2_{\text{loc}}(0,T;L^2(\Omega))^N. \tag{3.22}$$

Multiplying the equation of (P_n) by $(u_n^m - (\frac{1}{n})^m - u^m)$, integrating and using the regularizing effect, (3.5), (3.10) and the Vitali Convergence Theorem, we get

$$\int_\tau^t \int_\Omega (u_n)_s (u_n^m - (\frac{1}{n})^m - u^m) = - \int_\tau^t \int_\Omega \nabla u_n^m \cdot \nabla (u_n^m - u^m) + o(n).$$

Now, by (3.11) it follows that

$$\int_\tau^t \int_\Omega (u_n)_s (u_n^m - (\frac{1}{n})^m - u^m) =$$

$$= - \int_\tau^t \int_\Omega \| \nabla (u_n^m - u^m) \|^2 + \int_\tau^t \int_\Omega \nabla u^m \cdot \nabla (u_n^m - u^m) + o(n) =$$

$$= - \int_\tau^t \int_\Omega \| \nabla (u_n^m - u^m) \|^2 + o(n),$$

where the Landau symbol $o(n)$ as usual denotes any term converging to zero as $n \to \infty$. On the other hand,

$$\int_\tau^t \int_\Omega (u_n)_s (u_n^m - (\frac{1}{n})^m - u^m) =$$

$$= \frac{1}{m+1} \int_\Omega u_n(t)^{m+1} - u_n(\tau)^{m+1} + o(n) - \int_\tau^t \int_\Omega (u_n)_s u^m.$$

Since,

$$(u_n)_s \to u_s \quad \text{with respect to } \sigma(W', W),$$

using (3.15), we have that

$$\lim_{n\to\infty} \int_\tau^t \int_\Omega (u_n)_s u^m = -\frac{1}{m+1} \int_\Omega u(t)^{m+1} - u(\tau)^{m+1}.$$

Consequently,

$$\lim_{n\to\infty} \int_\tau^t \int_\Omega (u_n)_s (u_n^m - (\frac{1}{n})^m - u^m) = 0$$

and (3.22) holds.

Let us see now that

$$\|\nabla u_n^\alpha\|^q \to \|\nabla u^\alpha\|^q \quad \text{in } L^1(Q_T). \tag{3.23}$$

We first consider the case $\alpha \le m$. Taking $A := \{(x,t) \in Q_T \ : \ u(x,t) = 0\}$, by (3.22) we have (up to extraction of a subsequence, if necessary) that

$$\nabla u_n^\alpha \to \nabla u^\alpha \quad \text{a.e. in } Q_T \setminus A.$$

Moreover, if $E \subset Q_T \setminus A$ is measurable,

$$\int_E \|\nabla u_n^\alpha\|^q \le \left(\int_E \|\nabla u_n^\alpha\|^2 \right)^{\frac{q}{2}} \mu(E)^{\frac{2-q}{2}}.$$

Then, by (2.8), applying the Vitali Convergence Theorem, we have

$$\|\nabla u_n^\alpha\|^q \to \|\nabla u^\alpha\|^q \quad \text{in } L^1(Q_T \setminus A). \tag{3.24}$$

On the other hand, since $\alpha > \frac{m}{2}$, we take $\gamma > \frac{m}{2}$ such that

$$0 < \alpha - \gamma \le \frac{m}{q(\frac{2}{q})'}.$$

Hence

$$\|\nabla u_n^\alpha\|^q = (\frac{\alpha}{\gamma})^q (u_n)^{(\alpha-\gamma)q} \|\nabla u_n^\gamma\|^q.$$

Therefore, by Hölder inequality and (2.8), we obtain

$$\int_A \|\nabla u_n^\alpha\|^q \le (\frac{\alpha}{\gamma})^q \left[\int_A \left(u_n^{(\alpha-\gamma)q}\right)^{(\frac{2}{q})'}\right]^{\frac{1}{(\frac{2}{q})'}} \left[\int_A \|\nabla u_n^\gamma\|^2\right]^{\frac{q}{2}} \le$$

$$\le K_2 \left[\int_A \left(u_n^{(\alpha-\gamma)q}\right)^{(\frac{2}{q})'}\right]^{\frac{1}{(\frac{2}{q})'}} \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover, by the Stampacchia Theorem

$$\int_A \|\nabla u^\alpha\|^q = 0.$$

From here and (3.24), (3.23) follows for $\alpha \le m$. Consider now the case $\alpha > m$. By (3.22) it follows (up to extraction of a subsequence, if necessary) that

$$\nabla u_n^\alpha \to \nabla u^\alpha \quad \text{a.e. in } Q_T.$$

Let us see that

$$\lim_{\lambda_N(E)\to 0} \int_E \|\nabla u_n^\alpha\|^q = 0 \quad \text{uniformly in} \quad n \in \mathbb{N}. \tag{3.25}$$

In fact: for $k \in \mathbb{N}$, $k \ge 2$, we consider the function

$$G_k(r) = \begin{cases} 0, & \text{if } 0 \le r \le k-1 \\ r-(k-1), & \text{if } k-1 \le r \le k \\ 1, & \text{if } r > k. \end{cases}$$

Multiplying the equation of (P$_n$) by $G_k(u_n)$ and performing obvious manipulations it yields

$$\int_{\{u_n>k\}} \|\nabla u_n^\alpha\|^q \le \int_{Q_T} \|\nabla u_n^\alpha\|^q G_k(u_n) \le$$

$$\le \int_{\{u_n \ge k-1\}} u_n^p + \int_{\{u_{0,n}\ge k-1\}} u_{0,n}.$$

Hence

$$\lim_{k\to+\infty} \int_{\{u_n>k\}} \|\nabla u_n^\alpha\|^q = 0 \quad \text{uniformly in} \quad n \in \mathbb{N}.$$

On the other hand, for k fixed

$$\int_{\{u_n\le k\}\cap E} \|\nabla u_n^\alpha\|^q \le k^{(\alpha-m)q} \int_E \|\nabla u_n^m\|^q \le$$

$$\leq k^{(\alpha-m)q}\left(\int_{Q_T}\|\nabla u_n^m\|^2\right)^{\frac{q}{2}}\lambda_N(E)^{\frac{2-q}{2}},$$

which converges to 0 uniformly in $n \in \mathbb{N}$ when $\lambda_N(E) \to 0$. Consequently, (3.25) holds, and applying the Vitali Convergence Theorem, (3.23) follows.

Finally, since u_n is a smooth solution of problem (P_n), for any test function $\xi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T) \cap W^{1,\infty}(0,T;L^\infty(\Omega))$ with $\xi(T) = 0$, we have

$$-\int_{Q_T} u_n\xi_t - \int_\Omega u_{0,n}\xi(0) = \int_{Q_T}(u_n)_t\xi =$$

$$= -\int_{Q_T}\nabla u_n^m.\nabla\xi - \int_{Q_T}\|\nabla u_n^\alpha\|^q\xi + \int_{Q_T}(u_n - \frac{1}{n})^p\xi.$$

From here, passing to the limit when $n \to \infty$ we obtain that u is a weak solution of problem (I).

4. A model in population dynamics

As we said in the introduction, Ph. Souplet in [So_2] proposes a model in population dynamics in which the evolution equation satisfied by the density of the population is a nondegenerate semilinear equation. We are going to give arguments in support of a degenerate nonlinear partial differential equation of type (I) for the Souplet model. The model is as follows: Consider a population of a biological species, living on a territory represented by some domain $\Omega \subset \mathbb{R}^N$. The space density of the population at time $t \geq 0$ is denoted by $u(.,t)$. The evolution of this density is the result of three types of mechanisms: displacements, births and deaths. Respect to displacements, Souplet considers, for simplicity, that dispersal is due to random motion of individuals, i.e., the population flux is given by the constitutive relation

$$\phi = -\nabla u.$$

However, as was pointed out in the classical paper of M. E. Gurtin and R. C. MacCamy [GMc], there is ample evidence that for some species migration to avoid crowding, rather than random motion, is the primary cause of dispersal. So they propose the constitutive relation of the form

$$\phi = -\nabla u^m.$$

The population supply due to births is assumed to be proportional to the number of couples (or more generally of p-tuples). Hence, the corresponding contribution is given by

$$C_1 u^p.$$

In the population decline due to deaths, Souplet distinguishes between natural and accidental deaths. The natural deaths are assumed to be proportional to the number of individuals. So the corresponding contribution is given by

$$-C_2 u.$$

For the accidental deaths he supposes that the individual can be destroyed by some predators during their displacements. Hence, the density of predators is an increasing function D of the intensity of the flow of preys. Taking into account the diffusion law he proposes a contribution of the form

$$-C_3 \|\nabla u\| D(\|\nabla u\|).$$

Now, having in mind the diffusion law of Gurtin and MacCamy and also that the decline by accidental deaths should be a function of the density of predators and preys, it seems that the contribution by accidental deaths should be of the form

$$-C_3 u D(\|\nabla u^m\|).$$

Therefore, summing up the different contributions one obtains the equation

$$u_t = \Delta u^m + C_1 u^p - C_2 u - C_3 u D(\|\nabla u^m\|). \tag{4.1}$$

If we suppose that function $D(r)$ is of the form r^q, i.e., the predator's density is $\|\nabla u^m\|^q$, then the equation (4.1) takes the form

$$u_t = \Delta u^m + C_1 u^p - C_2 u - C_4 \|\nabla u^\alpha\|^q, \tag{4.2}$$

with

$$\alpha = \frac{qm+1}{q}.$$

Consequently, problem (I) corresponds to the above model, with no natural deaths and a nonviable enviroment in the boundary zone (since we have homogeneous Dirichlet's boundary conditions). Observe that since $p > 1$, our existence result is also true for equations of the form (4.2), i.e., also cover the case with natural deaths.

Finally, observe that in this particular case the assumptions (H) correspond to

$$m \geq 1, \quad 1 \leq q < 2 \quad \text{and} \quad 1 \leq p < qm + 1.$$

5. Appendix A

In this appendix we are going to see that for initial data in $L^1(\Omega)$, under condition (H) and assuming $m < \alpha q$, we can also prove the existence of a global weak solution of problem (I) in the following sense.

Definition. *Given* $0 \le u_0 \in L^1(\Omega)$, *by a weak solution of problem (I) on* Q_T *we mean a function* $u \in L^\infty(]\tau, T[\times\Omega)$ *for every* $0 < \tau < T$, *such that* $u^m, u^p \in L^1(Q_T)$, $\|\nabla u^\alpha\|^q \in L^1(Q_T)$ *and* u *satisfies the identity*

$$\int_{Q_T} (u_0 - u)\xi_t - u^m \Delta\xi + \|\nabla u^\alpha\|^q \xi - u^p \xi = 0$$

for any function $\xi \in L^\infty(0,T; W_0^{2,\infty}(\Omega)) \cap W^{1,\infty}(0,T; L^\infty(\Omega))$ *with* $\xi(T) = 0$.

We shall say that u *is a global weak solution of problem (I) if* u *is a weak solution on* Q_T *for all positive* T.

Proposition 5.1. *Assume that (H) holds and* $m < \alpha q$. *For every nonnegative initial datum* $u_0 \in L^1(\Omega)$ *there exists a global weak solution of problem (I).*

Proof. Working as in the proof of Theorem 3.1, we get u_n smooth solution of problem (P_n). Multiplying the equation of (P_n) by $T(u_n - 1/n)$, where $T(r) = \min\{|r|, 1\}\text{sign}(r)$, we obtain

$$\int_{Q_T} |\nabla (G(u_n - \frac{1}{n}))^\alpha|^q \le \int_\Omega u_{0,n} + C_1 + C_2 \int_{Q_T} (G(u_n - \frac{1}{n}))^p,$$

where $G(r) = r - T(r)$.
From here, using Poincaré and Young inequalities, it follows

$$\int_{Q_T} (G(u_n - \frac{1}{n}))^{\alpha q} \le C_3.$$

Consequently,

$$\int_{Q_T} u_n^{\alpha q} \le C_4.$$

Now, using the regularizing effect (Proposition 2.2) which also works for $u_0 \in L^1(\Omega)$ and proceeding as in the proof of Theorem 3.1, we can obtain the following convergences

$$u_n \to u \quad \text{a.e. in } Q_T,$$

$$\nabla u_n \to \nabla u \quad \text{a.e. in } Q_T.$$

Then, with a slight modification of the proof of (3.23) we can conclude that

$$\|\nabla u_n^\alpha\|^q \to \|\nabla u^\alpha\|^q \quad \text{in } L^1(Q_T).$$

From here the proof concludes.

Observe that the restriction $m < \alpha q$ is only needed in order to pass to the limit in the second order term. Of course, this assumption is not necessary if the datum $u_0 \in L^r(\Omega)$ with $r > 1$. Without this restriction, for $u_0 \in L^1(\Omega)$, the existence of a mild solution is obtained in the following sense.

Definition. *A measurable function u is a mild solution of problem (I) if for any $T > 0$,*

$$(i) \qquad -\|\nabla u^\alpha\|^q + u^p \in L^1(Q_T),$$

$$(ii) \qquad u(.,t) = S(t; u_0, -\|\nabla u^\alpha\|^q + u^p), \quad 0 \le t \le T,$$

where $S(t; u_0, f)$ is the mild solution in the sense of nonlinear semigroups (see [Cr]) of the problem

$$\begin{cases} u_t = \Delta u^m + f & \textit{in } Q_T \\ u = 0 & \textit{on } S_T \\ u(x,0) = u_0(x) \ge 0 & \textit{in } \Omega. \end{cases}$$

In fact, if u_n is the smooth solution of problem (P_n), by the above proposition, $u_n \to u$ in $L^1(Q_T)$ and if $F_n(x,t) = -\|\nabla u_n^\alpha\|^q + (u_n - 1/n)^p$,

$$F_n \to -\|\nabla u^\alpha\|^q + u^p \quad \text{in } L^1(Q_T).$$

Now, by [BG], there is a unique strong solution w_n of problem

$$\begin{cases} (w_n)_t = \Delta\psi_n(w_n) + F_n & \text{in } Q_T \\ w_n = 0 & \text{on } S_T \\ w_n(x,0) = u_{0,n}(x) \ge 0 & \text{in } \Omega, \end{cases}$$

where $\psi_n(r) = |r + 1/n|^m \operatorname{sing}(r + 1/n) - (1/n)^m$. Consequently, $w_n = u_n - 1/n$. From here, applying [BCS, Theorem I], $w_n \to v$ in $L^1(Q_T)$, where $v = S(t; u_0, -\|\nabla u^\alpha\|^q + u^p)$. Now, since $u_n \to u$ in $L^1(Q_T)$, $v = u$.

6. Appendix B

In the one-dimensional case, take for example $\Omega = (-1,1)$, we can prove more regularity for the solution given in Theorem 3.1: the gradient of u^m is bounded in $L^\infty_{loc}(Q)$ as in the case of the porous medium equation (see [Ar], [Be]). In general we are not able to prove that the solution is strong (i.e. u_t is a function) as in [Be] because we can not prove the second energy estimate for $q > 1$.

Proposition 6.1. *If $N = 1$, $\Omega = (-1,1)$ and $\alpha \geq m$ then the global solution given in Theorem 3.1 satisfies for any $\tau > 0$ and $\delta \in (0,1)$*

$$|(u^m)_x(x,t)| \leq C(\tau,\delta)u(x,t) \quad \forall\,(x,t) \in [-\delta,\delta] \times [\tau,\infty). \tag{6.1}$$

Proof. Let $c > 0$ and consider u a solution of the problem

$$u_t = (\varphi(u))_{xx} - F(\varphi(u)_x\, a(u)) + f(u), \tag{6.2}$$

where $\varphi,\ F,\ a,\ f$ are smooth functions and $0 < \varepsilon \leq u \leq c$. We set $p = \dfrac{\varphi(u)_x}{k(u)}$, where k is a smooth function, with $k(r) > 0$ for $r > 0$. We have

$$pk(u) = \varphi(u)_x \tag{6.3}$$

$$p_x k(u) + p^2 \frac{k'(u)k(u)}{\varphi'(u)} = (\varphi(u))_{xx} \tag{6.4}$$

$$p_{xx}k(u) + 3pp_x \frac{k'(u)k(u)}{\varphi'(u)} + p^3 (\frac{k'k}{\varphi'})'(u) \frac{k(u)}{\varphi'(u)} = (\varphi(u))_{xxx} \tag{6.5}$$

$$p_t k(u) + pk'(u)u_t = (\varphi'(u)u_t)_x. \tag{6.6}$$

(6.4), (6.5), (6.6) are obtained by differentiation of (6.2) with respect to x and t. We now omit for simplicity to write (u) for the functions depending on u. By (6.2) and (6.6) we get

$$p_t k + pk'\{(\varphi(u))_{xx} - F(p\beta) + f\} = \{\varphi'\,(\varphi(u))_{xx} - F(p\beta) + f\}_x \tag{6.7}$$

where $\beta(u) := k(u)a(u)$. Then using (6.4) and (6.5) we have

$$p_t k + pk'\{p_x k + p^2 \frac{k'k}{\varphi'} - F(p\beta) + f\} =$$

$$= \frac{\varphi''}{\varphi'} pk\{p_x k + p^2 \frac{k'k}{\varphi'} - F(p\beta) + f\} + \tag{6.8}$$

$$+\varphi'\{p_{xx}k + 3pp_x \frac{k'k}{\varphi'} + p^3 (\frac{k'k}{\varphi'})' \frac{k}{\varphi'} - F'(p\beta)p_x\beta - F'(p\beta)p^2 \frac{\beta'}{\varphi'} k + \frac{f'}{\varphi'} pk\}.$$

Then $L(p) = 0$, where L is the parabolic differential operator defined by

$$L(v) = v_t - \varphi' v_{xx} - v v_x (2k' + k\frac{\varphi''}{\varphi'}) - v^3 \frac{k''k}{\varphi'} +$$

$$+v(\frac{k'}{k} f - \frac{\varphi''}{\varphi'} f - f') + v_x (F'(v\beta)\beta \frac{\varphi'}{k}) \tag{6.9}$$

$$-v(F(v\beta)\frac{k'}{k} + \frac{\varphi''}{\varphi'} F(v\beta) + F'(v\beta)v\beta').$$

We take $\varphi(r) = r^m$, $f(r) = r^p$, $k(r) = r(e - r^{m-1})$ with $e = 1 + c^{m-1}$, $a(r) = (\alpha/m)\ r^{\alpha-m}$ with $\alpha \geq m$ by hypothesis and F satisfying $0 \leq F(\xi) \leq |\xi|^q$ and $|F'(\xi)| \leq q|\xi|^{q-1}$. There exists a constant C depending on c such that

$$|(2k' + k\frac{\varphi''}{\varphi'})(u)| \leq C, \quad |(\frac{k'}{k} f - \frac{\varphi''}{\varphi'} f - f')(u)| \leq C \tag{6.10}$$

$$|F(v\beta(u))\frac{k'(u)}{k(u)} + \frac{\varphi''(u)}{\varphi'(u)} F(v\beta(u)) + F'(v\beta(u))v\beta'(u)| \leq C|v|^q \tag{6.11}$$

$$|F'(v\beta(u))\beta(u)\frac{\varphi'(u)}{k(u)}| \leq C|v|^{q-1} \tag{6.12}$$

and

$$-\frac{k''k}{\varphi'}(u) \geq C. \tag{6.13}$$

Let $K > 0$ and consider $\zeta_\epsilon = (1 + \frac{1}{t^{1/2}})\frac{K\epsilon}{1-x^2}$ where $\epsilon = \pm 1$. Then (6.9), (6.10), (6.11), (6.12) and (6.13) yield

$$(1-x^2)^3 \epsilon L(\zeta_\epsilon) \geq C\{C'K^3(1 + \frac{1}{t^{1/2}})^3 - K^2(1 + \frac{1}{t^{1/2}})^2 -$$

$$-K(1 + \frac{1}{t^{1/2}}) - K^q(1 + \frac{1}{t^{1/2}})^q - K^{q+1}(1 + \frac{1}{t^{1/2}})^{q+1} - \frac{K}{t^{3/2}}\}, \tag{6.14}$$

where C and C' are positive constants. By hypothesis $q+1<3$, then we can choose K large enough such that

$$(1-x^2)^3 \epsilon L(\zeta_\epsilon) \geq 0. \tag{6.15}$$

We now apply the maximum principle to the solutions u_n of approximated problems, and we get for $p_n = \dfrac{\varphi_n(u_n)_x}{k(u_n)}$

$$-L_n(\xi_{-1}) \leq L_n(p_n) \leq L_n(\xi_1) \quad in \ Q_T$$

$$-\infty = \xi_{-1} \leq p_n \leq \xi_1 = +\infty \quad on \ \partial\Omega \times (0,T),$$

where L_n correspond to operator L with approximated coefficients. Passing to the limit we obtain for a.e. $(x,t) \in Q_T$

$$-K(1+\frac{1}{t^{1/2}})\frac{k(u)}{1-x^2} \leq (u^m)_x(x,t) \leq K(1+\frac{1}{t^{1/2}})\frac{k(u)}{1-x^2}$$

and (6.1) is proved.

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