A PDE approach
to
OPTIMAL MATCHING PROBLEMS

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Plan

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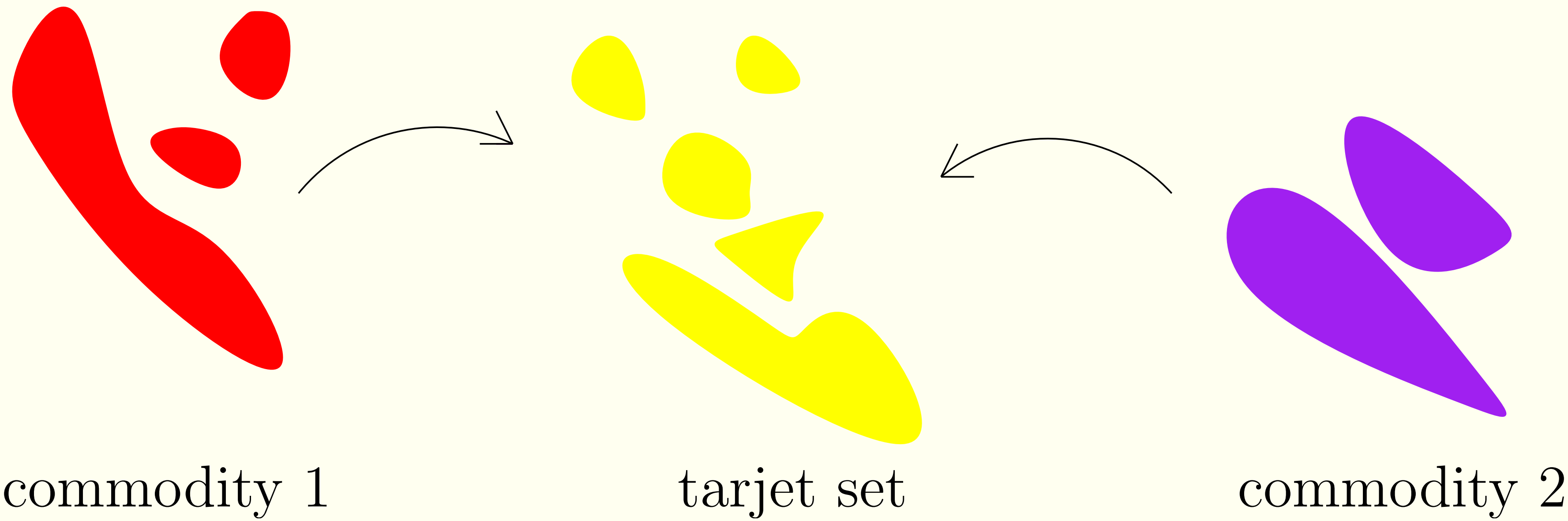
1. Optimal matching problems

We want to transport two commodities (modeled by two measures that encode the spacial distribution of each commodity) to a given location,

the target set,

where they will match, minimizing the total transport cost, given in terms of the Euclidean distance.


commodity 1
tarjet set
commodity 2

In mathematical terms.

We fix two non-negative compactly supported functions $f_1,\ f_2 \in L^\infty$, with supports $X_1,\ X_2$, respectively, satisfying

$$M_0 := \int_{X_1} f_1 = \int_{X_2} f_2 > 0.$$

We also consider a compact set Γ (the target set).

We take a large bounded smooth domain Ω. We assume

$$X_1 \cap \ X_2 = \emptyset,$$

$$\left(X_1 \cup X_2\right) \cap \Gamma = \emptyset,$$

$$\left(X_1 \cup X_2\right) \cup \Gamma \subset\subset \Omega.$$

Given a measure $\mu \in M^+(X)$ (say $X = \Omega$ or $X = \Omega \times \Omega$) and $H : X \to Y$ measurable, we define

$$H\#\mu(E) = \mu(H^{-1}(E)) \quad \text{for Borelian sets } E \subset Y.$$

The Wasserstein distance

for $\mu, \nu \in \mathcal{M}^+(\Omega)$, $\mu(\Omega) = \nu(\Omega)$,

$$W_1(\mu, \nu) := \inf_{\substack{\gamma \in \mathcal{M}^+(\Omega \times \Omega) \\ \pi_x\#\gamma = \mu \\ \pi_y\#\gamma = \nu}} \int_{\Omega\times\Omega} |x - y| d\gamma(x, y).$$

gives the optimal cost of transporting μ to ν.

Kantorovich Theorem

$$W_1(\mu_0, \mu_1) = \sup \left\{ \int_\Omega u \, d(\mu_0 - \mu_1) \ \ : \ u \in K_1 \right\},$$

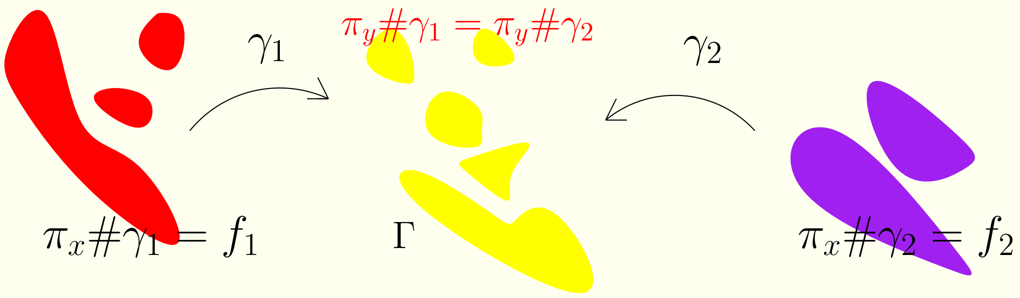
where

$$K_1 := \{u : \Omega \to \mathbb{R} \ : \ |u(x) - u(y)| \leq |x - y| \ \ \forall \, x, y \in \Omega\}.$$

A maximizer is called a Kantorovich potential.

The *optimal matching problem* consist in solving

$$\min_{\substack{(\gamma_1,\gamma_2)\in\mathcal{M}^+(\Omega\times\Omega)^2 \\ \pi_x\#\gamma_i = f_i \\ \pi_y\#\gamma_1 = \pi_y\#\gamma_2 \\ \mathrm{supp}(\pi_y\#\gamma_i)\subset\Gamma}} \left\{ \int_{\Omega\times\Omega} |x-y|\,d\gamma_1(x,y) + \int_{\Omega\times\Omega} |x-y|\,d\gamma_2(x,y) \right\}. \tag{1.1}$$



If (γ_1,γ_2) is a minimizer of (1.1), we shall call an *optimal matching measure to:*

$$\rho = \pi_y\#\gamma_1 = \pi_y\#\gamma_2.$$

Consider

$$\mathcal{M}(\Gamma, M_0) := \{\mu \in \mathcal{M}^+(\Omega) \, : \, \mathbf{supp}(\mu) \subset \Gamma, \; \mu(\Omega) = M_0\}$$

the set of all possible matching measures, then

$$\min_{\cdots} \left\{ \int_{\Omega\times\Omega} |x-y| d\gamma_1(x,y) + \int_{\Omega\times\Omega} |x-y| d\gamma_2(x,y) \right\}$$

$$= \inf_{\mu \in \mathcal{M}(\Gamma, M_0)} \left\{ W_1(f_1, \mu) + W_1(f_2, \mu) \right\} =: W^{\Gamma}_{f1, f_2} \, .$$

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There exist *optimal matching measures supported on the boundary of the target set.*

Evans and Gangbo approach

For $\mu_0 = f_+\mathcal{L}^N$ and $\mu_1 = f_-\mathcal{L}^N$, $f_+, f_- \in L^1(\Omega)$ smooth, Evans and Gangbo find a Kantorovich potential as a limit, as $p \to +\infty$, of solutions to a $p-$Laplace equation with Dirichlet boundary conditions in a large ball,

$$\begin{cases} -\Delta_p u_p = f_+ - f_- & \text{in } \ B(0,R), \\ u_p = 0 & \text{on } \ \partial B(0,R) \end{cases}$$

They prove:

u_p converge uniformly to $u^* \in K_1$ as $p \to +\infty$;

u^* is a Kantorovich potential;

and there exists $0 \leq a \in L^\infty(\Omega)$ (the transport density) such that

$$f_+ - f_- = -\mathrm{div}(a\nabla u^*) \quad \text{in} \;\; \mathcal{D}'(\Omega).$$

Furthermore $|\nabla u^*| = 1$ a.e. in the set $\{a > 0\}$.

We give a p-Laplacian approach (following E-G) of the OMP.

We get a matching measure
(that also encodes the location for the matching) and
a pair of Kantorovich potentials

by

taking limit as $p \to \infty$ in a variational $p-$Laplacian type system, which is nontrivially coupled.

For $p > N$ consider the variational problem

$$\min_{\substack{(v,w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v+w \geq 0 \text{ in } \Gamma}} \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p + \int_\Omega v f_1 + \int_\Omega w f_2. \tag{1.2}$$

Theorem 1.1 (MRT).

1. There exists a minimizer (v_p, w_p) of (1.2).

2. There exists a positive Radon measure h_p of mass M_0, supported on $\{x \in \Gamma : v_p(x) + w_p(x) = 0\}$, such that

$$\begin{cases} -\mathbf{div}\left(|\nabla v_p(x)|^{p-1}\nabla v_p(x)\right) = h_p - f_1 & \text{in } \Omega, \\ |\nabla v_p(x)|^{p-1}\nabla v_p(x) \cdot \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

and

$$\begin{cases} -\mathbf{div}\left(|\nabla w_p(x)|^{p-1}\nabla w_p(x)\right) = h_p - f_2 & \text{in } \Omega, \\ |\nabla w_p(x)|^{p-1}\nabla w_p(x) \cdot \eta = 0 & \text{on } \partial\Omega, \end{cases}$$

where η is the exterior normal vector on $\partial\Omega$.

Theorem 1.2 (MRT). There exists a subsequence $p_i \to +\infty$:

1. $\lim_{i\to\infty} v_{p_i} = v_\infty$ and $\lim_{i\to\infty} w_{p_i} = w_\infty$ uniformly, where (v_∞, w_∞) is a solution of the variational problem

$$\max_{\substack{v,w \in W^{1,\infty}(\Omega) \\ |\nabla v(x)|, |\nabla w(x)| \le 1 \text{ a.e.} \\ v+w \ge 0 \text{ in } \Gamma}} -\int_\Omega v f_1 - \int_\Omega w f_2 =: KW^{\Gamma}_{f_1,f_2}.$$

Observe:

$$-\int v f_1 - \int w f_2 = -\int v d\pi_x \# \gamma_1 - \int w \pi_x \# \gamma_2$$

$$= -\iint v(x) d\gamma_1(x,y) - \iint w(x) d\gamma_2(x,y)$$

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$$\le \iint (v(y) - v(x))d\gamma_1(x,y) + \iint (w(y) - w(x))d\gamma_2(x,y)$$

$$\le \iint |x-y|d\gamma_1(x,y) + \iint |x-y|d\gamma_2(x,y)$$

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3. $\lim_{i\to\infty} h_{p_i} = \rho$ weakly* as measures, with ρ a positive Radon measure of mass M_0 supported on

$$\{x \in \Gamma : v_\infty(x) + w_\infty(x) = 0\}.$$

4. ρ is an **optimal matching measure** for the matching problem.

5. We also have:
v_∞ is a Kantorovich potential for the transport of f_1 to ρ and
w_∞ is a Kantorovich potential for the transport of f_2 to ρ;

2. Optimal matching with constrains

Consider a more realistic case:

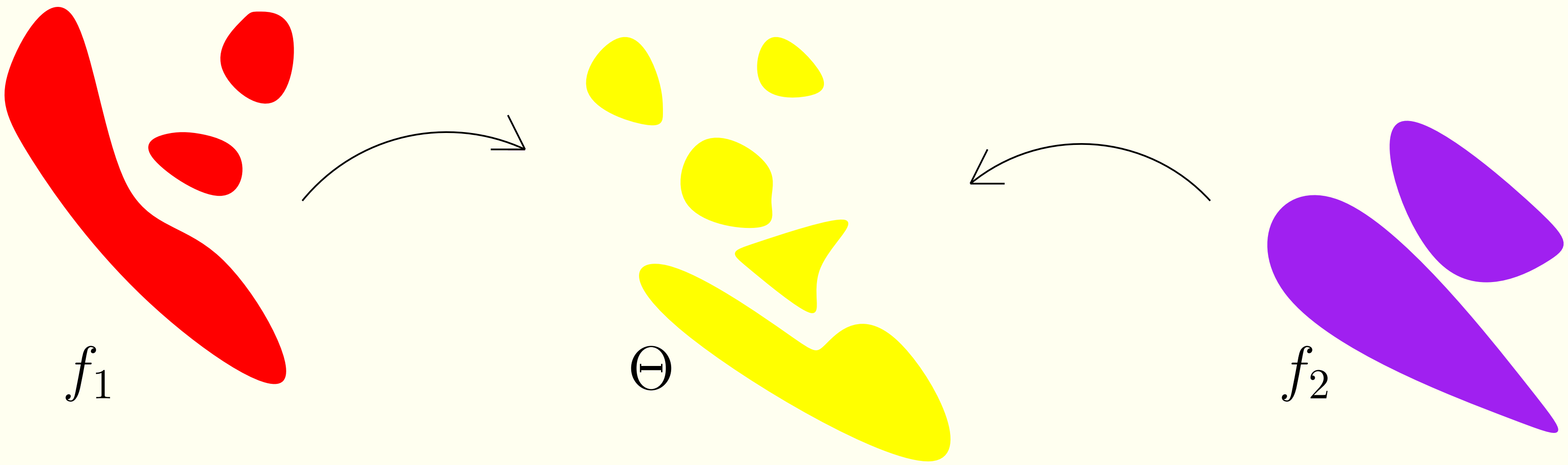
there are some constraints on the amount of material we can transport to points in the target.

This amount is represented with a nonnegative Radon measure Θ in Ω (with support Γ).

The restriction says:

for any set $E \subset \Omega$, the amount of material matched there does not exceeds $\int_E d\Theta$,

We need the condition $\int_\Omega d\Theta > M_0$.


f_1
Θ
f_2

Our aim now is to study

$$\inf_{\mu\in\mathcal{M}(\Theta,M_0)} \Big\{ W_1(f_1,\mu) + W_1(f_2,\mu) \Big\} =: W^{\Theta}_{f_1,f_2},$$

where

$$\mathcal{M}(\Theta, M_0) := \{\mu \in \mathcal{M}^+(\Omega) \;:\; \mu(\Omega) = M_0,\; \mu \le \Theta\}$$

is now the set of all possible optimal matching measures.
Ω will be a large convex domain.

Theorem 2.1 (MRT).

$$W^{\Theta}_{f_1,f_2} = \max_{\substack{v,w \in W^{1,\infty}(\Omega)\\ |\nabla v|(x), |\nabla w|(x) \le 1 \text{ a.e.}}} \left\{ -\int_\Omega v f_1 - \int_\Omega w f_2 - \int_\Omega (v+w)^- d\Theta \right\}. \tag{2.3}$$

We can obtain a pair of maximizers by taking limits as p goes to $+\infty$ of a pair of minimizers (v_p, w_p) of

$$\min_{v,w \in W^{1,p}(\Omega)} \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p + \int_\Omega v f_1 + \int_\Omega w f_2 + \int_\Omega (v+w)^- d\Theta, \tag{2.4}$$

and, also, a matching measure.

Theorem 2.2 (MRT).
Let (v_p, w_p) be minimizer functions of (2.4).

Set $\mathcal{V}_p := |\nabla v_p|^{p-2}\nabla v_p$ and $\mathcal{W}_p := |\nabla w_p|^{p-2}\nabla w_p$.

Then:

1. The distributions $\mathcal{V}_p^\eta$, $\mathcal{W}_p^\eta$ in $\mathbb{R}^N$ given by

$$\langle \mathcal{V}_p^\eta, \varphi \rangle := \int_\Omega \mathcal{V}_p \cdot \nabla\varphi + \int_\Omega f_1\,\varphi \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^N),$$
$$\langle \mathcal{W}_p^\eta, \varphi \rangle := \int_\Omega \mathcal{W}_p \cdot \nabla\varphi + \int_\Omega f_2\,\varphi \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^N).$$

are equal and are positive Radon measures supported on $\{v_p + w_p \leq 0\} \cap \Gamma$.

Formally:

$$\begin{cases} -\mathbf{div}\left(|\nabla v_p(x)|^{p-1}\nabla v_p(x)\right) = \mathcal{V}_p^\eta - f_1\,, \\ \\ -\mathbf{div}\left(|\nabla w_p(x)|^{p-1}\nabla w_p(x)\right) = \mathcal{V}_p^\eta - f_2\,. \end{cases}$$

$\mathcal{V}_p^\eta$ is a positive Radon measure supported on

$$\{v_p + w_p \leq 0\} \cap \Gamma.$$

2. There exist Radon measures $\mathcal{V}$, $\mathcal{W}$ in Ω and ρ in Γ, and a sequence $p_i \to +\infty$, such that

$$(v_{p_i}, w_{p_i}) \to (v_\infty, w_\infty) \quad \text{uniformly in } \Omega,$$

$$\mathcal{V}_{p_i} \to \mathcal{V} \quad \text{weakly* in the sense of measures in } \Omega,$$

$$\mathcal{W}_{p_i} \to \mathcal{W} \quad \text{weakly* in the sense of measures in } \Omega,$$

$$\mathcal{V}^\eta_{p_i} \to \rho \quad \text{weakly* in the sense of measures in } \Gamma,$$

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$$\mathcal{W}_{p_i} \to \mathcal{W} \quad \text{weakly* in the sense of measures in } \Omega,$$

$$\mathcal{V}_{p_i}^{\eta} \to \rho \quad \text{weakly* in the sense of measures in } \Gamma,$$

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Moreover,

$$0 \le \rho \le \Theta \llcorner \{v_\infty + w_\infty \le 0\},$$

and

$$\int_\Omega (v_\infty + w_\infty)^- d\rho = \int_\Omega (v_\infty + w_\infty)^- d\Theta.$$

From the above result we can infer that

$$-\mathrm{div}(\mathcal{V}) = \rho - f_1\,,$$

$$-\mathrm{div}(\mathcal{W}) = \rho - f_2\,,$$

and

$$\int_\Omega \nabla v_\infty \, d\mathcal{V} + \int_\Omega \nabla w_\infty \, d\mathcal{W}$$

$$= -\int_\Omega f_1\, v_\infty - \int_\Omega f_2\, w_\infty + \int_\Gamma (w_\infty + v_\infty) d\rho.$$

Theorem 2.3 (INT).

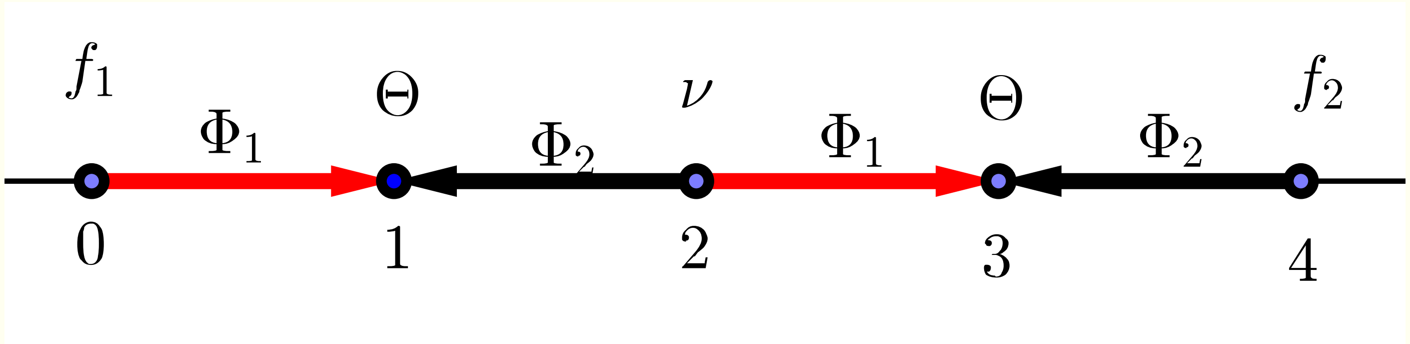
$$W_{f_1,f_2}^{\Theta} = \min_{\substack{\Phi_1, \Phi_2 \in \mathcal{M}_b(\overline{\Omega})^N \\ \nu \in \mathcal{M}_b^+(\overline{\Omega}) \\ -\nabla \cdot \Phi_i = \Theta - \nu - f_i}} \left\{ |\Phi_1|(\overline{\Omega}) + |\Phi_2|(\overline{\Omega}) \right\},$$

Let us call this minimizing problem as

minimal matching flow problem (MMF).

Theorem 2.4 (INT).
Let (Φ_1, Φ_2, ν) be an optimal solution for (MMF).
If $\nu \leq \Theta$ then $\Theta - \nu$ is an optimal matching measure and Φ_i is an optimal flow for transporting f_i onto ρ, $i = 1, 2$.

Then, the connection between both approaches lies in the condition $\nu \leq \Theta$ for an optimal solution (Φ_1, Φ_2, ν) of (MMF). Unfortunately, this does not hold in general.



However, consider the assumption

$$S(f_1, f_2) \cap \mathbf{supp}(\Theta) = \emptyset, \tag{H}$$

where $S(f_1, f_2) := \{[x, y] : x \in \mathbf{supp}(f_1),\ y \in \mathbf{supp}(f_2)\}$.

Theorem 2.5 (INT).
Let f_1, f_2, Θ be such that (H) holds.
Let (Φ_1, Φ_2, ν) be an optimal solution for (MMF).
Then $\Theta - \nu \geq 0$ and it is an optimal matching measure.

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Let f_1, f_2, Θ be such that (H) holds.
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Then $\Theta - \nu \geq 0$ and it is an optimal matching measure.

Theorem 2.6 (INT).
Let f_1, f_2, Θ be such that (H) holds, and $\Theta \in L^1$.
Then there exists a unique optimal matching measure:

$$\Theta \llcorner [u_1 + u_2 < 0],$$

for any maximizer (u_1, u_2) of the dual problem

To solve numerically the above problem:

For any $u=(u_1,u_2)\in V:=C^1(\overline{\Omega})\times C^1(\overline{\Omega})$, set

$$\mathcal{F}(u):=\int u_1 \mathrm{d}f_1+\int u_2 \mathrm{d}f_2-\int (u_1+u_2)\mathrm{d}\Theta$$

$$\Lambda(u):=(\nabla u_1,\nabla u_2,u_1+u_2),$$

and, for any $(p,q,s)\in Z:=C(\overline{\Omega})^N\times C(\overline{\Omega})^N\times C(\overline{\Omega})$, set

$$\mathcal{G}(p,q,s):=\begin{cases}0 & \text{if } |p(x)|\le 1, |q(x)|\le 1, s(x)\le 0 \ \ \forall x\in\overline{\Omega}\\ +\infty & \text{otherwise.}\end{cases}$$

Fenchel–Rockafellar:

$$\inf_{u\in V}\mathcal{F}(u)+\mathcal{G}(\Lambda u)=\sup_{\sigma\in Z^*}\left(-\mathcal{F}^*(-\Lambda^*\sigma)-\mathcal{G}^*(\sigma)\right). \tag{2.5}$$

Consider a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$.

For an integer $k \geq 1$, consider P_k the set of polynomials of degree less or equal than k.

Take $E_h \subset H^1(\Omega)$, the space of continuous functions on $\overline{\Omega}$ and belonging to P_k on each triangle of $\mathcal{T}_h$.

Denote by Y_h the space of vectorial functions such that their restrictions belong to $(P_{k-1})^N$ on each triangle of $\mathcal{T}_h$.

Set $V_h := E_h \times E_h$ and $Z_h := Y_h \times Y_h \times E_h$.

Let $f_{1,h}, f_{2,h}, \Theta_h \in E_h$ **be such that**

$f_{1,h}(\Omega) = f_{2,h}(\Omega) < \Theta_h(\Omega)$,

$f_{1,h} \rightharpoonup f_1$, **weakly* in** $\mathcal{M}_b(\overline{\Omega})$,

$f_{2,h} \rightharpoonup f_2$, **weakly* in** $\mathcal{M}_b(\overline{\Omega})$,

$\Theta_h \rightharpoonup \Theta$ **weakly* in** $\mathcal{M}_b(\overline{\Omega})$.

For any $(u_1, u_2) \in V_h$, set

$$\Lambda_h(u_1, u_2) := (\nabla u_1, \nabla u_2, u_1 + u_2) \in Z_h,$$

$$\mathcal{F}_h(u_1, u_2) := \langle u_1, f_{1,h} \rangle + \langle u_2, f_{2,h} \rangle - \langle u_1 + u_2, \Theta_h \rangle,$$

and for any $(p, q, s) \in Z_h$,

$$\mathcal{G}_h(p, q, s) := \begin{cases} 0 & \text{if } |p(x)| \le 1,\ |q(x)| \le 1,\ s(x) \le 0 \text{ a.e. } x \in \Omega, \\ +\infty & \text{otherwise}. \end{cases}$$

Theorem 2.7 (INT). Let $(u_{1,h}, u_{2,h}) \in V_h$ be an optimal solution to the finite-dimensional approximation problem

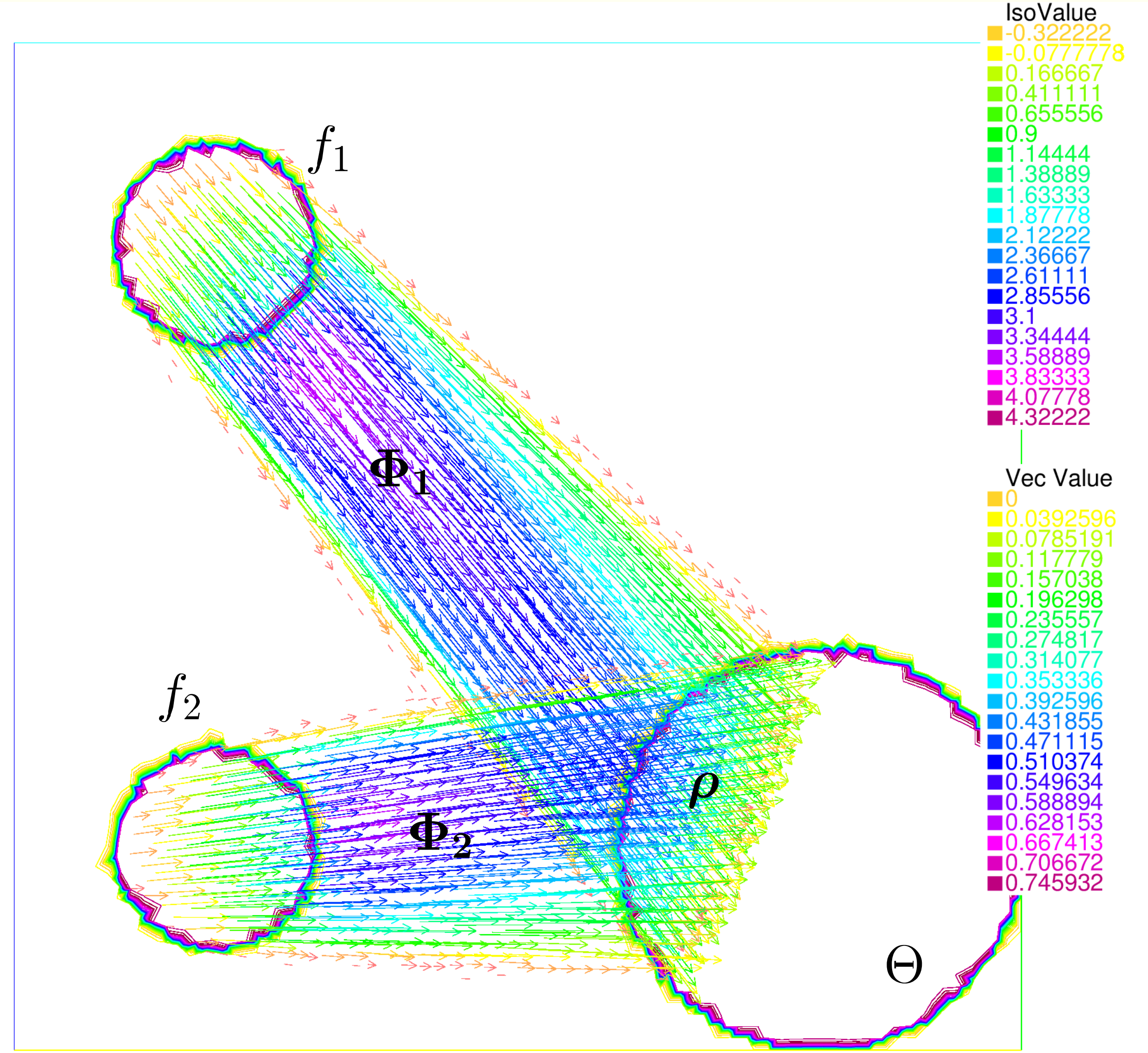
$$\inf_{(u_1,u_2)\in V_h} \mathcal{F}_h(u_1, u_2) + \mathcal{G}_h(\Lambda_h(u_1, u_2)). \tag{2.6}$$

such that $\displaystyle\int_\Omega u_{1,h} = \int_\Omega u_{2,h}$,

and let $(\Phi_{1,h}, \Phi_{2,h}, \nu_h)$ be an optimal dual solution to (2.6).
Then, up to a subsequence,
$(u_{1,h}, u_{2,h})$ converges uniformly to (u_1^*, u_2^*) an optimal solution of the dual maximization problem,
and $(\Phi_{1,h}, \Phi_{2,h}, \nu_h)$ converges weakly* to (Φ_1, Φ_2, ν) an optimal solution of (MMF).

We solve the finite-dimensional problem (2.6) by using the ALG2 method.

$f_1 = 4\chi_{[(x-0.2)^2+(y-0.8)^2<0.01]}$, $f_2 = 4\chi_{[(x-0.2)^2+(y-0.2)^2<0.01]}$,
$\Theta = 4\chi_{[(x-0.8)^2+(y-0.2)^2<0.04]}$.



$$f_1 = 4\chi_{[(x-0.1)^2+(y-0.9)^2<0.01]},\ f_2 = 4\chi_{[(x-0.7)^2+(y-0.3)^2<0.01]},$$
$$\Theta = 4\chi_{[(x-0.2)^2+(y-0.2)^2<0.04]} + 4\chi_{[(x-0.6)^2+(y-0.6)^2<0.0064]}.$$