Stabilization of Solutions of the Filtration Equation with absorption and non-linear flux.

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Abstract

This paper is primarily concerned with the large time behaviour of solutions of the initial boundary value problem

$$u_t = \Delta\phi(u) - \varphi(x,u) \quad \text{in} \quad \Omega \times (0,\infty)$$

$$-\frac{\partial\phi(u)}{\partial\eta} \in \beta(u) \quad \text{on} \quad \partial\Omega \times (0,\infty)$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega.$$

Problems of this sort arise in a number of areas of science; for instance, in models for gas or fluid flows in porous media and for the spread of certain biological populations.

0 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The Filtration Equation is the degenerated parabolic equation

$$(I) \qquad \begin{aligned} & u_t = \Delta\phi(u) \quad \text{in} \quad \Omega \times (0,\infty) \\ & -\frac{\partial\phi(u)}{\partial\eta} \in \beta(u) \quad \text{on} \quad \partial\Omega \times (0,\infty) \\ & u(x,0) = u_0(x) \quad \text{in} \quad \Omega \end{aligned}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function with $\phi(0) = 0$, $\partial/\partial\eta$ is the Neumann boundary operator and β is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta(0)$, we enphasize that this assumption is essential throughout the paper. This equation is very general. Different choices of β's lead to different boundary conditions. For instance, $\beta = \mathbb{R} \times \{0\}$ gives Neumann's condition, $\beta = \{0\} \times \mathbb{R}$ gives Dirichlet's condition and $\beta = \{0\}\times]-\infty,0] \cup [0,+\infty[\times\{0\}$ gives the unilateral boundary condition corresponding to variational inequalities introduced by J. L. Lions and G. Stampacchia [29]. Also, different choices of ϕ's correspond to equations that arise in many applications. For instance, if $\phi(r) = |r|^m sign(r)$, we have: for $m > 1$, the Porous Medium Equation, since it arises in the study of gas flows in homogeneous porous media ([35]); for $m = 1$ we recover the classical Equation of Heat Conduction and for $0 < m < 1$ we have the so-called Fast Diffussion Equation which occurs in the modelling of plasma ([12]).

N. D. Alikakos and R. Rostamian ([3]) proved that the solution $u(x,t)$ of the Porous Medium Equation with Neumann boundary condition stabilizes as $t \to \infty$ by converging to the average of the initial data $u_0 \in L^1(\Omega)$, *i.e.*,

$$u(x,t) \overset{L^1(\Omega)}{\longrightarrow} \frac{1}{\mu(\Omega)} \int_\Omega u_0(x)dx \quad \text{as } t \to \infty.$$

More generally, in [34] it is showed that for very general ϕ's and β's the solutions of problem (I) stabilize as $t \to \infty$ by converging to a constant function in $L^1(\Omega)$. The aim of this paper is to show that if we consider perturbations of equation (I) of the following type:

$$(II) \qquad \begin{aligned} &u_t = \Delta\phi(u) - \varphi(x,u) \quad \text{in } \ \Omega \times (0,\infty) \\ &-\frac{\partial \phi(u)}{\partial \eta} \in \beta(u) \quad \text{on } \ \partial\Omega \times (0,\infty) \\ &u(x,0) = u_0(x) \quad \text{in } \ \Omega, \end{aligned}$$

for some φ's the solutions of problem (II) stabilize as $t \to \infty$ by converging to a constant function in $L^1(\Omega)$. We want to mention that the techniques used here are different and easier than the one used in [34] for the unperturbed case.

Problems of type (II) arise in many applications. For instance, the linear case, *i.e.*, the case $\phi(r) = r$ for every $r \in \mathbb{R}$, corresponds to Semilinear Heat Equations. There is an extensive literature about the large time behaviour of the solutions of semilinear parabolic equations, see for instance the works of P. Baras and L. Véron [18], N. Chaffe [20], A. Gmira and L. Véron [25], M. W. Hirsch [26], F. J. Massey, III [30], H. Matano [31, 32] and P. L. Lions [27]. Now, our main interest lies in the nonlinear case, that is, when ϕ is a nonlinear function. Evolution equations with types of nonlinearities arise in modelling gas flow in porous media [6], and the spread of biological populations ([24], [37]). With respect to the stabilization of solutions of this type of problems see the works of D. Aronson, M. G. Crandall and L. A. Peletier [2] and M. Langlais and D. Phillips [28] (see also [1] and [13]).

There are some alternative approches to problem (II). We study this problem within the context of nonlinear semigroup theory. The Filtration Equation with a non-linear flux function was studied from the point of view of nonlinear semigroup theory in $L^1(\Omega)$ by Ph. Bénilan in [9] and more recently by Ph. Bénilan, M. G. Crandall and P. Sacks [8] (see also [10]). Here we will use the results of Ph. Bénilan and we will consider (II) as a perturbation (in the sense of accretive operators) of problem (I), so for us a solution of problem (II) will be a mild-solution obtained via the Crandall-Liggett exponential formula.

The plan of the paper is as follows: Some preliminary results and notation are collected in Section 1. In the second section we show that for some φ's, problem (II) is well-posed and governed by an order-preseving contraction semigroup in $L^1(\Omega)$ with relatively compact orbits. In the third section we show that the mild-solution of problem (II) stabilizes as $t \to \infty$ by converging to a constant function. Finally, in Section 4 we consider the case in which $\varphi(x, r) = 0$ a.e. if and only if $r = 0$. We prove that the solutions of problem (II) for these φ's stabilize as $t \to \infty$ by converging to zero, independently of the choice of β.

1 Preliminaries

In this section we give some of the notation and definitions used later. If $\Omega \subset \mathbb{R}^N$ is a Lebesgue measurable set, $\mu(\Omega)$ denotes its measure. The norm in $L^p(\Omega)$ is denoted by $\| \, . \, \|_p$, $1 \le p \le \infty$. If $k \ge 0$ is an integer and $1 \le p \le \infty$, $W^{k,p}(\Omega)$ is the Sobolev space of functions u on the open set $\Omega \subset \mathbb{R}^N$ for which $D^\alpha u$ belong to $L^p(\Omega)$ when $| \, \alpha \, | \le k$, with its usual norm. $W_0^{k,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Also, if $p = 2$ we write $H^k(\Omega)$ for $W^{k,2}(\Omega)$.

As we said in the introduction, our abstract framework is the theory of non-linear semigroups. We refer the reader to [4], [7], [21] and [23] for background material on non-linear contraction semigroups. From this theory we need the following:

Let X be a real Banach space. A mapping A from X into 2^X, the collection of all subsets of X, will be called an operator on X. The domain of A is denoted by $\mathcal{D}(A)$ and its range by $\mathcal{R}(A)$. An operator A in X is *accretive* if

$$\|x - \hat{x} + \lambda(y - \hat{y})\| \ge \|x - \hat{x}\| \quad for \; \lambda \ge 0, \; y \in Ax, \; \hat{y} \in A\hat{x} \tag{1}$$

From (1), it follows that for every $\lambda > 0$ the problem $x + \lambda Ax \ni z$ has at most one solution $x \in \mathcal{D}(A)$ for a given $z \in X$. Thus, we may define J_λ, the resolvent of A, for each $\lambda > 0$ by $J_\lambda = (I + \lambda A)^{-1}$ and $\mathcal{D}(J_\lambda) = \mathcal{R}(I + \lambda A)$. From (1), it readily follows that J_λ is a nonexpansive mapping, *i.e.*,

$$\|J_\lambda x - J_\lambda y\| \le \|x - y\| \quad \text{for} \;\; x, y \in \mathcal{D}(J_\lambda).$$

Let A be an accretive operator on X and consider the initial value problem

$$u' + Au \ni 0, \quad u(0) = u_0 \tag{2}$$

Discretizing the derivative in (2) and using an implicit difference scheme, we obtain for any partition $0 = t_0 < t_1 < \cdots < t_{n-1} \le T < t_n$ a system of difference relations

$$\frac{u_i - u_{i-1}}{h_{i-1}} + Au_i \ni 0, \; i = 1, 2, \ldots, n \tag{3}$$

where $h_{i-1} = t_i - t_{i-1}$. Using the resolvent of A, the values u_i in (3) are determined successively by

$$u_i = J_{h_{i-1}} u_{i-1}, \; i = 1, 2, \ldots, n \tag{4}$$

and therefore (3) has a solution if, and only if, $u_i \in \mathcal{R}(I + h_{i-1}A)$ for $i = 1, 2, \ldots, n$. This fact motivates the next definitions. Let A be an accretive operator in X. A is *m-accretive* if for every $\lambda > 0$, $\mathcal{R}(I + \lambda A) = X$. If A is m-accretive, then for every $u_0 \in \overline{\mathcal{D}(A)}$ and every partition $0 = t_0 < t_1 < \cdots < t_{n-1} \le T < t_n$, the relation (3) has a unique solution given by (4). The step function $v : [0, T] \to X$ defined by $v(0) = u_0$ and $v(t) = u_i$ for $t_{i-1} < t \le t_i$ is considered to be an approximate solution of (2). The Crandall-Liggett Theorem states that as $max(t_i - t_{i-1}) \to 0$ the approximate solution of (2) converges to a unique continuous function u on $[0, T]$. This function u is defined to be the *mild-solution* of (2) on $[0, T]$. To be more concrete we have:

Crandall-Liggett Theorem Let A be an m-accretive operator in X. Then, for any $u_0 \in \overline{\mathcal{D}(A)}$

$$e^{-tA} u_0 = \lim_{n \to \infty} J^n_{t/n} u_0$$

exists uniformly on compact subsets of $[0, \infty[$. Moreover, the family of operators e^{-tA}, $t > 0$, is a continuous semigroup of nonexpansive self-mappings of $\overline{\mathcal{D}(A)}$.

Many of the partial differential equations that can be studied by means of the Crandall-Liggett Theorem satisfy a *"comparison principle"*. This fact is equivalent to the order preserving property of the semigroup $(e^{-tA})_{t \ge 0}$. The operators which generate order-preserving semigroups are the following:

Let X be a Banach lattice and let A be an operator in X. A is called *T-accretive* if

$$\|\big(x - \hat{x} + \lambda(y - \hat{y})\big)^+\| \ge \|(x - \hat{x})^+\|, \quad \text{for } \lambda \ge 0, \; y \in Ax, \; \hat{y} \in A\hat{x}.- \tag{5}$$

It is clear that A is T-accretive if, and only if, its resolvents are T-contractions, *i.e.*,

$$\|(J_\lambda x - J_\lambda y)^+\| \le \|(x - y)^+\|, \quad \text{for } \lambda \ge 0, \; x, y \in \mathcal{D}(J_\lambda).- \tag{6}$$

Now, since every T-contraction is order-preserving, we have that if A is m-T-accretive then each e^{-tA} is order-preserving. In general, T-accretivity does not imply accretivity, but in some Banach spaces T-accretivity implies accretivity, this is the case for the spaces $L^p(\Omega)$ for $1 \le p \le \infty$.

If X is a Hilbert space, the notion of m-accretive operator coincides with that of maximal monotone operator (see [16] or [4]). An important class of monotone operators consists of gradients of convex functions. More precisely, let ψ be a convex lower semicontinuos function from the Hilbert space X into $]-\infty, +\infty]$. We assume that

$$\mathcal{D}(\psi) := \{u \in X \; : \; \psi(u) < \infty\} \ne \emptyset.$$

For $u \in \mathcal{D}(\psi)$, the set

$$\partial\psi(u) := \{f \in X \ : \psi(v) - \psi(u) \geq (f, v-u) \ \ \forall\ v \in \mathcal{D}(\psi)\}$$

is called the *subdifferential* of ψ at u. A result of G. Minty [33] (see also [16] or [4]) says that the operator $u \to \partial\psi(u)$ is maximal monotone.

We use some terminology and notations from classical topological dynamics: Let $(T(t))_{t\geq 0}$ be a continuous semigroup on a metric space X. The *orbit* or *trajectory* of $u \in X$, respect to $(T(t))_{t\geq 0}$, is the set

$$\gamma(u) = \{T(t)u : t \geq 0\},$$

and the ω-*limit set* of u is

$$\omega(u) = \{v \in X : v = \lim_{n\to\infty} T(t_n)u \ \text{ for some sequence } t_n \to \infty\}.$$

This set is possibly empty. Now, it is well-known that if $\gamma(u)$ is relatively compact, then $\omega(u)$ is a non empty, compact and connected subset of X. Furthemore, $\omega(u)$ is positive invariant under $T(t)$, *i.e.*, $T(t)\omega(u) \subset \omega(u)$ for any $t \geq 0$. An *equilibrium* or *stationary point* $u \in X$ is a point such that $\gamma(u) = \omega(u) = \{u\}$, or equivalently, $T(t)u = u$ for all $t \geq 0$.

2 The Filtration Equation with absorption

In this section we show that, for some φ's, problem (II) is well posed and is governed by an order-preserving contraction semigroup in $L^1(\Omega)$, *i.e.*, we associate with problem (II) an m-T-accretive operator in $L^1(\Omega)$. To do that, first we give a result about the m-T-accretivity of some perturbations of m-T-accretive operators in $L^1(\Omega)$. For accretive operators this result is an exercise in [7], we give here the proof for the sake of completeness.

Theorem 2.1 Let $X = L^1(\Omega, \mathcal{B}, \mu)$ and let A be m-T-accretive in X. Let $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying:
(a) For almost all $x \in \Omega$, $r \to \varphi(x, r)$ is continuous nondecreasing.
(b) For every $r \in \mathbb{R}$, $x \to \varphi(x, r)$ is in $L^1(\Omega, \mathcal{B}, \mu)$.

Let B_φ be the single-valued operator in X defined by $B_\varphi u(x) := \varphi(x, u(x))$ with $\mathcal{D}(B_\varphi) = \{u \in X \ : \varphi(., u(.)) \in X\}$. Then, $A + B_\varphi$ is T-accretive and closed in X. Moreover, if A satisfies
(H) There exists $\omega > 1$ and for every $M > 0$ there exists an $N > 0$ such that for $(u, v) \in A$, $|u| \leq M$ a.e. on $\{\ v \ sign\ u \geq \omega|u|\ \}$ implies $|u| \leq N$ a.e. on Ω.

Then, $A + B_\varphi$ is m-T-accretive in X.

Proof Since B_φ is s-T-accretive, we have $A + B_\varphi$ is T-accretive in X.

Let see now that $A + B_\varphi$ is closed. Let $(u_n, v_n) \in A + B_\varphi$ such that $(u_n, v_n) \to (u, v)$ in $X \times X$. Then, $v_n = a_n + B_\varphi u_n$ with $(u_n, a_n) \in A$. Given $n, m \in \mathbb{N}$, since A is accretive in X there exists $h \in L^\infty(\Omega)$, $|h| \leq 1$ and $h(u_n - u_m) = |u_n - u_m|$ a.e., such that

$$\int_\Omega (a_n - a_m)h \geq 0.$$

Now, $B_\varphi u_n - B_\varphi u_m = 0$ a.e. in $\{u_n = u_m\}$. Hence,

$$\|B_\varphi u_n - B_\varphi u_m\|_1 = \int_\Omega (B_\varphi u_n - B_\varphi u_m)h = \int_\Omega (v_n - v_m)h - \int_\Omega (a_n - a_m)h \leq$$

$$\leq \int_\Omega (v_n - v_m)h \leq \|v_n - v_m\|_1.$$

The above inequality implies that the sequence $(B_\varphi u_n)$ converges in X. We can assume that $u_n \to u$ a.e. Then, since $r \to \varphi(x,r)$ is continuous, we have that $B_\varphi u_n \to \varphi(.,u(.))$ a.e., and consequently, $u \in \mathcal{D}(B_\varphi)$ and $B_\varphi u_n \to B_\varphi u$ in X. Hence $(u_n, a_n) \to (u,a)$ in $X \times X$. Thus, since A is closed $(u,v) \in A + B_\varphi$, and consequently, $A + B_\varphi$ is closed.

Suppose now that A verifies (H) and let see that $A + B_\varphi$ verifies the rank condition $\mathcal{R}(I + A + B_\varphi) = X$, which implies $A + B_\varphi$ is m-accretive in X. We can assume that $\varphi(x,0) = 0$ (in other case we will work with $\psi(x,r) := \varphi(x,r) - \varphi(x,0)$). For every $N \in \mathbb{N}$ let $\varphi_N : \Omega \times \mathbb{R} \to \mathbb{R}$ the function

$$\varphi_N(x,r) := \begin{cases} \varphi(x,N) & \text{if } r \geq N \\ \varphi(x,r) & \text{if } |r| \leq N \\ \varphi(x,-N) & \text{if } r \leq -N, \end{cases}$$

and B_N the operator associated with φ_N, *i.e.*, $B_N u(x) := \varphi_N(x, u(x))$. As a consequence of (b) we have that $\mathcal{D}(B_N) = X$. On the other hand, it follows from (a) and the Dominated Convergence Theorem that the operator B_N is continuous. Therefore the operator $A + B_N$ is m-accretive in X (see for instance [7]). Since $\overline{\mathcal{R}(I + A + B_\varphi)} = \mathcal{R}(I + A + B_\varphi)$, in order to show that $\mathcal{R}(I + A + B_\varphi) = X$ it is enough to see that $L^\infty(\Omega) \cap X \subset R(I + A + B_\varphi)$. In fact: given $w \in L^\infty(\Omega) \cap X$ by (H) there exists $K > 0$ such that

(7) for $(u,v) \in A$, $|u| \leq \dfrac{\|w\|_\infty}{1+\omega}$ on $\{v \ sign\ u \geq \omega |u|\}$ implies $|u| \leq K$ a.e. on Ω.

We know that $w \in \mathcal{R}(I + A + B_N)$ for any $N \in \mathbb{N}$. Hence, there exists $(u_N, v_N) \in A$ such that $w = u_N + v_N + B_N u_N$. Set

$$C_N := \{v_N \ sign\ (u_N) \geq \omega |u_N|\}.$$

If $x \in C_N$, we have that

$$(v_N(x) + u_N(x) + B_N u_N(x)) sign\ u_N(x) \geq \omega |u_N(x)| + |u_N(x)| + B_N u_N(x)\ sign\ (u_N(x))$$

which implies that

$$C_N = \{x \in \Omega \ : \ w(x)\ sign - \ u_N(x) \geq (1+\omega)|u_N(x)| + B_N u_N(x)\ sign\ u_N(x)\}.$$

Since $B_N u_N(x)\ sign\ (u_N(x)) \geq 0$ for $x \in \Omega$, if $x \in C_N$ we have

$$|u_N(x)| \leq \frac{1}{1+\omega} w(x)\ sign\ u_N(x) \leq \frac{\|w\|_\infty}{1+\omega}.$$

Then, by (7) it follows that $|u_N(x)| \leq K$ a.e. on Ω. Thus, if $N > K$, $\varphi_N(x, u_N(x)) = \varphi(x, u_N(x))$ a.e. on Ω. Consequently, $w = u_N + v_N + B_\varphi u_N \in R(I + A + B_\varphi)$ and the proof concludes.

$\square$

Let $Lip(\mathbb{R})$ be the set of Lipschitz continuous maps from $\mathbb{R}$ into $\mathbb{R}$ and set

$$\mathbf{P}_0 := \{\ p \in Lip(\mathbb{R}),\ p \text{ nondecresing},\ p(0) = 0 \text{ and } \operatorname{supp}(p') \text{ compact}\}.$$

The following definition is due to Ph. Bénilan ([9]) and its precursor may be founded in the paper by H. Brézis and W. A. Strauss ([17]) on semilinear elliptic equations.

Definition We say that an operator A in $L^1(\Omega)$ verifies *property (M_0)* if for every $p \in \mathbf{P}_0$ and $(u, v) \in A$,

$$\int_\Omega p(u)v \geq 0.$$

Ph. Bénilan [9, Cor. 2.1, Cor. 2.2] shows that if A is an m-T-accretive operator in $L^1(\Omega)$ satisfying property (M_0), then its resolvent J_λ verifies:

$$\|J_\lambda u\|_p \leq \|u\|_p \ \text{ for } 1 \leq p \leq \infty,$$

and

$$-\|u^-\|_\infty \leq J_\lambda u \leq \|u^+\|_\infty.$$

Lemma 2.2 Every operator A in $L^1(\Omega)$ verifying property (M_0) satisfies condition (H). Concretely, it verifies

(H') for every $M > 0$ if $(u, v) \in A$ satisfies $|u| \leq M$ a.e. on $\{\ v \, sign\, u \geq 0\}$, then

$|u| \leq M$ a.e. on Ω.

Proof Suppose that $(u, v) \in A$ satisfies $|u| \leq M$ a.e. on $C := \{\ v \, sign\, u \geq 0\}$. For every $n \in \mathbb{N}$, set

$$p_n(r) := \begin{cases} 0 & \text{if } r \leq M \\ n(r - M) & \text{if } M \leq r \leq M + \frac{1}{n} \\ 1 & \text{if } r \geq M + \frac{1}{n}. \end{cases}$$

Since $p_n \in \mathbf{P}_0$ we have

$$\int_\Omega p_n(u)v \geq 0 \ \text{ for all } n \in \mathbb{N}. \tag{8}$$

By (8) and the Dominated Convergence Theorem it follows that

$$\int_{\{u > M\}} v \geq 0. \tag{9}$$

If $u \leq M$ a.e. on Ω does not hold, since $u \leq M$ a.e. on C, $\{u > M\} \subset \Omega \sim C$ and $\mu(\{u > M\}) > 0$. Hence, $v < 0$ a.e. on $\{u > M\}$ which contradicts (9).

The same argument taking

$$p_n(r) := \begin{cases} -1 & \text{if } r \leq -M - \frac{1}{n} \\ n(r+M) & \text{if } -M - \frac{1}{n} \leq r \leq -M \\ 0 & \text{if } r \geq -M, \end{cases}$$

shows that $-M \leq u$ a.e. on Ω. □

As consequence of the above lemma and Theorem 2.1 we have the following result.

Corollary 2.3 Let $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying (a) and (b). Let A be an m-T-accretive operator in $L^1(\Omega)$ verifying property (M_0). Then, $A + B_\varphi$ is m-T-accretive in $L^1(\Omega)$.

From now on, Ω will be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. The following definition is given in [17].

Definition Let $u \in W^{1,1}(\Omega)$, $v \in L^1(\Omega)$ and $w \in L^1(\partial\Omega)$. We say that u is a *weak solution* of the Neumann problem

$$\begin{cases} -\Delta u = v, & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = w, & \text{on } \partial\Omega \end{cases}$$

provided the following identity holds for all $f \in C^1(\overline{\Omega})$:

$$\int_\Omega \nabla u \cdot \nabla f = \int_\Omega vf + \int_{\partial\Omega} wf.$$

Let β be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$ and $\phi : \mathbb{R} \to \mathbb{R}$ a continuous increasing function with $\phi(0) = 0$. In order to study problem (I) from the point of view of nonlinear semigroup theory, Ph. Bénilan [9] defines the following operator in $L^1(\Omega)$:

$A_{\beta,\phi} = \{(u,v) \in L^1(\Omega) \times L^1(\Omega)$: there exists $w \in L^1(\partial\Omega)$ such that $h = \phi(u)$ is a weak solution of $-\Delta h = v$ in Ω, $\dfrac{\partial h}{\partial \eta} = w$ on $\partial\Omega$; and $-w(x) \in \beta(u(x))$ a.e. on $\partial\Omega\}$.

In the definition of $A_{\beta,\phi}$ we understand the trace of u on $\partial\Omega$ as $u_{|_{\partial\Omega}} = \phi^{-1}\left(\phi(u)_{|_{\partial\Omega}}\right)$, which makes sense since $\phi(u) \in W^{1,1}(\Omega)$ (Theorem 4.2 of [36]).

In the following theorem we summarize all the results we need about $A_{\beta,\phi}$, given in [9].

Theorem 2.4 The operator $A_{\beta,\phi}$ verifies the following properties:

(i) $A_{\beta,\phi}$ is m-T-accretive in $L^1(\Omega)$.

(ii) $\mathcal{D}(A_{\beta,\phi})$ is dense in $L^1(\Omega)$.

(iii) $\|J_\lambda f\|_p \leq \|f\|_p$ for $1 \leq p \leq \infty$, being $J_\lambda = (I + \lambda A_{\beta,\phi})^{-1}$.

(iv) $A_{\beta,\phi}$ verifies property (M_0).

(v) If $B \subset L^1(\Omega)$ is bounded, then $\{\phi(J_\lambda f) : f \in B\}$ is a bounded subset of $W^{1,q}(\Omega)$

for $1 \leq q < \frac{N}{N-1}$.

About the well-posedness of problem (II) we have the following result.

Theorem 2.5 Let $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying (a) and (b). Let β be a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta(0)$ and $\phi : \mathbb{R} \to \mathbb{R}$ a continuous increasing function with $\phi(0) = 0$. Then, $A_{\beta,\phi}+B_\varphi$ is m-T-accretive in $L^1(\Omega)$ and $\mathcal{D}(A_{\beta,\phi} + B_\varphi)$ is dense in $L^1(\Omega)$. Moreover, if $\varphi(x, 0) = 0$ a.e., then, $A_{\beta,\phi} + B_\varphi$ verifies property (M_0).

Proof The m-T-accretivity of $A_{\beta,\phi} + B_\varphi$ is a consequence of Corollary 2.3 and the above theorem. Since Ω is bounded we have $L^\infty(\Omega) \subset \mathcal{D}(B)$. Therefore, $C_0^\infty(\Omega) \subset D(A_{\beta,\phi} + B_\varphi)$ and consequently $D(A_{\beta,\phi} + B_\varphi)$ is dense in $L^1(\Omega)$.

Finally, if $\varphi(x, 0) = 0$ a.e., we have that $B_\varphi u$ and u have the same sign. Hence, since $A_{\beta,\phi}$ verifies property (M_0), it follows that $A_{\beta,\phi} + B_\varphi$ verifies property (M_0).

□

As a consequence of the Crandall-Liggett Theorem and the above Theorem we have that for every initial data $u_0 \in L^1(\Omega)$ the problem (II) has a mild-solution given by

$$u(x, t) = \big(S(t)u_0\big)(x),$$

being $\big(S(t)\big)_{t\geq 0}$ the order-preserving contraction semigroup generated by $A_{\beta,\phi}+ B_\varphi$.

In order to prove the stabilization theorem we need the orbits to be relatively compact. Now, it is not possible to obtain this result from the compacity of the semigroup because it is known that if $\phi(r) = |r|^m \, sign(r)$ and β corresponds to the Dirichlet boundary condition then, $e^{-tA_{\beta,\phi}} : L^1(\Omega) \to L^1(\Omega)$ is compact if $m > \frac{N-2}{N}$ $(N \geq 3)$ (see [5]), but for $0 < m \leq \frac{N-2}{N}$, even the resolvents are not compact (see [11]). However in [34] it is showed that the orbits of the semigroup generated by $A_{\beta,\phi}$ are relatively compact in $L^1(\Omega)$. The next theorem is a generalization of this result.

Theorem 2.6 Let $S(t)$ be the semigroup generated by $A_{\beta,\phi} + B_\varphi$ and J_λ its resolvent. If φ satisfies (a), (b) and $\varphi(x, 0) = 0$ a.e. Then,

(i) $J_\lambda(B)$ is a relatively compact subset of $L^1(\Omega)$ if B is a bounded subset of $L^\infty(\Omega)$.

(ii) For every $u_0 \in L^1(\Omega)$ the orbit $\gamma(u_0) = \{S(t)u_0 : t \geq 0\}$ is a relatively compact subset of $L^1(\Omega)$.

Proof (i): Let B a bounded subset of $L^\infty(\Omega)$. Take $(f_n) \subset B$ and let $u_n := J_\lambda f_n$. Set $M := \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. Since $A_{\beta,\phi} + B_\varphi$ satisfies property (M_0), $\|u_n\|_\infty \leq M$ for every $n \in \mathbb{N}$. Consequently,

$$|\varphi(x, u_n(x))| \leq \varphi(x, M) + \varphi(x, -M) \quad \text{a.e. for all } n \in \mathbb{N}.$$

Hence, $(f_n - \lambda B_\varphi u_n)$ is a bounded sequence in $L^1(\Omega)$. Now,

$$u_n = (I + \lambda A_{\beta,\phi})^{-1}(f_n - \lambda B_\varphi u_n).$$

Therefore, by (v) in Theorem 2.4, we have that $\phi(u_n)$ is a bounded sequence in $W^{1,q}(\Omega)$ for $1 \leq q < N/N-1$. Then, by the Rellich-Kondrakov Theorem we have that $\phi(u_n)$ is a relatively compact subset of $L^1(\Omega)$. Now, since ϕ^{-1} is continuous, there exists a subsequence (u_{n_k}) such that

$$\lim_{k\to\infty} u_{n_k} = v \quad \text{a.e.}$$

Now, since $\|u_n\|_\infty \leq M$ for every $n \in \mathbb{N}$, applying the Dominated Convergence Theorem we have

$$\lim_{k\to\infty} u_{n_k} = v \quad \text{in} \quad L^1(\Omega).$$

Consequently, $J_\lambda(B)$ is a relatively compact subset of $L^1(\Omega)$.

(ii): Consider first $u_0 \in \mathcal{D}(A_{\beta,\phi} + B_\varphi) \cap L^\infty(\Omega)$. Then, since

$$\|S(t)u_0\|_\infty \leq \|u_0\|_\infty \quad \text{for all} \quad t \geq 0,$$

as a consequence of (i), we have that $J_\lambda(\gamma(u_0))$ is a relatively compact subset of $L^1(\Omega)$ for all $\lambda > 0$. Moreover,

$$\|S(t)u_0 - J_\lambda S(t)u_0\|_1 \leq \lambda \inf\{\|v\|_1 : v \in (A_{\beta,\phi} + B_\varphi)u_0\}.$$

Hence, $\gamma(u_0)$ is relatively compact in $L^1(\Omega)$.

On the other hand, it is easy to see that $\mathcal{D}(A_{\beta,\phi} + B_\varphi) \cap L^\infty(\Omega)$ is dense in $L^1(\Omega)$. Thus, given $u_0 \in L^1(\Omega)$ and $\epsilon > 0$, there exists $v_0 \in \mathcal{D}(A_{\beta,\phi} + B_\varphi) \cap L^\infty(\Omega)$ such that $\|u_0 - v_0\|_1 < \epsilon$. So we have,

$$\sup_{t\geq 0} \inf_{s\geq 0} \|S(t)u_0 - S(s)v_0\|_1 \leq \sup_{t\geq 0} \|S(t)u_0 - S(t)v_0\|_1 \leq \|u_0 - v_0\|_1 < \epsilon.$$

From where it follows that $\gamma(u_0)$ is relatively compact in $L^1(\Omega)$. □

3 The stabilization results

In this section we stablish that the mild-solutions of problem (II) stabilize as $t \to \infty$ by converging to a constant function. We use Lyapunov's method for semigroups of nonlinear contractions introduced by A. Pazy [38]. In order to apply this method we need some regularity results for the elliptic problem associated to (II), similar to the one obtained by Ph. Bénilan [9] for the elliptic problem associated to (I). To obtain this results we need the following.

Let β be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$. Consider in $L^2(\Omega)$ the operator A^2_β with domain

$$\mathcal{D}(A^2_\beta) = \{u \in H^2(\Omega) \ : \ -\frac{\partial u}{\partial \eta} \in \beta(u) \ \text{ a.e. on } \partial\Omega\},$$

$$A_\beta^2 u = -\Delta u, \quad u \in \mathcal{D}(A_\beta^2).$$

H. Brézis in [14] (see also [4, **?**]) shows that A_β^2 coincides with $\partial\psi_1$, being $\psi_1 : L^2(\Omega) \to [0, +\infty]$ the convex lower semicontinuos function defined by

$$\psi_1(u) := \begin{cases} 1/2 \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} j(u), & u \in H^1(\Omega), \;\; j(u) \in L^1(\partial\Omega) \\ +\infty, & \text{otherwise} \end{cases}$$

where $j(r) := \int_0^r \beta^0(s)\, ds$, $\beta^0(s) = min\{r \,:\, r \in \beta(s)\}$.

Assume now that $x \to \varphi(x, r)$ is in $L^2(\Omega)$ for all $r \in \mathbb{R}$. We define in $L^2(\Omega)$ the operator B_φ^2 as

$$B_\varphi^2 := \{(u, v) \in L^2(\Omega) \times L^2(\Omega) \;:\; v = \varphi(., u(.)) \;\text{ a.e.}\}.$$

Take $\psi_0 : L^2(\Omega) \to [0, +\infty]$, defined by

$$\psi_0(u) := \begin{cases} \int_\Omega g(x, u(x))\, dx, & x \to g(x, u(x)) \in L^1(\Omega) \\ +\infty, & \text{otherwise} \end{cases}$$

where $g(x, r) := \int_0^r \varphi(x, s)\, ds$. It is easy to see that $L^\infty(\Omega) \subset \mathcal{D}(\psi_0)$ and ψ_0 is convex. Moreover, by Fatou's lemma ψ_0 is lower semicontinuos. Hence, $\partial\psi_0$ is a maximal monotone graph in $L^2(\Omega)$. We have the following result.

Lemma 3.1 Under the above conditions we have.
(i) $\partial\psi_0 = B_\varphi^2$.
(ii) $\partial(\psi_0 + \psi_1) = A_\beta^2 + B_\varphi^2$.

Proof (i): Let see that $B_\varphi^2 \subset \partial\psi_0$. Let $u \in \mathcal{D}(B_\varphi^2)$ be. We must prove that

$$\psi_0(v) - \psi_0(u) \geq \int_\Omega \varphi(x, u(x))(v(x) - u(x)) \;\text{ for all } v \in \mathcal{D}(\psi_0).$$

Since $g(x, u(x)) \leq \varphi(x, u(x))u(x)$ and u and $\varphi(., u)$ are in $L^2(\Omega)$, it follows that $u \in \mathcal{D}(\psi_0)$. Thus,

$$\psi_0(v) - \psi_0(u) = \int_\Omega \int_{u(x)}^{v(x)} \varphi(x, s)\, dsdx =$$

$$= \int_\Omega \varphi(x, z(x))(v(x) - u(x)) \geq \int_\Omega \varphi(x, u(x))(v(x) - u(x)),$$

since $z(x)$ is between $u(x)$ and $v(x)$.

On the other hand, it is easy to see that B_φ^2 is monotone. Thus, since $\partial\psi_0$ is maximal monotone, to prove that $\partial\psi_0 = B_\varphi^2$, it is enough to show that given $v \in L^2(\Omega)$ there exists $u \in L^2(\Omega)$ such that $u + B_\varphi^2 u = v$. In fact: Fix $x \in \Omega$. Since the mapping $r \to r + \varphi(x, r)$ is one to one from $\mathbb{R}$ onto $\mathbb{R}$, there exists a unique $u(x)$ such that $u(x) + \varphi(x, u(x)) = v(x)$. Since $x \to \varphi(x, r)$ is measurable for all $r \in \mathbb{R}$, it is not difficult to see that u is measurable. Moreover, since $|u| \leq |v|$, it follows that $u \in L^2(\Omega)$, and the proof of (i) finishes.

(ii): Obviously, $\psi_0+\psi_1$ is convex and lower semicontinuous. Hence $\partial(\psi_0+\psi_1)$ is a maximal monotone graph in $L^2(\Omega)$. Thus, since $\partial\psi_0+\partial\psi_1$ is monotone and $\partial\psi_0+\partial\psi_1 \subset \partial(\psi_0+\psi_1)$, we must only prove the rank conditon

$$\mathcal{R}(I+\partial\psi_0+\partial\psi_1) = L^2(\Omega).$$

To do that it is enough to prove

$$\partial\psi_0+\partial\psi_1 \text{ is closed in} - L^2(\Omega) \tag{10}$$

and

$$L^\infty(\Omega) \subset \mathcal{R}(I+\partial\psi_0+\partial\psi_1). \tag{11}$$

Let $(u_n, f_n) \in \partial\psi_0+\partial\psi_1$ with $u_n \to u$ and $f_n \to f$. Then, $f_n = A^2_\beta u_n + B^2_\varphi u_n$. Now,

$$\|A^2_\beta u_n - A^2_\beta u_m\|_2 = \big(A^2_\beta u_n - A^2_\beta u_m, \frac{u_n-u_m}{\|u_n-u_m\|_2}\big) =$$

$$= \big(f_n - f_m, \frac{u_n-u_m}{\|u_n-u_m\|_2}\big) + \big(B^2_\varphi u_m - B^2_\varphi u_n, \frac{u_n-u_m}{\|u_n-u_m\|_2}\big) \leq$$

$$\leq \big(f_n - f_m, \frac{u_n-u_m}{\|u_n-u_m\|_2}\big) \leq \|f_n - f_m\|_2.$$

From here, since A^2_β is closed, it follows that $(u,f) \in \partial\psi_0+\partial\psi_1$ and, consequently, (10) holds. Finally, let $u \in L^\infty(\Omega)$. Since $A_{\beta,1}+B_\varphi$ is m-accretive in $L^1(\Omega)$ by Theorem 2.5, there exists $v \in L^1(\Omega)$ such that $u = v + A_{\beta,1}v + B_\varphi v$. Moreover, $\|v\|_\infty \leq \|u\|_\infty$, hence, $B_\varphi v \in L^2(\Omega)$ and consequently $v \in \mathcal{D}(A^2_\beta)$ and $u = v + A^2_\beta v + B^2_\varphi v$

□

Using the above lemma and slight modifications of the arguments in the proof of [9, **?**], we have the following result.

Proposition 3.2 Let $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying (a), $\varphi(x,0) = 0$ a. e. and
(b') For every $r \in \mathbb{R}$, $x \to \varphi(x,r)$ is in $L^2(\Omega)$.
Let β be a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta(0)$ and $\phi : \mathbb{R} \to \mathbb{R}$ a continuous increasing function with $\phi(0) = 0$. Then, given $v \in L^\infty(\Omega)$ there exists $u \in C(\overline{\Omega})$ with $\phi(u) \in H^2(\Omega)$ verifying

$$u - \Delta\phi(u) + \varphi(.,u) = v \quad \text{a. e.} \quad \text{in } \Omega$$

$$-\frac{\partial\phi(u)}{\partial\eta} \in \beta(u) \quad \text{a. e.} \quad \text{in } \partial\Omega.$$

Proof Let $k := \|v\|_\infty$ be. Define

$$\phi_1(r) := \begin{cases} \phi(k) + r - k, & r \geq k \\ \phi(r), & |r| < k \\ \phi(-k) + r + k, & r \leq -k, \end{cases}$$

$\beta_1 := \beta \circ \phi^{-1}$, $\varphi_1(x,r) := \varphi(x, \phi^{-1}(r))$. By the above lemma $A^2_{\beta_1} + B^2_{\varphi_1}$ is a maximal monotone graph in $L^2(\Omega)$. Moreover, $A^2_{\beta_1} + B^2_{\varphi_1} \subset A_{\beta_1,1} + B_{\varphi_1}$ and by Theorem 2.5 the operator $A_{\beta_1,1} + B_{\varphi_1}$ is T-accretive and verifies property (M_0). Then, by the proof of Proposition 2.5 of [9], there exists $h \in L^2(\Omega)$ such that

$$\phi_1^{-1}(h) + A^2_{\beta_1} h + B^2_{\varphi_1} h = v \quad \text{a. e.} \quad \text{in } \Omega.$$

Thus, if $u := \phi_1^{-1}(h)$, we have that

$$u + A^2_{\beta_1} \phi_1(u) + \varphi(., \phi^{-1}\phi_1(u)) = v \quad \text{a. e.} \quad \text{in } \Omega.$$

Now, by the property (M_0), $\|u\|_\infty \leq \|v\|_\infty = k$, so $\phi_1(u) = \phi(u)$ and consequently we obtain that

$$u - \Delta\phi(u) + \varphi(., u) = v \quad \text{a. e.} \quad \text{in } \Omega$$

$$-\frac{\partial \phi(u)}{\partial \eta} \in \beta(u) \quad \text{a. e. in } \partial\Omega.$$

Finally, since $(h, v + h - \phi_1^{-1}(h) - B^2_{\varphi_1} h) \in I + A^2_{\beta_1}$, it follows from [14, Theorem 1.10] that $\phi(u) = h \in H^2(\Omega)$ and

$$\|\phi(u)\|_{H^2(\Omega)} = \|h\|_{H^2(\Omega)} \leq C\|v + h - \phi_1^{-1}(h) - B^2_{\varphi_1} h\|_2 \leq$$

$$\leq C_1\big(\sup_{|r|\leq k} |\phi(r) - r| + \|v\|_\infty + \|\varphi(., k)\|_2\big).$$

□

Now we come to the main result.

Theorem 3.3 Let β be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$ and $\phi : \mathbb{R} \to \mathbb{R}$ a continuous increasing function with $\phi(0) = 0$. Suppose φ verifies (a), (b') and $\varphi(x, 0) = 0$ a. e. Let $u_0 \in L^1(\Omega)$. Then, if $u(x,t)$ is the mild-solution of problem (II) there exists a constant K, $K \in \beta^{-1}\{0\} \cap \{s \in \mathbb{R} \ : \ \varphi(x,s) = 0 \text{ a.e}\}$ such that

$$\|u(., t) - K\|_1 \to 0 \quad \text{as } t \to \infty.$$

Moreover, if $u_0 \in L^\infty(\Omega)$ then

$$\|u(., t) - K\|_p \to 0 \quad \text{as } t \to \infty$$

for any $p \in [1, \infty[$.

Proof Suppose first that $u_0 \in L^\infty(\Omega)$. Let $S(t)$ be the semigroup generated by $A_{\beta,\phi} + B_\varphi$ and J_λ, its resolvent. Let $\mathcal{V} : L^1(\Omega) \to]-\infty, +\infty]$ defined by

$$\mathcal{V}(u) = \begin{cases} \int_\Omega j(u), & \text{if } j(u) \in L^1(\Omega) \\ \\ +\infty, & \text{if } j(u) \notin L^1(\Omega) \end{cases}$$

being $j(r) = \int_0^r \phi(s)ds$. Since ϕ is increasing, it is easy to see that j is continuous and convex, hence $\mathcal{V}$ is lower semicontinuous (see [15, pag. 160]). On the other hand, as a consequence of [9, Proposition 2.3] we have that

$$\int_\Omega j(J^n_{t/n} f) \leq \int_\Omega j(f) \quad \text{for } f \in L^1(\Omega),\ t > 0 \ \text{ and } \ n \in \mathbb{N}.$$

Now, by the Crandall-Liggett Theorem, since $\mathcal{V}$ is lower semicontinuous, we have

$$\mathcal{V}\big(S(t)f\big) \leq \liminf_{n\to\infty} \mathcal{V}\big(J^n_{t/n} f\big) \leq \mathcal{V}(f), \quad \text{for } t \geq 0.$$

Therefore, $\mathcal{V}$ is a Liapunov functional for the semigroup $\big(S(t)\big)_{t\geq 0}$.

Let $\mathcal{W} : L^1(\Omega) \to]-\infty, +\infty]$ defined by

$$\mathcal{W}(u) = \begin{cases} \frac{1}{2}\int_\Omega \big(\nabla\phi(u)\big)^2, & \text{if } \big(\nabla\phi(u)\big)^2 \in L^1(\Omega) \\ \\ +\infty, & \text{if } \big(\nabla\phi(u)\big)^2 \notin L^1(\Omega) \end{cases}$$

Since $u_0 \in L^\infty(\Omega)$, $u_0 \in \mathcal{D}(\mathcal{V})$, and it follows from Proposition 3.2 that $J_\lambda u_0 \in \mathcal{D}(\mathcal{V}) \cap \mathcal{D}(\mathcal{W})$. Our next step will be to prove

$$\mathcal{V}(J_\lambda u_0) + \lambda\mathcal{W}(J_\lambda u_0) - \mathcal{V}(u_0) \leq 0. \tag{12}$$

Since ϕ is continuous and increasing, it easy to see that

$$\frac{1}{\lambda}\big(\mathcal{V}(J_\lambda u_0) - \mathcal{V}(u_0)\big) \leq \int_\Omega \frac{1}{\lambda}(J_\lambda u_0 - u_0)\ \phi(J_\lambda u_0). \tag{13}$$

On the other hand, if $\psi : L^2(\Omega) \to]-\infty, +\infty]$ is defined by

$$\psi(u) = \frac{1}{2}\int_\Omega \big(\nabla u\big)^2 + \int_{\partial\Omega}\int_0^u \beta\circ\phi^{-1}(s)\ ds + \int_\Omega\int_0^u \varphi(x, \phi^{-1}(s))\ ds,$$

when the integrals are finite, and ∞ otherwise, it follows from Lemma 3.1 that

$$\partial\psi = A^2_{\beta\circ\phi^{-1}} + B^2_\vartheta,$$

being $\vartheta(x,s) = \varphi(x, \phi^{-1}(s))$.

Since $\big(J_\lambda u_0, \frac{1}{\lambda}(u_0 - J_\lambda u_0)\big) \in A_{\beta,\phi} + B_\varphi$ and $\frac{1}{\lambda}(u_0 - J_\lambda u_0) \in L^\infty(\Omega)$, from Proposition 3.2 it follows that

$$-\Delta\phi(J_\lambda u_0) + \varphi(., J_\lambda u_0) = \frac{1}{\lambda}(u_0 - J_\lambda u_0) \ \text{ a. e. } \ \text{ in } \ \Omega$$

$$-\frac{\partial\phi(J_\lambda u_0)}{\partial\eta} \in \beta(J_\lambda u_0) \ \text{ a. e. } \ \text{ in } \ \partial\Omega,$$

which implies that

$$\big(\phi(J_\lambda u_0), \frac{1}{\lambda}(u_0 - J_\lambda u_0)\big) \in A^2_{\beta\circ\phi^{-1}} + B^2_\vartheta = \partial\psi.$$

Then, by the definition of subdifferential and (13) we obtain

$$\frac{1}{\lambda}\big(\mathcal{V}(J_\lambda u_0)-\mathcal{V}(u_0)\big)\le\int_\Omega\frac{1}{\lambda}(J_\lambda u_0-u_0)\ \phi(J_\lambda u_0)\le$$

$$\le\psi(0)-\psi(\phi(J_\lambda u_0))=-\psi(\phi(J_\lambda u_0)\le-\mathcal{W}(J_\lambda u_0),$$

and (12) holds.

Replacing u_0 by $J_\lambda^{k-1}u_0$ in (12) we find

$$\mathcal{V}(J_\lambda^k u_0)+\lambda\mathcal{W}(J_\lambda^k u_0)-\mathcal{V}\big(J_\lambda^{k-1}u_0\big)\le 0.$$

Summing these inequalities from $k=1$ to $k=n$ and choosing $\lambda=t/n$, we get

$$\mathcal{V}(J_{\frac{t}{n}}^n u_0)+\sum_{k=1}^n\frac{t}{n}\mathcal{W}(J_{\frac{t}{n}}^k u_0)-\mathcal{V}(u_0)\le 0. \tag{14}$$

Next we define a piecewise constant function $F_n(\tau)=\mathcal{W}(J_{\frac{t}{n}}^k u_0)$ for $(k-1)t/n<\tau\le kt/n$. Then

$$\sum_{k=1}^n\frac{t}{n}\mathcal{W}(J_{\frac{t}{n}}^k u_0)=\int_0^t F_n(\tau)\ d\tau.$$

On the other hand, by the Crandall-Liggett Theorem and the Dominated Convergence Theorem it follows, taking a subsequence if necessary, that

$$\lim_{n\to\infty}\phi(J_{\frac{t}{n}}^k u_0)=\phi\big(S(\tau)u_0\big)\quad\text{in}\quad L^2(\Omega).$$

Now, it is easy to see that the functional

$$\mathcal{U}(u)=\begin{cases}\frac{1}{2}\int_\Omega\big(\nabla u\big)^2, & \text{if } \big(\nabla u\big)^2\in L^1(\Omega)\\[2ex] +\infty, & \text{if } \big(\nabla u\big)^2\notin L^1(\Omega)\end{cases}$$

is lower semicontinuous in $L^2(\Omega)$. Hence,

$$\mathcal{W}\big(S(\tau)u_0\big)=\mathcal{U}\big(\phi\big(S(\tau)u_0\big)\big)\le\liminf_{n\to\infty}\mathcal{U}\big(\phi(J_{\frac{t}{n}}^k u_0)\big)=$$

$$=\liminf_{n\to\infty}\mathcal{W}\big((J_{\frac{t}{n}}^k u_0)\big)=\liminf_{n\to\infty}F_n(\tau).$$

Now, by Fatou's lemma, we obtain

$$\int_0^t\mathcal{W}\big(S(\tau)u_0\big)\ d\tau\le\int_0^t\liminf_{n\to\infty}F_n(\tau)\ d\tau\le\liminf_{n\to\infty}\int_0^t F_n(\tau)\ d\tau.$$

Hence,

$$\int_0^t\mathcal{W}\big(S(\tau)u_0\big)\ d\tau\le\liminf_{n\to\infty}\sum_{k=1}^n\frac{t}{n}\mathcal{W}(J_{\frac{t}{n}}^k u_0). \tag{15}$$

Passing to the limit as $n \to \infty$ in (14) and taking into account (15) and the lower semicontinuity of $\mathcal{V}$, we find

$$\mathcal{V}\big(S(t)u_0\big) + \int_0^t \mathcal{W}\big(S(\tau)u_0\big)\ d\tau - \mathcal{V}(u_0) \leq 0,$$

From where it follows that

$$\int_0^\infty \mathcal{W}\big(S(\tau)u_0\big)\ d\tau \leq \mathcal{V}(u_0).$$

Thus, there exists a sequence $t_n \to \infty$, such that $\mathcal{W}\big(S(t_n)u_0\big) \to 0$ as $n \to \infty$.

Now by Theorem 2.6, there exists a subsequence (t_{n_k}) such that

$$\lim_{k\to\infty} S(t_{n_k})u_0 = v \in \omega(u_0).$$

Hence, by the Dominated Convergence Theorem,

$$\lim_{k\to\infty} \phi\big(S(t_{n_k})u_0\big) = \phi(v) \ \text{ in } L^2(\Omega)$$

and by the lower semicontinuity of $\mathcal{U}$, it follows that

$$\mathcal{W}(v) \leq \liminf_{k\to\infty} \mathcal{W}\big(S(t_{n_k})u_0\big) = 0.$$

Therefore, v is a constant K. If $K = 0$, since 0 is an equilibrium, $\omega(u_0) = \{0\}$. Suppose $K > 0$. Then, since $\|S(t)K\|_\infty \leq \|K\|_\infty = K$,

$$0 \leq S(t)K \leq K. \tag{16}$$

Now, since $S(t)K,\ K \in \omega(u_0)$ and $\mathcal{V}$ is a Liapunov functional, it follows from the invariance principle of Dafermos [22, **?**] that $\mathcal{V}\big(S(t)K\big) = \mathcal{V}(K)$. Consequently, by (16) and the definition of $\mathcal{V}$, $S(t)K = K$ for all $t \geq 0$, hence we get $\omega(u_0) = \{K\}$ and the proof for the case $u_0 \in L^\infty(\Omega)$ concludes. Now, since $L^\infty(\Omega)$ is dense in $\overline{\mathcal{D}(A_{\beta,\phi} + B_\varphi)} = L^1(\Omega)$ and $S(t)$ is a T-contraction, from the above we obtain easily the conclusion in the general case $u_0 \in L^1(\Omega)$.

As K is an equilibrium, it follows that $K \in \beta^{-1}(0) \cap \{s \in \mathbb{R} \ : \ \varphi(x,s) = 0 \text{ a.e}\}$.

Finally, if $u_0 \in L^\infty(\Omega)$, since $\|u(.,t)\|_\infty \leq \|u_0\|_\infty$ and $u(.,t) \to K$ in $L^1(\Omega)$, we obtain by the Dominated Convergence Theorem that

$$\lim_{t\to\infty} \|u(.,t) - K\|_p = 0$$

for any $p \in [1, \infty[$. □

To finish this section we shall see that for the Neumann boundary problem (*i.e.*, problem (II) with $\beta = \mathbb{R} \times \{0\}$) we can be more precise about the stabilization constant of the above theorem. If $v \in L^1(\Omega)$, we denote by $\overline{v}$ the average of v, *i.e.*,

$$\overline{v} := \frac{1}{\mu(\Omega)} \int_\Omega v(x)\ dx.$$

Theorem 3.4 Suppose φ and ϕ verifies the assumptions of Theorem 3.3. Let

$$b := \text{ess inf}_{x\in\Omega} \sup\{r \in \mathbb{R} \ : \ \varphi(x,r) = 0\}$$

and $0 \leq u_0 \in L^1(\Omega)$. If $u(x,t)$ is the mild-solution of the Neumann boundary problem

$$\begin{aligned} u_t &= \Delta\phi(u) - \varphi(x,u) \quad \text{in} \ \ \Omega \times (0,\infty) \\ \frac{\partial \phi(u)}{\partial \eta} &= 0 \quad \text{on} \ \ \partial\Omega \times (0,\infty) \\ u(x,0) &= u_0(x) \quad \text{in} \ \ \Omega, \end{aligned}$$

Then, there exists a constant K such that

$$\lim_{t\to\infty} u(.,t) = K \quad \text{in} \ \ L^1(\Omega)$$

and

$$\overline{\inf\{u_0, b\}} \leq K \leq \inf\{\overline{u_0}, b\}.$$

Proof By Theorem 3.3, $K \leq b$. Let see that $K \leq \overline{u_0}$. Since $\|u(.,t)\|_1 \leq \|u_0\|_1$ and $u_0 \geq 0$, we have

$$\int_\Omega u(x,t)\ dx \leq \int_\Omega u_0(x)\ dx.$$

Then from Theorem 3.3 it follows that

$$\int_\Omega K\ dx \leq \int_\Omega u_0(x)\ dx,$$

and consequently $K \leq \overline{u_0}$.

Let $u_b := \inf\{u_0, b\}$ be. To finish the proof let see that $\overline{u_b} \leq K$. In fact: Take $v_b := J_\lambda u_b$. Since $u_b \leq b$, we have $v_b \leq J_\lambda b = b$, from where it follows that $B_\varphi v_b = 0$. Hence,

$$u_b = \big(I + \lambda(A_{\beta,\phi} + B_\varphi)\big)v_b = (I + \lambda A_{\beta,\phi})v_b.$$

Thus, $J_\lambda u_b = (I + \lambda A_{\beta,\phi})^{-1}u_b$ and by the Crandall-Liggett Theorem,

$$e^{-t(A_{\beta,\phi}+B_\varphi)}u_b = e^{-tA_{\beta,\phi}}u_b.$$

Now, by [34, **?**]

$$\lim_{t\to\infty} e^{-tA_{\beta,\phi}}u_b = \overline{u_b}.$$

Finally, since $u_b \leq u_0$ we have

$$K \geq \lim_{t\to\infty} e^{-t(A_{\beta,\phi}+B_\varphi)}u_b = \overline{u_b}.$$

□

Remark 3.5 E. A. Carl [19] has observed that arctic ground squirrels migrate from densely populated areas into sparsely populated areas, even when the latter provides a less favorable habitat. Gurtin and MacCamy [24] give a possible model for the spatial diffusion of such biological species. This model leads to the non-linear partial differential equation

$$u_t = \Delta\phi(u) + \sigma(x,u)$$

with ϕ non-linear and increasing. Here, $u(x,t)$ is the population density and σ is the population supply by births and deaths.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $\Sigma \subset \Omega$ such that $\mu(\Sigma) > 0$. If we suppose that Σ is a death-dominant region, i.e., $\sigma(x,u) = \lambda(x)u$ for $x \in \Sigma$ with the Malthusian parameter $\lambda(x) < 0; \Omega \sim \Sigma$ is a stable region, i.e., $\sigma(x,u) = 0$ for $x \in \Omega \sim \Sigma$, and the population is confined in Ω (for instance if Ω is an island). Then the Gurtin-MacCamy model leads to the problem

$$(III) \qquad \begin{aligned} &u_t = \Delta\phi(u) - \varphi(x,u) \quad \text{in } \Omega \times (0,\infty) \\ &-\frac{\partial\phi(u)}{\partial\eta} = 0 \quad \text{on } \partial\Omega \times (0,\infty) \\ &u(x,0) = u_0(x) \quad \text{in } \Omega, \end{aligned}$$

where

$$\varphi(x,u) = \begin{cases} -\lambda(x)u & \text{if } x \in \Sigma \\ 0 & \text{if } x \in \Omega \sim \Sigma. \end{cases}$$

As a consequence of Theorem 3.4, if $u(x,t)$ is the solution of problem (III), there exists a constant K such that

$$\lim_{t\to\infty} u(.,t) = K \quad \text{in } L^1(\Omega)$$

and

$$\overline{\inf\{u_0, b\}} \le K \le \inf\{\overline{u_0}, b\},$$

where

$$b := \text{ess inf}_{x\in\Omega} \sup\{r \in \mathbb{R} \ : \ \varphi(x,r) = 0\}.$$

Now, in our case, $b = 0$. Hence $K = 0$ and consequently the population stabilizes as $t \to \infty$ by converging to zero. This shows that the Gurtin-MacCamy model is a good model for populations for which migration to avoid crowding, rather than random motion, is the primary cause of dispersal.

4 Stabilization to zero

In general, by comparison arguments, if the solution of problem (I) stabilizes to zero, then the solution of problem (II) also stabilizes to zero. In this section we show that, although problem (I) does not stabilize to zero, the solutions of problem (II) stabilize to zero when $\varphi(x,r)=0$ a.e. if and only if $r=0$. This fact is a consequence of Theorem 3.3 but here we will obtain it under a weaker assumption on φ.

We need to know the equilibrium points of the semigroup generated by $A_{\beta,\phi}+B_{\varphi}$. For some φ's we have the following result.

Lemma 4.1 Assume φ verifies (a), (b) and

(c) $\varphi(x,r)=0$ a.e. if and only if $r=0$.

Then,

$$(A_{\beta,\phi}+B_{\varphi})^{-1}0=\{0\}.$$

i.e., 0 is the only equlibrium point of $\big(S(t)\big)_{t\geq 0}$.

Proof Obviously, $(0,0)\in A_{\beta,\phi}+B_{\varphi}$. Let $(u,0)\in A_{\beta,\phi}+B_{\varphi}$, then $0=v_1+v_2$ with $(u,v_1)\in A_{\beta,\phi}$ and $(u,v_1)\in B_{\varphi}$, so $v_2=B_{\varphi}u$ and consequently $v_1=-B_{\varphi}u$. Hence, by assumption on φ we have

$$C:=\{x\in\Omega\ :v_1\ sign\ u\geq 0\}=\{x\in\Omega\ :u=0\}\ \text{ a.e.}$$

Thus given $\epsilon>0$, $|u|\leq\epsilon$ a.e. on C, from where it follows by (H') that $|u|\leq\epsilon$ a.e. on Ω. Therefore, $u=0$ a.e. on Ω. □

Remark 4.2 Observe that in the above lemma we can change $A_{\beta,\phi}$ by an m-T-accretive operator A in $L^1(\Omega)$ verifying (H') and such that $(0,0)\in A$.

In the above lemma the assumption (c) is necessary. In fact: Take $\Omega=]0,1[$, $r>0$ and $\varphi:\Omega\times\mathbb{R}\to\mathbb{R}$ defined by

$$\varphi(x,r):=\begin{cases} r-x, & \text{if } r>x\\ 0, & \text{if } r\leq x.\end{cases}$$

Then φ satisfies (a) and (b). However, if $u(x)=x$ for all $x\in\Omega$, $u\in(A+B_{\varphi})^{-1}0$, being $A=-\Delta$ in $L^1(\Omega)$.

Now we come to the main result of this section.

Theorem 4.3 Let $\varphi:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfying (a), (b) and (c). Let β be a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0\in\beta(0)$ and $\phi:\mathbb{R}\to\mathbb{R}$ a continuous increasing function with $\phi(0)=0$. Let $u_0\in L^1(\Omega)$. If $u(x,t)$ is the mild-solution of problem

$$u_t=\Delta\phi(u)-\varphi(x,u)\quad\text{in }\ \Omega\times(0,\infty)$$
$$-\frac{\partial\phi(u)}{\partial\eta}\in\beta(u)\quad\text{on }\ \partial\Omega\times(0,\infty)$$
$$u(x,0)=u_0(x)\quad\text{in }\ \Omega,$$

Then,

$$\lim_{t\to\infty}u(.,t)=0\quad\text{in }\ L^1(\Omega).$$

Proof As $L^\infty(\Omega)$ is dense in $L^1(\Omega)$ and the semigroup generated by $A_{\beta,\phi}+B_{\varphi}$ is a contraction semigroup in $L^1(\Omega)$, we can consider $u_0\in L^\infty(\Omega)$. Take $q_1:=-\|u_0\|_\infty$ and $q_2:=\|u_0\|_\infty$. Then, for all $\lambda>0$ we have

$$J_\lambda q_2 = \big(I + \lambda(A_{\beta,\phi} + B_\varphi)\big)^{-1} q_2 = (I + \lambda A_{\beta,\phi})^{-1}\big(q_2 - \lambda B_\varphi(J_\lambda q_2)\big) \leq$$

$$\leq (I + \lambda A_{\beta,\phi})^{-1} q_2 \leq \|(I + \lambda A_{\beta,\phi})^{-1} q_2\|_\infty \leq \|q_2\|_\infty = q_2,$$

since $(I + \lambda A_{\beta,\phi})^{-1}$ is order-preserving and (iii) of Theorem 2.4. The same argument shows that $J_\lambda q_1 \geq q_1$. By the Crandall-Liggett Theorem we have $q_1 \leq S(t) q_1$ and $q_2 \geq S(t) q_2$ for all $t \geq 0$. Then it follows from [26, **?**] that $\omega(q_1)$ and $\omega(q_2)$ only contain equilibrium points. Consequently, by the above lemma, we have that $\omega(q_1) = \omega(q_2) = \{0\}$. From here, since $q_1 \leq u_0 \leq q_2$, it follows that $\omega(u_0) = \{0\}$ and the proof concludes. □

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