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On the notions of solution to nonlinear elliptic problems: results and developments

edited by Angelo Alvino, Anna Mercaldo,
François Murat, and Ireneo Peral

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1. Introduction

The purpose of this survey is to present some recent results given by the authors about existence and uniqueness of solutions for a degenerate elliptic problem with nonlinear boundary condition of the form

$$(S^{\gamma,\beta}_{\mu_1,\mu_2}) \quad \begin{cases} -\mathrm{div}\ \mathbf{a}(x,Du) + \gamma(u) \ni \mu_1 & \text{in } \Omega \\ \\ \mathbf{a}(x,Du)\cdot\eta + \beta(u) \ni \mu_2 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, the function $\mathbf{a} : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray-Lions conditions, η is the unit outward normal on $\partial\Omega$, $\mu_1 = \mu_1 \llcorner \partial\Omega$, $\mu_2 = \mu_2 \llcorner \Omega$ are measures and γ and β are maximal monotone graphs in $\mathbb{R}^2$ (see, e.g., [23]), $0 \in \gamma(0) \cap \beta(0)$. General nonlinear diffusion operators of Leray-Lions type, different from the Laplacian, appear when one deals with non-Newtonian fluids (see, e.g., [7]).

The nonlinearities γ and β satisfy rather general assumptions. In particular, they may be multivalued and this allows to include the Dirichlet condition (taking β to be the monotone graph D defined by $D(0) = \mathbb{R}$) and the non homogeneous Neumann boundary condition (taking β to be the monotone graph N defined by $N(r) = 0$ for all $r \in \mathbb{R}$) as well as many other nonlinear fluxes on the boundary that occur in some problems in Mechanic and Physics (see, e.g., [34] or [22]). For instance, in the Signorini problem (see, e.g., [36], [37], [28]) which appears in elasticity and corresponds to the monotone graph

$$\beta(r) = \begin{cases} \emptyset & \text{if } r < 0 \\ ]-\infty, 0] & \text{if } r = 0 \\ 0 & \text{if } r > 0, \end{cases}$$

in problems of optimal control of temperature and in the modelling of semipermeability (see [34]), which corresponds in some cases to the monotone graph

$$\beta(r) = \begin{cases} \emptyset & \text{if } r < a \\ ]-\infty, 0] & \text{if } r = a \\ 0 & \text{if } r \in]a,b[\\ [0,+\infty[& \text{if } r = b \\ \emptyset & \text{if } r > b, \end{cases}$$

where $a < 0 < b$.

Note also that, since γ may be multivalued, problems of type $(S^{\gamma,\beta}_{\mu_1,\mu_2})$ appears in various phenomena with changes of state like multiphase Stefan problem (cf. [30]) and in the weak formulation of the mathematical model of the so called Hele Shaw problem (cf. [32] and [35]). In the case in which $D(\gamma) \neq \mathbb{R}$ we are dealing with obstacle problems, also called unilateral problems in the literature. Obstacle problems appear in different physical context, for instance, in deformation of membrane constrained by an obstacle, in bending of elastic isotropic homogeneous plat over an obstacle and in cavitation problems in hydrodynamic lubrication. Notice also that some free boundary problems fall into this scope by using Baiocchi transformation (see [8]), for more details concerning physical applications we refer to [44] or [34].

In the particular case $\mathbf{a}(x,\xi) = \xi$, the problem $(S^{\gamma,\beta}_{\mu_1,\mu_2})$ reads

$$(L^{\gamma,\beta}_{\mu_1,\mu_2}) \quad \begin{cases} -\Delta u + \gamma(u) \ni \mu_1 & \text{in } \Omega \\ \\ \partial_\eta u + \beta(u) \ni \mu_2 & \text{on } \partial\Omega, \end{cases}$$

where $\partial_\eta u$ simply denotes the outward normal derivative of u. For this kind of problems in the homogeneous case, $\mu_2 \equiv 0$, the pioneering works are the paper by H. Brezis ([22]), in which problem $(L^{\gamma,\beta}_{\mu_1,0})$ is studied for γ the identity, β a maximal monotone graph and $\mu_1 \in L^2(\Omega)$, and the paper by H. Brezis and W.Strauss ([27]), in which problem $(L^{\gamma,\beta}_{\mu_1,0})$ is studied for $\mu_1 \in L^1(\Omega)$ and γ, β continuous nondecreasing functions from $\mathbb{R}$ into $\mathbb{R}$ with $\gamma' \geq \epsilon > 0$. These works were extended by Ph. Bénilan, M. G. Crandall and P. Sacks in [17] where they study problem $(S^{\gamma,\beta}_{\mu_1,0})$ for any γ and β maximal monotone graphs in $\mathbb{R}^2$ such that $0 \in \gamma(0)$ and $0 \in \beta(0)$, and prove, between other results, that, for any $\mu_1 \in L^1(\Omega)$ satisfying the range condition

$$\inf\{\mathrm{Ran}(\gamma)\}\mathrm{meas}(\Omega) + \inf\{\mathrm{Ran}(\beta)\}\mathrm{meas}(\partial\Omega) < \int_\Omega \mu_1$$
$$< \sup\{\mathrm{Ran}(\gamma)\}\mathrm{meas}(\Omega) + \sup\{\mathrm{Ran}(\beta)\}\mathrm{meas}(\partial\Omega),$$

there exists a unique, up to a constant for u, named weak solution, $[u,z,w] \in W^{1,1}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$, $z(x) \in \gamma(u(x))$ *a.e.* in Ω, $w(x) \in \beta(u(x))$ *a.e.* in

$\partial\Omega$, such that

$$\int_\Omega Du \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_\Omega \mu_1 v,$$

for all $v \in W^{1,\infty}(\Omega)$.

For p-Laplacian type equations, an important work in the L^1-theory is [12], where problem

$$(D_\phi^\gamma) \quad \begin{cases} -\text{div } \mathbf{a}(x, Du) + \gamma(u) \ni \mu_1 & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is studied for any γ maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \gamma(0)$. It is proved that, for any $\mu_1 \in L^1(\Omega)$, there exists a unique, named entropy solution, $[u, z] \in \mathcal{T}_0^{1,p}(\Omega) \times L^1(\Omega)$, $z(x) \in \gamma(u(x))$ *a.e.* in Ω, such that

$$\int_\Omega \mathbf{a}(., Du) \cdot DT_k(u - v) + \int_\Omega zT_k(u - v) \leq \int_\Omega \mu_1 T_k(u - v) \quad \forall k > 0,$$

for all $v \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ (see Section 2 for the definition of $\mathcal{T}_0^{1,p}(\Omega)$).

In [2], [4] and [5] the results of [17] and [12] are extended by proving the existence and uniqueness of weak, entropy, renormalized or generalized weak solutions for the general non homogeneous problem $(S_{\mu_1,\mu_2}^{\gamma,\beta})$ depending on the data μ_1, μ_2. The arguments of the proofs are very connected to the nature of the nonlinearities γ and β. Grosso modo the following cases are studied,

(A) $D(\gamma) = \mathbb{R}$, either $D(\beta) = \mathbb{R}$ or $\text{div}\, \mathbf{a}(x, Du) = \Delta_p(u)$, and μ_1, μ_2 integrable data;
$\mu_2 \equiv 0$, either $D(\beta) = \mathbb{R}$ or $\text{div}\, \mathbf{a}(x, Du) = \Delta_p(u)$, and μ_1 integrable data (no conditions on γ);

(B) $\mathbb{R} \neq \overline{D(\gamma)} \subset D(\beta)$ and μ_1, μ_2 integrable data (an obstacle problem);

(C) $D(\gamma) = D(\beta) = \mathbb{R}$ and $\mu_1 + \mu_2$ a diffuse measure, that is, it does not charge sets of zero $p-$capacity.

The main interest in this study is that we are dealing with general nonlinear operators $-\text{div } \mathbf{a}(x, Du)$ with nonhomogeneous boundary conditions, which is quite different from the homogeneous case $\mu_2 = 0$, and general nonlinearities γ and β. As in [17], a range condition relating the average of μ_1 and μ_2 to the range of β and γ is necessary for existence of weak solution and entropy solution (see Remarks 3.1 and 5.2).

However, in contrast to the smooth homogeneous case $\mu_2 = 0$, even for $\mathbf{a}$ corresponding to the Laplacian, for the nonhomogeneous case this range condition is not sufficient for the existence of weak solution. The intersection of the domains of β and γ creates an obstruction phenomena for the existence of these solutions. Even if $D(\beta) = \mathbb{R}$ it does not exist weak solution as the following example shows. Let γ be such that $D(\gamma) = [0,1]$, $\beta = \mathbb{R} \times \{0\}$, and let $\mu_1 \in L^1(\Omega)$, $\mu_2 \leq 0$ *a.e.* in Ω, and $\mu_2 \in L^1(\partial\Omega)$, $\mu_2 \leq 0$ *a.e.* in $\partial\Omega$. If there exists $[u,z,w]$ weak solution of problem $(L^{\gamma,0}_{\mu_1,\mu_2})$ (see Definition 3.1), then $z \in \gamma(u)$, therefore $0 \leq u \leq 1$ *a.e.* in Ω, $w = 0$, and it holds that for any $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \mathbf{a}(x, Du) Dv + \int_\Omega zv = \int_\Omega \mu_1 v + \int_{\partial\Omega} \mu_2 v.$$

Taking $v = u$, as $u \geq 0$, we get

$$0 \leq \int_\Omega \mathbf{a}(x, Du) Du + \int_\Omega zu = \int_\Omega \mu_1 u + \int_{\partial\Omega} \mu_2 u \leq 0.$$

Therefore, we obtain that $\int_\Omega |Du|^p = 0$, so u is constant and

$$\int_\Omega zv = \int_\Omega \mu_2 v + \int_{\partial\Omega} \mu_1 v,$$

for any $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, and in particular, for any $v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$. Consequently, $\mu_1 = z$ *a.e.* in Ω, and μ_2 must be 0 *a.e.* in $\partial\Omega$.

In general, for obstacle problems the existence of weak solution, in the usual sense, fails to be true for nonhomogeneous boundary conditions, so a new concept of solution has to be introduced.

For the case where the data are Radon measures, the problem is again different. There is a large literature on elliptic problems with measure data, mainly for the homogeneous Dirichlet problem and $\gamma \equiv 0$, that is, for the problem

$$(S^{0,D}_{\mu,0}) \quad \begin{cases} -\mathrm{div}\ \mathbf{a}(x,Du) = \mu & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In the linear case, existence and uniqueness of solutions of $(S^{0,D}_{\mu,0})$ was obtained by G. Stampacchia [45] by duality techniques. In the nonlinear case the first attempt to solve problem $(S^{0,D}_{\mu,0})$ was done by L. Boccardo and T. Gallouët, who proved in [18] and [19] the existence of weak solutions of $(S^{0,D}_{\mu,0})$ under the assumption $p > 2 - \frac{1}{N}$. In the case $1 < p \leq 2 - \frac{1}{N}$, even for the particular case $\mu \in L^1(\Omega)$, the definition of weak solution is not enough in order to get uniqueness. It was necessary to find some extra conditions on the distributional solutions of $(S^{0,D}_{\mu,0})$ in order to ensure both existence and uniqueness. This was done by Ph. Bénilan et alt. for the case of measures in $L^1(\Omega)$, by introducing the concept of *entropy solution* in [12], and by P. L. Lions and F. Murat in an unpublished paper where the concept of *renormalized solution* was introduced. For diffuse measures, that is, for measures in $L^1(\Omega) + W^{-1,p'}(\Omega)$, the problem was solved by L. Boccardo, T. Gallouët and L. Orsina in [20], and for general measures by G. Dal Maso et alt. in [31].

The study of the homogeneous Dirichlet problem for the Laplacian and $\gamma \not\equiv 0$ was initiated by Ph. Bénilan and H. Brezis in 1975 (see [13]) for the particular case $\gamma(r) = g_p(r) := |r|^{p-1}r$. They proved the existence of weak solutions of problem

$$(L^{\gamma,D}_{\mu,0}) \quad \begin{cases} -\Delta u + \gamma(u) = \mu & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for any measure μ if $p < \frac{N}{N-2}$ $(N \geq 2)$, and non existence if $p \geq \frac{N}{N-2}$ $(N \geq 3)$ for $\mu = \delta_a$, with $a \in \Omega$. Problem $(L^{\gamma,D}_{\mu,0})$ was also studied by P. Baras and M. Pierre [9]. Recently it has been studied by H. Brezis, M. Marcus and A. C. Ponce in [25] in the case of a continuous nondecreasing nonlinearity $\gamma : \mathbb{R} \to \mathbb{R}$, $\gamma(0) = 0$ (see also [46], [10] for the particular case $\gamma(r) = e^r - 1$). The same problem has been studied by H. Brezis and A. C. Ponce [26] in the case $\mathrm{Dom}(\gamma) \neq \mathbb{R}$ closed. The case $\mathrm{Dom}(\gamma) \neq \mathbb{R}$ open has been studied by L. Dupaigne, A. C. Ponce and A. Porreta [33].

The study of nonlinear equations involving measures as boundary condition was initiated by A. Gmira and L. Veron [38]. They proved the existence of weak solutions of problem

$$(GV) \quad \begin{cases} -\Delta u + |u|^{q-1}u = 0 & \text{in } \Omega \\ \\ u = \mu & \text{on } \partial\Omega, \end{cases}$$

for any Radon measure μ on $\partial\Omega$ in the subcritical case $1 < q < \frac{N+1}{N-1}$. In the supercritical case, $q \geq \frac{N+1}{N-1}$, this is no longer true; for instance, the problem has no solution if the measure μ is concentrated at a single point. M. Marcus and L. Veron in [42] characterized the Radon measures μ on $\partial\Omega$ for which problem (GV) has solution in the supercritical case, these measures are those that are absolutely continuous respect to the Bessel capacity $C_{\frac{2}{q},q'}$ on $\partial\Omega$. In the last years an extensive study of removable singularities and boundary traces for this type of problems has been done by M. Marcus and L. Veron (see [43] and the references therein).

The study of reduced measures initiated in [25] by H. Brezis, M. Marcus and A. C. Ponce for problem $(L^{\gamma,D}_{\mu,0})$ has been developed in [26] by H. Brezis and A. C. Ponce for problems of the form

$$(BP) \begin{cases} -\Delta u + \gamma(u) = 0 & \text{in } \Omega \\ \\ u = \mu & \text{on } \partial\Omega, \end{cases}$$

where $\gamma : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function with $\gamma(r) = 0$ for all $r \leq 0$. In that paper the authors make the observation that in all the above problems the equation in Ω is nonlinear but the boundary conditions is the usual Dirichlet boundary condition, being also interesting to investigate problems with nonlinear boundary conditions of type

$$(L^{g_1,\beta}_{0,\mu}) \quad \begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \\ \dfrac{\partial u}{\partial \eta} + \beta(u) \ni \mu & \text{on } \partial\Omega, \end{cases}$$

where β is a maximal monotone graph in $\mathbb{R}^2$.

2. Preliminaries

Throughout this article, $\Omega \subset \mathbb{R}$ is a bounded domain with boundary $\partial\Omega$ of class C^1, $p > 1$, γ and β are maximal monotone graphs in $\mathbb{R}^2$ such that $0 \in \gamma(0) \cap \beta(0)$ and $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

(H_1) there exists $\Lambda > 0$ such that $\mathbf{a}(x, \xi) \cdot \xi \geq \Lambda |\xi|^p$ for *a.e.* $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$,

(H_2) there exists $\sigma > 0$ and $\varrho \in L^{p'}(\Omega)$ such that $|\mathbf{a}(x, \xi)| \leq \sigma(\varrho(x) + |\xi|^{p-1})$ for *a.e.* $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $p' = \frac{p}{p-1}$,

(H_3) $(\mathbf{a}(x, \xi_1) - \mathbf{a}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0$ for *a.e.* $x \in \Omega$ and for all $\xi_1, \xi_2 \in \mathbb{R}^N$, $\xi_1 \neq \xi_2$.

The hypotheses $(H_1 - H_3)$ are classical in the study of nonlinear operators in divergence form (cf., [41]). The model example of a function $\mathbf{a}$ satisfying these hypotheses is $\mathbf{a}(x, \xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p-Laplacian operator $\Delta_p(u) = \mathrm{div}(|Du|^{p-2}Du)$.

We denote by $\mathcal{L}^N$ the $N-$dimensional Lebesgue measure of $\mathbb{R}^N$ and by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure.

For $1 \leq p < +\infty$, $L^p(\Omega)$ and $W^{1,p}(\Omega)$ denote respectively the standard Lebesgue space and Sobolev space, and $W_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. For $u \in W^{1,p}(\Omega)$, we denote by u or $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense and by $W^{\frac{1}{p'},p}(\partial\Omega)$ the set $\tau(W^{1,p}(\Omega))$. Recall that $\mathrm{Ker}(\tau) = W_0^{1,p}(\Omega)$.

We write

$$\mathrm{sign}_0(r) := \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0, \end{cases} \qquad \mathrm{sign}_0^+(r) := \begin{cases} 1 & \text{if } r > 0, \\ 0. & \text{if } r \leq 0, \end{cases}$$

and for $k > 0$,

$$T_k(s) = \sup(-k, \inf(s, k)).$$

In [12], the authors introduce the set

$$\mathcal{T}^{1,p}(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable such that } T_k(u) \in W^{1,p}(\Omega) \quad \forall k > 0\}.$$

They also prove that given $u \in \mathcal{T}^{1,p}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$DT_k(u) = v\chi_{\{|v|<k\}} \quad \forall k > 0.$$

This function v will be denoted by Du. It is clear that if $u \in W^{1,p}(\Omega)$, then $v \in L^p(\Omega)$ and $v = Du$ in the usual sense.

As in [6], $\mathcal{T}^{1,p}_{tr}(\Omega)$ denotes the set of functions u in $\mathcal{T}^{1,p}(\Omega)$ satisfying the following conditions, there exists a sequence u_n in $W^{1,p}(\Omega)$ such that

(a) u_n converges to u *a.e.* in Ω,

(b) $DT_k(u_n)$ converges to $DT_k(u)$ in $L^1(\Omega)$ for all $k > 0$,

(c) there exists a measurable function v on $\partial\Omega$, such that u_n converges to v *a.e.* in $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [6]. In the sequel, the trace of $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$ or u. Let us recall that in the case where $u \in W^{1,p}(\Omega)$, $tr(u)$ coincides with the trace of u, $\tau(u)$, in the usual sense, and the space $\mathcal{T}^{1,p}_0(\Omega)$, introduced in [12] to study (D^γ_ϕ), is equal to $\mathrm{Ker}(tr)$. Moreover, for every $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ and every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$, and, if $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $u - \phi \in \mathcal{T}^{1,p}_{tr}(\Omega)$ and $tr(u-\phi) = tr(u) - \tau(\phi)$.

We denote

$$V^{1,p}(\Omega) := \Big\{\phi \in L^1(\Omega) : \exists M > 0 \text{ such that}$$
$$\int_\Omega |\phi v| \le M\|v\|_{W^{1,p}(\Omega)} \ \forall v \in W^{1,p}(\Omega)\Big\}$$

and

$$V^{1,p}(\partial\Omega) := \Big\{\psi \in L^1(\partial\Omega) : \exists M > 0 \text{ such that}$$
$$\int_{\partial\Omega} |\psi v| \le M\|v\|_{W^{1,p}(\Omega)} \ \forall v \in W^{1,p}(\Omega)\Big\}.$$

$V^{1,p}(\Omega)$ is a Banach space endowed with the norm

$$\|\phi\|_{V^{1,p}(\Omega)} := \inf\Big\{M > 0 : \int_\Omega |\phi v| \le M\|v\|_{W^{1,p}(\Omega)} \ \forall v \in W^{1,p}(\Omega)\Big\},$$

and $V^{1,p}(\partial\Omega)$ is a Banach space endowed with the norm

$$\|\psi\|_{V^{1,p}(\partial\Omega)} := \inf\Big\{M > 0 : \int_{\partial\Omega} |\psi v| \leq M\|v\|_{W^{1,p}(\Omega)}\ \forall v \in W^{1,p}(\Omega)\Big\}.$$

Observe that, Sobolev embeddings and Trace theorems imply, for $1 \leq p < N$,

$$L^{p'}(\Omega) \subset L^{(Np/(N-p))'}(\Omega) \subset V^{1,p}(\Omega)$$

and

$$L^{p'}(\partial\Omega) \subset L^{((N-1)p/(N-p))'}(\partial\Omega) \subset V^{1,p}(\partial\Omega).$$

Also,

$$V^{1,p}(\Omega) = L^1(\Omega) \ \text{ and } \ V^{1,p}(\partial\Omega) = L^1(\partial\Omega) \ \text{ when } \ p > N,$$

$$L^q(\Omega) \subset V^{1,N}(\Omega) \ \text{ and } \ L^q(\partial\Omega) \subset V^{1,N}(\partial\Omega) \ \text{ for any } \ q > 1.$$

For an open bounded set U of $\mathbb{R}^N$, the p-capacity relative to U, $C_p(.,U)$, is defined in the following way. For any compact subset K of U,

$$C_p(K,U) = \inf\left\{\int_U |Du|^p \ ; \ u \in \mathcal{C}_c^\infty(U), \ u \geq \chi_K\right\},$$

where χ_K is the characteristic function of K; we will use the convention that $\inf\emptyset = +\infty$. The p-capacity of any open subset $O \subset U$ is defined by

$$C_p(O,U) = \sup\left\{C_p(K) \ ; \ K \subset O \text{ compact}\right\}.$$

Finally, the p-capacity of any Borel set $A \subset U$ is defined by

$$C_p(A,U) = \inf\left\{C_p(O) \ ; \ O \subset A \text{ open}\right\}.$$

A function u defined on U is said to be cap$_p$-quasi-continuous in $A \subset U$ if for every $\varepsilon > 0$, there exists an open set $B_\epsilon \subseteq U$ with $C_p(B_\epsilon, U) < \varepsilon$ such that the restriction of u to $A \setminus B_\epsilon$ is continuous. It is well known that every function in $W^{1,p}(U)$ has a cap$_p$-quasi-continuous representative, whose values are defined cap$_p$-quasi everywhere in U, that is, up to a subset of U of zero p-capacity. When we are dealing with the pointwise values of a function $u \in W^{1,p}(U)$, we always identify u with its cap$_p$-quasi-continuous representative.

Let us remark that if $u \in \mathcal{T}^{1,p}(\Omega)$, then u has a cap$_p$-quasi-continuous representative, which will be denoted equally by u; the cap$_p$-quasi-continuous representative can be infinite on a set of positive p-capacity (see [31]). If in addition the function $u \in \mathcal{T}^{1,p}(\Omega)$ is assumed to satisfy the estimate

$$\int_\Omega |DT_k(u)|^p \, dx \leq C(k+1) \qquad \forall\, k > 0,$$

where C is independent of k, then the cap_p-quasi-continuous representative of u is cap_p-quasi every where finite (see [31]).

Since we are considering Ω to be a bounded domain in $\mathbb{R}^N$ with $\partial\Omega$ of class C^1, Ω is an extension domain (see [24]), so we can fix an open bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, and there exists a bounded linear operator $E : W^{1,p}(\Omega) \to W_0^{1,p}(U_\Omega)$ for which

(i) $E(u) = u$ a.e in Ω for each $u \in W^{1,p}(\Omega)$,

(ii) $\|E(u)\|_{W_0^{1,p}(U_\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$, where C is a constant depending only on p and Ω.

We call $E(u)$ an extension of u to U_Ω. If $u \in W^{1,p}(\Omega)$, $1 < p \leq \infty$, it is possible to give a pointwise definition of the trace $\tau(u)$ of u on $\partial\Omega$ in the following way (see [47]), as $E(u) \in W_0^{1,p}(U_\Omega)$, every point of U_Ω, except possibly a set of zero $p-$capacity, is a Lebesgue point of $E(u)$. Since $p > 1$, the sets of zero $p-$capacity are of $\mathcal{H}^{N-1}$-measure zero and therefore $E(u)$ is defined $\mathcal{H}^{N-1}$-almost everywhere on $\partial\Omega$, so $\tau(u) = E(u)$ on $\partial\Omega$. This definition is independent of the open set U_Ω and also of the extension $E(u)$. From now on U_Ω will be a fix open bounded subset of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$. We denote $\tau(u)$ by u in the rest of the paper.

Given $u \in \mathcal{T}^{1,p}(\Omega)$ there exists $\overline{u} \in \mathcal{T}_0^{1,p}(U_\Omega)$ such that

$$T_k(\overline{u}) = E[T_k(u)] \qquad \text{for all } k > 0.$$

For U an open subset of $\mathbb{R}^N$, we set by $\mathcal{M}_b(U)$ the space of all Radon measures in U with bounded total variation. We recall that for a measure $\mu \in \mathcal{M}_b(U)$ and a Borel set $A \subset U$, the measure $\mu \llcorner A$ is defined by $(\mu \llcorner A)(B) = \mu(B \cap A)$ for any Borel set $B \subset U$. If a measure $\mu \in \mathcal{M}_b(U)$ is such that $\mu = \mu \llcorner A$ for a certain Borel set A, the measure μ is said to be concentrated on A. For $\mu \in \mathcal{M}_b(U)$, we denote by μ^+, μ^- and $|\mu|$ the positive part, negative part and the total variation of the measure μ, respectively. By $\mu = \mu_a + \mu_s$ we denote the Radon-Nikodym decomposition of μ relatively to $\mathcal{L}^N$. For simplicity, we write also μ_a for its density respect to $\mathcal{L}^N$, that is, for the function $f \in L^1(U)$ such that $\mu_a = f\mathcal{L}^N \llcorner U$.

We denote by $\mathcal{M}_b^p(U)$ the space of all diffuse Radon measures in U, i.e., measures which do not charge sets of zero $p-$capacity. In [20] it is proved that $\mu \in \mathcal{M}_b(U)$ belongs to $\mathcal{M}_b^p(U)$ if and only if it belongs to $L^1(U) + W^{-1,p'}(U)$, where $W^{-1,p'}(U) = [W_0^{1,p}(U)]^*$. Moreover, if $u \in W^{1,p}(U)$ and $\mu \in \mathcal{M}_b^p(U)$, then u is measurable with respect to μ. If u further belongs to $L^\infty(U)$, then u belongs to $L^\infty(U, d\mu)$, hence to $L^1(U, d\mu)$.

Let ϑ be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. For $r \in \mathbb{N}$, the *Yosida approximation* ϑ_r of ϑ is given by $\vartheta_r = r(I - (I + \frac{1}{r}\vartheta)^{-1})$. The function ϑ_r is maximal monotone and Lipschitz. We recall the definition of the *main section* ϑ^0 of ϑ

$$\vartheta^0(s) := \begin{cases} \text{the element of minimal absolute value of } \vartheta(s) & \text{if } \vartheta(s) \neq \emptyset, \\ \\ +\infty \qquad \text{if } [s, +\infty) \cap \mathrm{Dom}(\vartheta) = \emptyset, \\ \\ -\infty \qquad \text{if } (-\infty, s] \cap \mathrm{Dom}(\vartheta) = \emptyset. \end{cases}$$

We have that $|\vartheta_r|$ is increasing in r, if $s \in \mathrm{Dom}(\vartheta)$, $\vartheta_r(s) \to \vartheta^0(s)$ as $r \to +\infty$, and if $s \notin \mathrm{Dom}(\theta)$, $|\vartheta_r(s)| \to +\infty$ as $r \to +\infty$. If $0 \in \mathrm{Dom}(\vartheta)$, $j_\vartheta(r) = \int_0^r \vartheta^0(s)ds$ defines a convex lower semi-continuous function such that $\vartheta = \partial j_\vartheta$. If j_ϑ^* is the Legendre transformation of j_ϑ then $\vartheta^{-1} = \partial j_\vartheta^*$.

We set

$$\vartheta(r+) := \inf \vartheta(]r, +\infty[), \quad \vartheta(r-) := \sup \vartheta(] - \infty, r[)$$

for $r \in \mathbb{R}$, where we use the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. It is easy to see that

$$\vartheta(r) = [\vartheta(r-), \vartheta(r+)] \cap \mathbb{R} \quad \text{ for } \quad r \in \mathbb{R}.$$

Moreover,

$$J(\vartheta) := \{\theta \in \mathrm{Dom}(\vartheta) : \vartheta(r-) < \vartheta(r+)\}$$

is a countable set.

In [15] the following relation for $u, v \in L^1(\Omega)$ is defined,

$$u \ll v \quad \text{if}$$
$$\int_\Omega (u-k)^+ \le \int_\Omega (v-k)^+ \text{ and } \int_\Omega (u+k)^- \le \int_\Omega (v+k)^- \text{ for any } k>0.$$

We finish this section with the following definition.

Definition 2.1 ([6]). *We say that* $\mathbf{a}$ *is smooth when, for any* $\phi \in L^\infty(\Omega)$ *such that there exists a bounded weak solution* u *of the homogeneous Dirichlet problem*

$$(D) \qquad \begin{cases} -\,div\ \mathbf{a}(x, Du) = \phi & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

there exists $\psi \in L^1(\partial\Omega)$ *such that* u *is also a weak solution of the Neumann problem*

$$(N) \qquad \begin{cases} -\,div\ \mathbf{a}(x, Du) = \phi & in\ \Omega \\ \mathbf{a}(x, Du)\cdot\eta = \psi & on\ \partial\Omega. \end{cases}$$

Functions $\mathbf{a}$ corresponding to linear operators with smooth coefficients and p-Laplacian type operators are smooth (see [22] and [40]). The smoothness of the Laplacian operator is even stronger than this, in fact, there is a bounded linear mapping $T : L^1(\Omega) \to L^1(\partial\Omega)$, such that the weak solution of (D) for $\phi \in L^1(\Omega)$ is also a weak solution of (N) for $\psi = T(\phi)$ (see [17]).

3. Integrable data

In this section we deal with integrable data, so we rewrite $\mu_1 = \phi$ and $\mu_2 = \psi$ in order to denote functions. Let us begin by giving the different concepts of solutions we use.

Definition 3.1. *Let* $\phi \in L^1(\Omega)$ *and* $\psi \in L^1(\partial\Omega)$. *A triple of functions* $[u, z, w] \in W^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ *is a weak solution of problem* $(S^{\gamma,\beta}_{\phi,\psi})$ *if* $z(x) \in \gamma(u(x))$ *a.e. in* Ω, $w(x) \in \beta(u(x))$ *a.e. in* $\partial\Omega$, *and*

$$\int_\Omega \mathbf{a}(x, Du)\cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_{\partial\Omega} \psi v + \int_\Omega \phi v,$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

In general, as it is remarked in [12], for $1 < p \leq 2 - \frac{1}{N}$, there exists $f \in L^1(\Omega)$ such that the problem

$$u \in W^{1,1}_{loc}(\Omega), \ u - \Delta_p(u) = f \quad \text{in} \quad \mathcal{D}'(\Omega),$$

has no solution. In [12], to overcome this difficulty and to get uniqueness, it was introduced a new concept of solution, named entropy solution. Following these ideas, we introduce the following concept of solution.

Definition 3.2. *Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$. A triple of functions $[u, z, w] \in \mathcal{T}^{1,p}_{tr}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is an entropy solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$ and*

$$\begin{aligned} &\int_\Omega \mathbf{a}(x, Du) \cdot DT_k(u - v) + \int_\Omega zT_k(u - v) + \int_{\partial\Omega} wT_k(u - v) \\ &\qquad \leq \int_{\partial\Omega} \psi T_k(u - v) + \int_\Omega \phi T_k(u - v) \quad \forall k > 0, \end{aligned} \tag{3.1}$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

Obviously, every weak solution is an entropy solution and an entropy solution with $u \in W^{1,p}(\Omega)$ is a weak solution.

Remark 3.1 - If we take $v = T_h(u) \pm 1$ as test function in (3.1) and let h go to $+\infty$, we get that

$$\int_\Omega z + \int_{\partial\Omega} w = \int_{\partial\Omega} \psi + \int_\Omega \phi.$$

Then necessarily ϕ and ψ must satisfy

$$\mathcal{R}^-_{\gamma,\beta} \leq \int_{\partial\Omega} \psi + \int_\Omega \phi \leq \mathcal{R}^+_{\gamma,\beta},$$

where

$$\mathcal{R}^+_{\gamma,\beta} := \sup\{\mathrm{Ran}(\gamma)\}\mathrm{meas}(\Omega) + \sup\{\mathrm{Ran}(\beta)\}\mathrm{meas}(\partial\Omega)$$

and

$$\mathcal{R}^-_{\gamma,\beta} := \inf\{\mathrm{Ran}(\gamma)\}\mathrm{meas}(\Omega) + \inf\{\mathrm{Ran}(\beta)\}\mathrm{meas}(\partial\Omega).$$

In general, we will suppose $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$ and write $\mathcal{R}_{\gamma,\beta} :=]\mathcal{R}^-_{\gamma,\beta}, \mathcal{R}^+_{\gamma,\beta}[$.

The following result holds for entropy solutions.

Lemma 3.1. *Let $[u, z, w]$ be an entropy solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$. Then, for all $h > 0$,*

$$\lambda \int_{\{h<|u|<h+k\}} |Du|^p \leq k \int_{\partial\Omega \cap \{|u| \geq h\}} |\psi| + k \int_{\Omega \cap \{|u| \geq h\}} |\phi|.$$

Respect to uniqueness we have the following results.

Theorem 3.1 ([2]). *Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$, and let $[u_1, z_1, w_1]$ and $[u_2, z_2, w_2]$ be entropy solutions of problem $(S^{\gamma,\beta}_{\phi,\psi})$. Then, there exists a constant $c \in \mathbb{R}$ such that*

$$u_1 - u_2 = c \quad a.e. \text{ in } \Omega,$$

$$z_1 - z_2 = 0 \quad a.e. \text{ in } \Omega.$$

$$w_1 - w_2 = 0 \quad a.e. \text{ in } \partial\Omega.$$

Moreover, if $c \neq 0$, there exists a constant $k \in \mathbb{R}$ such that $z_1 = z_2 = k$.

Since every weak solution is an entropy solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$, the same result is true for weak solutions. Nevertheless we have the following general result for weak solutions, in fact, a contraction principle between sub and super weak solutions. As usual, we understand weak sub and supersolution in the following way. A triple of functions $[u, z, w] \in W^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is a *weak subsolution* (resp. *supersolution*) of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if $z(x) \in \gamma(u(x))$ *a.e.* in Ω, $w(x) \in \beta(u(x))$ *a.e.* on $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} w\, v \leq (\text{resp. } \geq) \int_\Omega \phi v + \int_{\partial\Omega} \psi v,$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, $v \geq 0$.

Theorem 3.2 ([4]). *Let $\phi_1, \phi_2 \in L^1(\Omega)$, $\psi_1, \psi_2 \in L^1(\partial\Omega)$. If $[u_1, z_1, w_1]$ is a weak subsolution of $(S^{\gamma,\beta}_{\phi_1,\psi_1})$ and $[u_2, z_2, w_2]$ is a weak supersolution of $(S^{\gamma,\beta}_{\phi_2,\psi_2})$, then*

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \leq \int_\Omega (\phi_1 - \phi_2)^+ + \int_{\partial\Omega} (\psi_1 - \psi_2)^+.$$

Respect to the existence of weak solutions we obtain the following results.

Theorem 3.3 ([2]). *Assume $D(\gamma) = \mathbb{R}$, either $D(\beta) = \mathbb{R}$ or* $\mathbf{a}$ *smooth, and $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$.*
For any $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$ with

$$\int_\Omega \phi + \int_{\partial\Omega} \psi \in \mathcal{R}_{\gamma,\beta},$$

there exists a weak solution $[u, z, w]$ of problem $(S^{\gamma,\beta}_{\phi,\psi})$.

In the special case $\mathcal{R}^-_{\gamma,\beta} = \mathcal{R}^+_{\gamma,\beta}$, that is, when $\gamma(r) = \beta(r) = 0$ for any $r \in \mathbb{R}$, existence and uniqueness of weak solutions are also obtained.

Theorem 3.4 ([2]). *For any $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$ with*

$$\int_\Omega \phi + \int_{\partial\Omega} \psi = 0,$$

there exists a unique (up to a constant) weak solution $u \in W^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}\, \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ \\ \mathbf{a}(x, Du) \cdot \eta = \psi & \text{on } \partial\Omega \end{cases}$$

in the sense that

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv = \int_{\partial\Omega} \psi v + \int_\Omega \phi v,$$

for all $v \in W^{1,p}(\Omega)$.

In the case $\psi = 0$ we have the following result without imposing any condition on γ, in the same line to the one obtained by Bénilan, Crandall and Sack in [17] for the Laplacian operator and L^1-data.

Theorem 3.5 ([2]). *Assume $D(\beta) = \mathbb{R}$ or* $\mathbf{a}$ *smooth, and $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$.*

(i) For any $\phi \in V^{1,p}(\Omega)$ such that $\int_\Omega \phi \in \mathcal{R}_{\gamma,\beta}$, there exists a weak solution $[u, z, w]$ of problem $(S^{\gamma,\beta}_{\phi,0})$, with $z << \phi$.

(ii) For any $[u_1, z_1, w_1]$ weak solution of problem $(S^{\gamma,\beta}_{\phi_1,0})$, $\phi_1 \in V^{1,p}(\Omega)$, $\int_\Omega \phi_1 \in \mathcal{R}_{\gamma,\beta}$, and any $[u_2, z_2, w_2]$ weak solution of problem $(S^{\gamma,\beta}_{\phi_2,0})$, $\phi_2 \in V^{1,p}(\Omega)$, $\int_\Omega \phi_2 \in \mathcal{R}_{\gamma,\beta}$, we have that

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \le \int_\Omega (\phi_1 - \phi_2)^+.$$

For Dirichlet boundary condition we have the following result.

Theorem 3.6 ([2]). *Assume $D(\beta) = \{0\}$. For any $\phi \in V^{1,p}(\Omega)$, there exists a unique $[u, z] = [u_{\phi,\psi}, z_{\phi,\psi}] \in W_0^{1,p}(\Omega) \times V^{1,p}(\Omega)$, $z \in \gamma(u)$ a.e. in Ω, such that*

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv = \int_\Omega \phi v,$$

for all $v \in W_0^{1,p}(\Omega)$.

Moreover, if $\phi_1, \phi_2 \in V^{1,p}(\Omega)$, then

$$\int_\Omega (z_{\phi_1,\psi_1} - z_{\phi_2,\psi_2})^+ \le \int_\Omega (\phi_1 - \phi_2)^+.$$

Let us now state the existence results of entropy solutions for data in L^1.

Theorem 3.7 ([2, 5]). *Assume $D(\gamma) = \mathbb{R}$ and either $D(\beta) = \mathbb{R}$ or* $\mathbf{a}$ *smooth. Then,*

(i) for any $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$, there exists an entropy solution $[u, z, w]$ of problem $(S^{\gamma,\beta}_{\phi,\psi})$.

(ii) For any $[u_1, z_1, w_1]$ entropy solution of problem $(S^{\gamma,\beta}_{\phi_1,\psi_1})$, $\phi_1 \in L^1(\Omega)$, $\psi_1 \in L^1(\partial\Omega)$, and any $[u_2, z_2, w_2]$ entropy solution of problem $(S^{\gamma,\beta}_{\phi_2,\psi_2})$, $\phi_2 \in L^1(\Omega)$, $\psi_2 \in L^1(\partial\Omega)$, we have that

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \le \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+.$$

As a consequence of the previous results we have the following corollary (see [17, Propostion C (iv)] for the Laplacian).

Corollary 3.1. $\mathbf{a}$ *is smooth if and only if for any $\phi \in L^1(\Omega)$ there exists $T(\phi) \in L^1(\partial\Omega)$ such that the entropy solution u of*

$$\begin{cases} -\mathrm{div}\, \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is an entropy solution of

$$\begin{cases} -\mathrm{div}\ \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ \\ \mathbf{a}(x, Du) \cdot \eta = T(\phi) & \text{on } \partial\Omega. \end{cases}$$

Moreover, the map $T : L^1(\Omega) \to L^1(\partial\Omega)$ satisfies

$$\int_\Omega (T(\phi_1) - T(\phi_2))^+ \leq \int_\Omega (\phi_1 - \phi_2)^+,$$

for all $\phi_1, \phi_2 \in L^1(\Omega)$, and $T\left(V^{1,p}(\Omega)\right) \subset V^{1,p}(\partial\Omega)$.

In the homogeneous case, without any condition on γ, we also get the following result.

Theorem 3.8 ([2, 5]). *Assume $D(\beta) = \mathbb{R}$ or $\mathbf{a}$ is smooth. Then,*

(i) for any $\phi \in L^1(\Omega)$, there exists an entropy solution $[u, z, w]$ of problem $(S^{\gamma,\beta}_{\phi,0})$, with $z << \phi$.

(ii) For any $[u_1, z_1, w_1]$ entropy solution of problem $(S^{\gamma,\beta}_{\phi_1,0})$, $\phi_1 \in L^1(\Omega)$, and any $[u_2, z_2, w_2]$ entropy solution of problem $(S^{\gamma,\beta}_{\phi_2,0})$, $\phi_2 \in L^1(\Omega)$, we have that

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \leq \int_\Omega (\phi_1 - \phi_2)^+.$$

In order to obtain the existence results the main idea is to consider the approximate problems

$$(S^{\gamma_{m,n},\beta_{m,n}}_{\phi_{m,n},\psi_{m,n}}) \quad \begin{cases} -\mathrm{div}\ \mathbf{a}(x, Du) + \gamma_{m,n}(u) \ni \phi_{m,n} & \text{in } \Omega \\ \\ \mathbf{a}(x, Du) \cdot \eta + \beta_{m,n}(u) \ni \psi_{m,n} & \text{on } \partial\Omega, \end{cases}$$

where $\gamma_{m,n}$ and $\beta_{m,n}$ are approximations of γ and β given by

$$\gamma_{m,n}(r) = \gamma(r) + \frac{1}{m} r^+ - \frac{1}{n} r^-$$

and

$$\beta_{m,n}(r) = \beta(r) + \frac{1}{m} r^+ - \frac{1}{n} r^-$$

respectively, $m, n \in \mathbb{N}$, and

$$\phi_{m,n} = \sup\{\inf\{m, \phi\}, -n\}$$

and

$$\psi_{m,n} = \sup\{\inf\{m, \psi\}, -n\}$$

are approximations of ϕ and ψ respectively. For these approximate problems we obtain existence of weak solutions with appropriated estimates and monotone properties, which allow us to pass to the limit.

4. An obstacle problem

Let us now suppose that

$$\mathbb{R} \neq \overline{D(\gamma)} \subset D(\beta).$$

Observe that if $D(\gamma)$ is not bounded we are dealing with a one obstacle problem and with a two obstacle problem if $D(\gamma)$ is bounded.

We want to stress, as remarked in the introduction, that for this kind of problems the existence of weak solution, in the usual sense, fails to be true for nonhomogeneous boundary conditions. We introduce a new notion of solution for which existence and uniqueness can be obtained for nonhomogeneous boundary conditions in the case $D(\gamma) \neq \mathbb{R}$.

To give an idea of this new concept of solution, let us consider again the problem $(L^{\gamma,0}_{\phi,\psi})$ treated in the introduction, with $\phi \in L^{p'}(\Omega)$, $\psi \in L^{p'}(\partial\Omega)$ and $D(\gamma) = [0, 1]$. In order to get existence, it is usual to use the approximate problems

$$(L^{\gamma_r,0}_{\phi,\psi}) \quad \begin{cases} -\Delta u_r + \gamma_r(u_r) = \phi & \text{in } \Omega \\ \\ \partial_\eta u_r = \psi & \text{on } \partial\Omega, \end{cases}$$

where γ_r is the Yosida approximation of γ. Now, by the results in the previous section, the estimates we can obtain are, essentially, $\{\gamma_r(u_r)\}$ bounded in $L^1(\Omega)$ and $\{u_r\}$ bounded in $H^1(\Omega)$. Therefore we have to pass to the limit weakly-star in the space of measures for $\gamma_r(u_r)$ and weakly in $H^1(\Omega)$ for u_r. And the standard analysis for this kind of problems allows us to obtain a couple $[u, \mu]$, where $u \in H^1(\Omega)$, μ is a diffuse Radon measure in $\mathbb{R}^N$ concentrated in $\overline{\Omega}$ and

$$\int_\Omega Du \cdot Dv + \int_\Omega v\,d\mu + \int_{\partial\Omega} v\,d\mu = \int_\Omega \phi v + \int_{\partial\Omega} \psi v \quad \forall\, v \in H^1(\Omega) \cap L^\infty(\Omega).$$

In a first step, we prove that the Radon-Nykodym decomposition of μ relatively to the Lebesgue measure, $\mu = \mu_a + \mu_s$, is such that $\mu_a \in \gamma(u)$ a.e. in Ω and μ_s is concentrated on $\{x \in \overline{\Omega}\ ;\ u(x) = 0\} \cup \{x \in \overline{\Omega}\ ;\ u(x) = 0\}$ with

$$\mu_s \leq 0 \quad \text{on} \quad \{x \in \overline{\Omega}\ ;\ u(x) = 0\} \qquad \text{and} \qquad \mu_s \geq 0 \quad \text{on} \quad \{x \in \overline{\Omega}\ ;\ u(x) = 1\}.$$

Afterwards, an accurate analysis allows us to prove moreover that μ_s is concentrated on $\partial\Omega$ and it is absolutely continuous with respect to an integrable function on the boundary, which implies that $\mu_s \in L^1(\partial\Omega)$. So, $[u, \mu_a, \mu_s]$ is a weak solution, in the usual sense, of the problem

$$\begin{cases} -\Delta u + \gamma(u) \ni \phi & \text{in } \Omega \\ \\ Du \cdot \eta + \partial \mathbb{I}_{[0,1]}(u) \ni \psi & \text{on } \partial\Omega, \end{cases}$$

where, for an interval $I \subset \mathbb{R}$, $\partial \mathbb{I}_I$ denotes the subdifferential of the indicator function of I,

$$\mathbb{I}_I(r) = \begin{cases} 0 & \text{if } r \in I, \\ \\ +\infty & \text{if } r \notin I, \end{cases}$$

which is the maximal monotone graph defined by $D(\partial \mathbb{I}_I) = I$ and $\partial \mathbb{I}_I(r) = 0$ for $r \in \operatorname{int}(I)$. For instance, if $\overline{\mathcal{D}(\gamma)} = [a, b]$, then

$$\partial \mathbb{I}_{\overline{\mathcal{D}(\gamma)}}(r) = \begin{cases}]-\infty, 0] & \text{if} \quad r = a \\ \\ 0 & \text{if} \quad a < r < b \\ \\ [0, +\infty[& \text{if} \quad r = b. \end{cases}$$

In other words, the boundary condition needs to be fullfield in the following sense

$$\begin{cases} \partial_\eta u = \psi & \text{on } [0 < u < 1] \\ \partial_\eta u \geq \psi & \text{on } [u = 1] \\ \partial_\eta u \leq \psi & \text{on } [u = 0]. \end{cases} \tag{4.1}$$

The last two conditions of (4.1) disappear whenever the data ϕ and ψ are such that the sets $[u = 1]$ and $[u = 0]$ are negligeable. This is the case, for instance, if $\psi \equiv 0$.

After a complete analysis of the general problem $(S^{\gamma,\beta}_{\phi,\psi})$ when $\overline{D(\gamma)} \subset D(\beta)$, we get the right notion of solution which coincides with the concept of weak solution for the problem

$$\begin{cases} -\mathrm{div}\ \mathbf{a}(x, Du) + \gamma(u) \ni \phi & \text{in } \Omega \\ \\ \mathbf{a}(x, Du) \cdot \eta + \beta(u) + \partial \mathbb{I}_{\overline{D(\gamma)}}(u) \ni \psi & \text{on } \partial\Omega. \end{cases}$$

Definition 4.1. *Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$. A triple of functions $[u, z, w] \in W^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is a generalized weak solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x)) + \partial \mathbb{I}_{\overline{D(\gamma)}}(u(x))$ a.e. on $\partial\Omega$, and*

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} w\, v = \int_\Omega \phi v + \int_{\partial\Omega} \psi v,$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

It is clear that a weak solution is a generalized weak solution. Moreover, thanks to the results in Section 3, the two concepts coincide in the case (A). Let us show the existence and uniqueness results of solutions in the sense of Definition 4.1 for $(S^{\gamma,\beta}_{\phi,\psi})$ in the case $\mathbb{R} \neq \overline{D(\gamma)} \subset D(\beta)$.

The main result about existence is divided in two statements. The statement (i) corresponds to the existence of generalized weak solutions for the one or two obstacle problem and regular data. Observe that for the one obstacle problem $\mathcal{R}_{\gamma,\beta}$ can be different from $\mathbb{R}$ and for the two obstacle problem $\mathcal{R}_{\gamma,\beta} = \mathbb{R}$. The statement (ii) is for the two obstacle problem and L^1-data.

Theorem 4.1 ([4]). *Assume $\mathbb{R} \neq \overline{D(\gamma)} \subset D(\beta)$. Then,*

(i) for any $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$ such that

$$\int_\Omega \phi + \int_{\partial\Omega} \psi \in \mathcal{R}_{\gamma,\beta},$$

there exists a generalized weak solution $[u, z, w]$ of problem $(S^{\gamma,\beta}_{\phi,\psi})$;

(ii) if $D(\gamma)$ is bounded, the existence of a generalized weak solution $[u,z,w]$ of problem $(S^{\gamma,\beta}_{\phi,\psi})$ holds to be true for any $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$.

Since $[u,z,w]$ is a generalized weak solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if and only if $[u,z,w]$ is a weak solution of problem $(S^{\gamma,\beta_\gamma}_{\phi,\psi})$, where $\beta_\gamma := \beta + \partial\mathbb{I}_{\overline{D(\gamma)}}$, the uniqueness of a generalized weak solution is a consequence of Theorems 3.1 and 3.2.

Theorem 4.2. *Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$. Let $[u_1,z_1,w_1]$ and $[u_2,z_2,w_2]$ be generalized weak solutions of $(S^{\gamma,\beta}_{\phi,\psi})$. Then, there exists a constant $c \in \mathbb{R}$ such that*

$$u_1 - u_2 = c \quad \textit{a.e. in } \Omega,$$

$$z_1 - z_2 = 0 \quad \textit{a.e. in } \Omega$$

and

$$w_1 - w_2 = 0 \quad \textit{a.e. in } \partial\Omega.$$

If $c \neq 0$, $z_1 = z_2$ is constant.

Moreover, given $[u_1,z_1,w_1]$ a generalized weak solution of $(S^{\gamma,\beta}_{\phi_1,\psi_1})$ and $[u_2,z_2,w_2]$ a generalized weak solution of $(S^{\gamma,\beta}_{\phi_2,\psi_2})$,

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \leq \int_\Omega (\phi_1 - \phi_2)^+ + \int_{\partial\Omega} (\psi_1 - \psi_2)^+.$$

By the above result we have that $[0,\phi,\psi]$ is the unique generalized weak solution of the paradigmatic problem $(L^{\gamma,0}_{\phi,\psi})$.

Thanks to the example $(L^{\gamma,0}_{\phi,\psi})$ it is clear that, for a generalized weak solution, $w \notin \beta(u)$ in general. However we show that for the homogeneous case $\psi \equiv 0$, $w \in \beta(u)$.

In order to prove the existence results we define

$$\mathcal{M}^p_b(\overline{\Omega}) := \left\{ \mu \in \mathcal{M}_b(U_\Omega) \cap W^{-1,p'}(U_\Omega) \ : \ \mu \text{ is concentrated on } \overline{\Omega} \right\}.$$

This definition is independent of the open set U_Ω, and for $u \in W^{1,p}(\Omega)$ and $\mu \in \mathcal{M}^p_b(\overline{\Omega})$ we have

$$\langle \mu, \tilde{u} \rangle = \int_\Omega u \, d\mu + \int_{\partial\Omega} u \, d\mu.$$

Observe also that if $\mu \in \mathcal{M}_b^p(\overline{\Omega})$ then $\mu \in \mathcal{M}_b^p(U_\Omega)$.

We also define, for $u \in W^{1,p}(\Omega)$ and $\mu \in \mathcal{M}_b^p(\overline{\Omega})$,

$$\mu \in \gamma(u)$$

if the following conditions are satisfied,

(i) $\mu_a \in \gamma(u)$ $\mathcal{L}^N - a.e.$ in Ω,

(ii) μ_s is concentrated on the set $\{x \in \overline{\Omega} : u = \gamma^{(i)}\} \cup \{x \in \overline{\Omega} : u = \gamma^{(s)}\}$,

(iii) $\mu_s \leq 0$ on $\{x \in \overline{\Omega} : u = \gamma^{(i)}\}$ and $\mu_s \geq 0$ on $\{x \in \overline{\Omega} : u = \gamma^{(s)}\}$,

where $\gamma^{(i)} = \inf \mathrm{Dom}(\gamma)$ and $\gamma^{(s)} = \sup \mathrm{Dom}(\gamma)$.

Let us point out the similitude of the above concept with the definition of $\mu \in \gamma(u)$ for $\mu \in \mathcal{M}_b^p(\Omega)$ given in [9], from where we have taken some ideas for our proofs.

Let $u \in W^{1,p}(\Omega)$ and $\mu \in \mathcal{M}_b^p(\overline{\Omega})$. Recalling that the set $\{x \in \overline{\Omega} : u = \pm\infty\}$ has zero p-capacity relative to U_Ω and that $\mu \in \mathcal{M}_b^p(U_\Omega)$, we have that

$$\mu_s = 0 \quad \text{on } \{x \in \overline{\Omega} : u = \pm\infty\}.$$

In particular, if $D(\gamma) = \mathbb{R}$, then

$$\mu \in \gamma(u) \quad \text{if and only if } \ \mu \in L^1(\Omega) \ \text{ and } \ \mu \in \gamma(u) \text{ a.e. in } \Omega.$$

Initially we prove the following proposition by approximating the problem by other one to which we can apply the results in Section 3. Concretely, we consider approximate problems replacing β by $\tilde{\beta}$, being $\tilde{\beta}$ the maximal monotone graph defined by

$$\tilde{\beta}(s) = \begin{cases} \beta(s) & \text{if } \ s \in]\gamma^{(i)}, \gamma^{(s)}[, \\ \beta^0(\gamma^{(i)}) & \text{if } \ s < \gamma^{(i)} \quad (\gamma^{(i)} \text{ finite}), \\ \beta^0(\gamma^{(s)}) & \text{if } \ s > \gamma^{(s)} \quad (\gamma^{(s)} \text{ finite}). \end{cases}$$

And replacing γ by γ^r as follows, in the case domain of γ is bounded, i.e. $\gamma^{(i)}$ and $\gamma^{(s)}$ are both finite, for every $r \in \mathbb{N}$ we take $\gamma^r = \gamma_r$ to be the Yosida approximation of γ, and in the case $D(\gamma)$ is not bounded, we consider that

$\gamma^{(i)} = -\infty$ and $\gamma^{(s)}$ is finite (the other case, $\gamma^{(i)}$ finite and $\gamma^{(s)} = +\infty$, being similar), for every $r \in \mathbb{N}$ we take γ^r the maximal monotone graph defined by

$$\gamma^r(s) = \begin{cases} \gamma(s) & \text{if} \quad s < 0, \\ \gamma_r(s) & \text{if} \quad s > 0. \end{cases}$$

It is clear that in the last case we are regularizing just the positive part, the regularization of the negative part is not necessary since it is everywhere defined.

Proposition 4.1 ([4]). *Assume $\mathbb{R} \neq \overline{D(\gamma)} \subset D(\beta)$. Then, for any $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$ such that*

$$\int_\Omega \phi + \int_{\partial\Omega} \psi \in \mathcal{R}_{\gamma,\beta},$$

there exists $[u, \mu, w] \in W^{1,p}(\Omega) \times M_b^p(\overline{\Omega}) \times V^{1,p}(\partial\Omega)$ such that $w \in \beta(u)$ a.e. in $\partial\Omega$, $\mu \in \gamma(u)$ and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \langle \mu, \tilde{v} \rangle + \int_{\partial\Omega} wv = \int_\Omega \phi v + \int_{\partial\Omega} \psi v \quad \forall\, v \in W^{1,p}(\Omega).$$

Next we show that μ_s is concentrated on $\partial\Omega$ and is absolutely continuous respect to an integrable function in order to get Theorem 4.1. To this end, the following technical result is established.

Lemma 4.1. *Let $\eta \in W^{1,p}(\Omega)$, $\nu \in M_b^p(\overline{\Omega})$ and $\lambda \in \mathbb{R}$ be such that $\eta \leq \lambda$ (resp. $\eta \geq \lambda$) a.e. Ω. If*

$$-\mathrm{div}\ \mathbf{a}(x, D\eta) = \nu$$

in the sense that $\displaystyle\int_\Omega \mathbf{a}(x, D\eta) \cdot D\xi = \int_{\overline{\Omega}} \xi \, d\nu$, *for any $\xi \in W^{1,p}(\Omega)$, then*

$$\int_{\{x \in \overline{\Omega} : \eta(x) = \lambda\}} \xi \, d\nu \geq 0$$

(resp.

$$\int_{\{x \in \overline{\Omega} : \eta(x) = \lambda\}} \xi \, d\nu \leq 0)$$

for any $\xi \in W^{1,p}(\Omega)$, $\xi \geq 0$.

5. Measure data

We define

$$\mathfrak{M}_b^p(\overline{\Omega}) := \left\{ \mu \in \mathcal{M}_b^p(U_\Omega) \ : \ \mu \text{ is concentrated on } \overline{\Omega} \right\}.$$

This definition is independent of the open set U_Ω. Note that for $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^p(\overline{\Omega})$, we have

$$\langle \mu, E(u) \rangle = \int_\Omega u \, d\mu + \int_{\partial\Omega} u \, d\mu;$$

on the other hand, there exists $f \in L^1(U_\Omega)$ and $F \in (L^{p'}(U_\Omega))^N$ such that $\mu = f + \operatorname{div}(F)$, therefore, we also can write

$$\langle \mu, E(u) \rangle = \int_{U_\Omega} f E(u) \, dx - \int_{U_\Omega} F \cdot DE(u) \, dx.$$

Note that, if $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$ then $f\mathcal{L}^N \llcorner \Omega + g\mathcal{H}^{N-1} \llcorner \partial\Omega$ is a diffuse measure concentrated in $\overline{\Omega}$. Now, if $p > N - k$, $1 \le k < N - 1$, and $\mathbb{M}$ is a k-rectifiable subset of $\partial\Omega$, then $\mathcal{H}^k \llcorner \mathbb{M}$ is a diffuse measure concentrated in $\partial\Omega$ which is not an L^1 function in $\partial\Omega$ (see, [39, Theorem 2.26] or [47, Theorem 2.6.16]).

Definition 5.1. *Let μ_1, μ_2 measures, $\mu_1 = \mu_1 \llcorner \Omega$ and $\mu_2 = \mu_2 \llcorner \partial\Omega$, such that $\mu_1 + \mu_2 \in \mathfrak{M}_b^p(\overline{\Omega})$. A triple of functions $[u, z, w] \in W^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is a weak solution of problem $(S^{\gamma,\beta}_{\mu_1,\mu_2})$ if $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$ and*

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv \, dx + \int_\Omega zv \, dx + \int_{\partial\Omega} wv \, d\mathcal{H}^{N-1} = \int_\Omega v \, d\mu_1 + \int_{\partial\Omega} v \, d\mu_2$$

for all $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

As we pointed out before the concept of weak solution for this type of problems is not enough in order to get uniqueness. It is necessary to find some extra conditions on the distributional solutions in order to ensure both existence and uniqueness. This was done by introducing the concepts of entropy and renormalized solutions (see, e.g., [12]). For our problem these concepts are the following.

Definition 5.2. *Let μ_1, μ_2 measures, $\mu_1 = \mu_1 \llcorner \Omega$ and $\mu_2 = \mu_2 \llcorner \partial\Omega$, such that $\mu_1 + \mu_2 \in \mathfrak{M}_b^p(\overline{\Omega})$. A triple of functions $[u, z, w] \in \mathcal{T}_{tr}^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is an entropy solution of problem $(S_{\mu_1,\mu_2}^{\gamma,\beta})$ if $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$ and*

$$\begin{aligned} &\int_\Omega \mathbf{a}(x, Du) \cdot DT_k(u - v)\, dx + \int_\Omega z T_k(u - v)\, dx \\ &\qquad + \int_{\partial\Omega} w T_k(u - v)\, d\mathcal{H}^{N-1} \\ &\leq \int_\Omega T_k(u - v)\, d\mu_1 + \int_{\partial\Omega} T_k(u - v)\, d\mu_2 \qquad \forall k > 0, \end{aligned} \tag{5.1}$$

for all $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Definition 5.3. *Let μ_1, μ_2 measures, $\mu_1 = \mu_1 \llcorner \Omega$ and $\mu_2 = \mu_2 \llcorner \partial\Omega$, such that $\mu_1 + \mu_2 \in \mathfrak{M}_b^p(\overline{\Omega})$. A triple of functions $[u, z, w] \in \mathcal{T}_{tr}^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is a renormalized solution of problem $(S_{\mu_1,\mu_2}^{\gamma,\beta})$ if $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, and the following conditions hold*

(a) for every $h \in W^{1,\infty}(\mathbb{R})$ with compact support we have

$$\begin{aligned} &\int_\Omega \mathbf{a}(x, Du) \cdot Du\, h'(u)\varphi\, dx + \int_\Omega \mathbf{a}(x, Du) \cdot D\varphi\, h(u)\, dx \\ &\qquad + \int_\Omega z\, h(u)\varphi\, dx + \int_{\partial\Omega} w\, h(u)\varphi\, d\mathcal{H}^{N-1} \\ &= \int_\Omega h(u)\varphi\, d\mu_1 + \int_{\partial\Omega} h(u)\varphi\, d\mu_2 \qquad \forall k > 0, \end{aligned} \tag{5.2}$$

for all $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $h(u)\varphi \in W^{1,p}(\Omega)$,

(b)

$$\lim_{n \to +\infty} \int_{\{n \leq |u| \leq n+1\}} \mathbf{a}(x, Du) \cdot Du\, dx = 0.$$

Remark 5.1 - Every term in (5.2) is well defined. This is clear for the right hand side since $h(u)\varphi$ belongs to $L^\infty(\overline{\Omega}, \mu_1 + \mu_2)$, and thus to $L^1(\overline{\Omega}, \mu_1 + \mu_2)$.

On the other hand, since $\mathrm{supp}(h) \subset [-k,k]$ for some $k>0$, the two first terms of the left hand side can be written as

$$\int_\Omega \mathbf{a}(x, DT_k(u)) \cdot DT_k(u)\, h'(u)\varphi\, dx + \int_\Omega \mathbf{a}(x, DT_k(u)) \cdot D\varphi\, h(u)\, dx,$$

and both integrals are well defined in view of (H_2), since both φ and $T_k(u)$ belong to $W^{1,p}(\Omega)$. Moreover, it is not difficult to see that the product $DT_k(u)\, h'(u)$ coincides with the gradient of the composite function $h(u) = h(T_k(u))$ almost everywhere (see [21]).

The nexus relating both concepts of solutions is the following one.

Lemma 5.1. *Let μ_1, μ_2 measures, $\mu_1 = \mu_1 \llcorner \Omega$ and $\mu_2 = \mu_2 \llcorner \partial\Omega$, such that $\mu_1 + \mu_2 \in \mathfrak{M}_b^p(\overline{\Omega})$. Let $[u,z,w]$ be an entropy solution of problem $(S^{\gamma,\beta}_{\mu_1,\mu_2})$. Then,*

$$\lim_{h\to+\infty} \int_{\{x\in\Omega : h<|u(x)|<h+k\}} |Du|^p = 0, \qquad \forall\, k>0.$$

Theorem 5.1. *Let μ_1, μ_2 measures, $\mu_1 = \mu_1 \llcorner \Omega$ and $\mu_2 = \mu_2 \llcorner \partial\Omega$, such that $\mu_1 + \mu_2 \in \mathfrak{M}_b^p(\overline{\Omega})$. Then, $[u,z,w]$ is an entropy solution of problem $(S^{\gamma,\beta}_{\mu_1,\mu_2})$ if and only if $[u,z,w]$ is a renormalized solution of problem $(S^{\gamma,\beta}_{\mu_1,\mu_2})$.*

Remark 5.2 - Assume that $[u,z,w]$ is an entropy solution of problem $(S^{\gamma,\beta}_{\mu_1,\mu_2})$. If we take $v = T_h(u) \pm 1$ as test functions in (5.1) and let h go to $+\infty$, we get that

$$\int_\Omega z + \int_{\partial\Omega} w = \mu_1(\Omega) + \mu_2(\partial\Omega).$$

Then, since $z(x) \in \gamma(u(x))$ *a.e.* in Ω and $w(x) \in \beta(u(x))$ *a.e.* in $\partial\Omega$, necessarily μ_1 and μ_2 must satisfy

$$\mathcal{R}^-_{\gamma,\beta} \leq \mu_1(\Omega) + \mu_2(\partial\Omega) \leq \mathcal{R}^+_{\gamma,\beta}.$$

For weak solutions a contraction principle is proved in Theorem 5.3. Respect to uniqueness for entropy solutions we have the following general result.

Theorem 5.2 ([5]). *Let μ_1, μ_2 measures, $\mu_1 = \mu_1 \llcorner \Omega$ and $\mu_2 = \mu_2 \llcorner \partial\Omega$, such that $\mu_1 + \mu_2 \in \mathfrak{M}_b^p(\overline{\Omega})$. Let $[u_1,z_1,w_1]$ and $[u_2,z_2,w_2]$ be entropy solutions of problem $(S^{\gamma,\beta}_{\mu_1,\mu_2})$. Then, there exists a constant $c \in \mathbb{R}$ such that*

$$u_1 - u_2 = c \quad \textit{a.e. in } \Omega,$$

$$z_1 - z_2 = 0 \quad a.e.\ in\ \Omega.$$

$$w_1 - w_2 = 0 \quad a.e.\ in\ \partial\Omega.$$

Moreover, if $c \neq 0$, there exists a constant $k \in \mathbb{R}$ such that $z_1 = z_2 = k$.

In order to get the existence of solutions we use the following key results.

Lemma 5.2 ([5]). *Let Ω be a bounded domain in $\mathbb{R}^N$ with $\partial\Omega$ of class C^1. Given $\mu \in \mathfrak{M}_b^p(\overline{\Omega})$, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset C_c(\Omega)$,*

$$\psi_n \rightharpoonup \mu \qquad \text{as measures},$$

such that, for any $\{v_n\}_{n\in\mathbb{N}} \in W^{1,p}(\Omega)$ with $v_n \to v$ weakly in $W^{1,p}(\Omega)$ and all $k > 0$,

$$\lim_{n\to\infty} \int_\Omega T_k(v_n)\psi_n = \int_{\overline{\Omega}} T_k(v)\, d\mu.$$

Moreover, if $\mu = f + \operatorname{div} F$, $f \in L^{p'}(U_\Omega)$, $F \in L^{p'}(U_\Omega)^N$, then

$$\left| \int_\Omega v_n \psi_n \right| \leq C_1 \|f\|_{L^{p'}(U_\Omega)} \|v_n\|_{L^p(\Omega)} + C_2 \|F\|_{(L^{p'}(U_\Omega))^N} \left(\|v_n\|_{L^p(\Omega)} + \|Dv_n\|_{L^p(\Omega)} \right).$$

Grosso modo, we rectify $\partial\Omega$ using local maps $\{(G_i, U_i)\}_{i=0,1,\ldots,k}$ and introduce a partition of unit $\{\theta_i\}_{i=0,1,\ldots,k}$ subordinate to $\partial\Omega$ and $U_1, \ldots U_k$ (see [24]). Letting $d_\epsilon(x', x_N) = (x', (1-\epsilon)x_N + \epsilon)$ and writing $d_n^i = G_i \circ d_{\frac{1}{n}} \circ {G_i}^{-1}$, $i = 1, ..., n$, we take

$$\psi_n = \left(\sum_{i=1}^k d_n^i \# \sigma_i + \sigma_0 \right) * \rho_{\frac{1}{2n}}, \tag{5.3}$$

where ρ_ϵ is a mollifier with support in $\overline{B(0,\epsilon)}$ and $d_n^i \# \nu$ is the push-forward measure.

Lemma 5.3 ([1]). *Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\Omega)$, $\{z_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$, $\{w_n\}_{n\in\mathbb{N}} \subset L^1(\partial\Omega)$ such that, for every $n \in \mathbb{N}$, $z_n \in \gamma(u_n)$ a.e. in Ω and $w_n \in \beta(u_n)$ a.e. in $\partial\Omega$. Let us suppose that* *(i)* *if* $\mathcal{R}^+_{\gamma,\beta} = +\infty$, *there exists* $M > 0$ *such that*

$$\int_\Omega z_n^+ dx + \int_{\partial\Omega} w_n^+ d\sigma < M \qquad \forall n \in \mathbb{N};$$

(ii) *if* $\mathcal{R}^+_{\gamma,\beta} < +\infty$, *there exists* $M \in \mathbb{R}$ *such that*

$$\int_\Omega z_n dx + \int_{\partial\Omega} w_n d\sigma < M < \mathcal{R}^+_{\gamma,\beta}$$

and

$$\lim_{L\to+\infty}\left(\int_{\{x\in\Omega: z_n(x)<-L\}} |z_n| dx + \int_{\{x\in\partial\Omega: w_n(x)<-L\}} |w_n| d\sigma\right) = 0$$

uniformly in $n \in \mathbb{N}$. *Then, there exists a constant* $C = C(M)$ *such that*

$$\|u_n^+\|_{L^p(\Omega)} \le C\left(\|Du_n^+\|_{L^p(\Omega)} + 1\right) \qquad \forall n \in \mathbb{N}.$$

The next theorem gives the existence results of weak and entropy solutions. A contraction principle for weak sub and super solutions also holds.

Theorem 5.3 ([5]). *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set with boundary* $\partial\Omega$ *of class* C^1. *Assume Dom*$(\gamma) =$ *Dom*$(\beta) = \mathbb{R}$ *and* $J(\gamma)$ *and* $J(\beta)$ *are bounded. Then,*

(i) *for any measures* μ_1, μ_2 *such that* $\mu_1 = \mu_1 \llcorner \Omega$, $\mu_2 = \mu_2 \llcorner \partial\Omega$, $\mu = \mu_1 + \mu_2 = f + \operatorname{div}(F)$, $f \in L^{p'}(U_\Omega)$, $F \in (L^{p'}(U_\Omega))^N$, *and* $\mu(\overline{\Omega}) \in \mathcal{R}_{\gamma,\beta}$, *there exists a weak solution* $[u, z, w]$ *of problem* $(S^{\gamma,\beta}_{\mu_1,\mu_2})$.

(ii) *If* $[u, z, w]$ *is a weak subsolution of problem* $(S^{\gamma,\beta}_{\mu_1,\mu_2})$, μ_1, μ_2 *measures,* $\mu_1 = \mu_1 \llcorner \Omega$ *and* $\mu_2 = \mu_2 \llcorner \partial\Omega$ *such that* $\mu = \mu_1 + \mu_2 \in \mathfrak{M}^p_b(\overline{\Omega})$, *and* $[\tilde{u}, \tilde{z}, \tilde{w}]$ *is a weak supersolution of problem* $(S^{\gamma,\beta}_{\tilde{\mu}_1,\tilde{\mu}_2})$, $\tilde{\mu}_1, \tilde{\mu}_2$ *measures,* $\tilde{\mu}_1 = \tilde{\mu}_1 \llcorner \Omega$ *and* $\tilde{\mu}_2 = \tilde{\mu}_2 \llcorner \partial\Omega$ *such that* $\tilde{\mu} = \tilde{\mu}_1 + \tilde{\mu}_2 \in \mathfrak{M}^p_b(\overline{\Omega})$, *then*

$$\int_\Omega (z - \tilde{z})^+ + \int_{\partial\Omega} (w - \tilde{w})^+ \le (\mu - \tilde{\mu})^+(\overline{\Omega}). \tag{5.4}$$

(iii) *For any measures* μ_1, μ_2 *such that* $\mu_1 = \mu_1 \llcorner \Omega$, $\mu_2 = \mu_2 \llcorner \partial\Omega$, $\mu = \mu_1 + \mu_2 \in \mathfrak{M}^p_b(\overline{\Omega})$, *and* $\mu(\overline{\Omega}) \in \mathcal{R}_{\gamma,\beta}$, *there exists an entropy solution* $[u, z, w]$ *of problem* $(S^{\gamma,\beta}_{\mu_1,\mu_2})$.

For the existence results we first suppose that $\mathcal{R}_{\gamma,\beta} = \mathbb{R}$ and consider a sequence of approximated problems, replacing ψ by ψ_n, given in (5.3), to which we can apply Theorem 3.5. Afterwards we use monotonicity arguments to deal with the other cases.

The proof of the contraction principle is as follows. Let $[u, z, w]$ be a weak subsolution of problem $(S^{\gamma,\beta}_{\mu_1,\mu_2})$ and let $[\tilde{u}, \tilde{z}, \tilde{w}]$ be a weak supersolution of problem $(S^{\gamma,\beta}_{\tilde{\mu}_1,\tilde{\mu}_2})$. Then,

$$\begin{aligned}\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv \leq \int_{\overline{\Omega}} v\, d\mu \\ \forall\, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega), \quad v \geq 0,\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\int_\Omega \mathbf{a}(x, D\tilde{u}) \cdot Dv + \int_\Omega \tilde{z}v + \int_{\partial\Omega} \tilde{w}v \geq \int_{\overline{\Omega}} v\, d\tilde{\mu} \\ \forall\, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega),\ v \geq 0.\end{aligned} \tag{5.6}$$

Consider $\rho \in W^{1,p}(\Omega)$, $0 \leq \rho \leq 1$. Taking as test function $v = \frac{T_k^+}{k}(u - \tilde{u} + k\rho)$ in (5.5) and in (5.6), we have

$$\begin{aligned}&\int_\Omega \mathbf{a}(x, Du) \cdot D\left(\frac{T_k^+}{k}(u - \tilde{u} + k\rho)\right) + \int_\Omega z \frac{T_k^+}{k}(u - \tilde{u} + k\rho) \\ &\quad + \int_{\partial\Omega} w \frac{T_k^+}{k}(u - \tilde{u} + k\rho) \leq \int_{\overline{\Omega}} \left(\frac{T_k^+}{k}(u - \tilde{u} + k\rho)\right) d\mu\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \mathbf{a}(x, D\tilde{u}) \cdot D\left(\frac{T_k^+}{k}(u - \tilde{u} + k\rho)\right) + \int_\Omega \tilde{z} \frac{T_k^+}{k}(u - \tilde{u} + k\rho) \\ &\quad + \int_{\partial\Omega} \tilde{w} \frac{T_k^+}{k}(u - \tilde{u} + k\rho) \geq \int_{\overline{\Omega}} \left(\frac{T_k^+}{k}(u - \tilde{u} + k\rho)\right) d\tilde{\mu}\end{aligned}$$

Therefore, having in mind the monotonicity of $\mathbf{a}$, we get

$$\int_\Omega (z-\tilde{z})\frac{T_k^+}{k}(u-\tilde{u}+k\rho) + \int_{\partial\Omega}(w-\tilde{w})\frac{T_k^+}{k}(u-\tilde{u}+k\rho)$$

$$+\int_{\{0<u-\tilde{u}+k\rho<k\}}(\mathbf{a}(x,Du)-\mathbf{a}(x,D\tilde{u}))\cdot D\rho$$

$$\leq \int_{\overline{\Omega}}\left(\frac{T_k^+}{k}(u-\tilde{u}+k\rho)\right)\,d(\mu-\tilde{\mu}) \leq (\mu-\tilde{\mu})^+(\overline{\Omega}).$$

Taking limit when k goes to 0 in the above expression, having in mind that $Du_1 = Du_2$ where $u_1 = u_2$, we obtain that

$$\int_\Omega (z-\tilde{z})\mathrm{sign}_0^+(u-\tilde{u})\chi_{\{u\neq\tilde{u}\}} + \int_\Omega (z-\tilde{z})\rho\chi_{\{u=\tilde{u}\}}$$

$$\int_{\partial\Omega}(w-\tilde{w})\mathrm{sign}_0^+(u-\tilde{u})\chi_{\{u\neq\tilde{u}\}} + \int_{\partial\Omega}(w-\tilde{w})\rho\chi_{\{u=\tilde{u}\}} \tag{5.7}$$

$$\leq (\mu-\tilde{\mu})^+(\overline{\Omega}).$$

By approximation we can suppose that (5.7) holds for every $0\leq\rho\in W^{1,1}(\Omega)\cap L^\infty(\Omega)$. It is easy to see that there exist $0\leq\rho_n\in W^{1,1}(\Omega)\cap L^\infty(\Omega)$ such that $\rho_n = \mathrm{sign}_0^+(w-\tilde{w})$ $\mathcal{H}^{N-1}$ a.e. on $\partial\Omega$ and

$$\rho_n \to \mathrm{sign}_0^+(z-\tilde{z}) \qquad \text{in} \quad L^1(\Omega).$$

Then, taking $\rho=\rho_n$ in (5.7) and sending $n\to+\infty$, we get the contraction principle (5.4).

Remark 5.3 - We also get the following monotonicity result for entropy solutions.
Let $[u,z,w]$ be an entropy solution of problem $(S^{\gamma,\beta}_{\mu_1,\mu_2})$, μ_1,μ_2 measures, $\mu_1 = \mu_1 \llcorner \Omega$ and $\mu_2 = \mu_2\llcorner\partial\Omega$ such that $\mu = \mu_1+\mu_2 \in \mathfrak{M}_b^p(\overline{\Omega})$ and $\mu(\overline{\Omega})\in\mathcal{R}_{\gamma,\beta}$, and $[\tilde{u},\tilde{z},\tilde{w}]$ an entropy solution of problem $(S^{\gamma,\beta}_{\tilde{\mu}_1,\tilde{\mu}_2})$, $\tilde{\mu}_1,\tilde{\mu}_2$ measures, $\tilde{\mu}_1=\tilde{\mu}_1\llcorner\Omega$ and $\tilde{\mu}_2=\tilde{\mu}_2\llcorner\partial\Omega$ such that $\tilde{\mu}=\tilde{\mu}_1+\tilde{\mu}_2\in\mathfrak{M}_b^p(\overline{\Omega})$ and $\tilde{\mu}(\overline{\Omega})\in\mathcal{R}_{\gamma,\beta}$. If $\mu_1\leq\mu_2$, then $z_1\leq z_2$ *a.e.* in Ω and $w_1\leq w_2$ *a.e.* in $\partial\Omega$.

Remark 5.4 - We point out that the cases of Hele Shaw problem, which corresponds to

$$\gamma(r) = \begin{cases} 0 & \text{if } \ r < 0, \\ [0,1] & \text{if } \ r = 0, \\ 1 & \text{if } \ r > 0, \end{cases}$$

and the multiphase Stefan problem, which corresponds to

$$\gamma(r) = \begin{cases} r-1 & \text{if } \ r < 0, \\ [-1,0] & \text{if } \ r = 0, \\ r & \text{if } \ r > 0, \end{cases}$$

are included in the above existence and uniqueness results.

In the particular case of Dirichlet boundary condition, that is, for β the monotone graph $D = \{0\} \times \mathbb{R}$, which corresponds to the problem

$$(S^{\gamma,D}_{\mu,0}) \quad \begin{cases} -\text{div } \mathbf{a}(x, Du) + \gamma(u) \ni \mu & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where γ is a maximal monotone graph in $\mathbb{R}^2$ and μ a diffuse measure in Ω, we also have existence and uniqueness of entropy solutions.

6. Applications

One of the main applications of these results is the study of doubly nonlinear evolution problems of elliptic-parabolic type and degenerate parabolic problems of Stefan or Hele-Shaw type, with nonhomogeneous boundary conditions and/or dynamical boundary conditions.

The results obtained have an interpretation in terms of accretive operators. Indeed, we can define the (possibly multivalued) operator $\mathcal{B}^{\gamma,\beta}$ in $X := L^1(\Omega) \times L^1(\partial\Omega)$ as

$$\mathcal{B}^{\gamma,\beta} := \Big\{ ((v,w),(\hat{v},\hat{w})) \in X \times X \ : \ \exists u \in \mathcal{T}^{1,p}_{tr}(\Omega), \\ \text{with } [u,v,w] \text{ an entropy solution of } (S^{\gamma,\beta}_{v+\hat{v},w+\hat{w}}) \Big\}.$$

Then, under certain assumptions, $\mathcal{B}^{\gamma,\beta}$ is an m-T-accretive operator in X. Therefore, by the theory of Evolution Equations Governed by Accretive Operators (see, [11], [16] or [29]), for any $(v_0, w_0) \in \overline{D(\mathcal{B}^{\gamma,\beta})}^X$ and any $(f,g) \in L^1(0,T;L^1(\Omega)) \times L^1(0,T;L^1(\partial\Omega))$, there exists a unique mild-solution of the problem

$$V' + \mathcal{B}^{\gamma,\beta}(V) \ni (f,g), \qquad V(0) = (v_0, w_0),$$

which rewrites, as an abstract Cauchy problem in X, the following degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions

$$(DP) \quad \begin{cases} v_t - \operatorname{div} \mathbf{a}(x, Du) = f, \ v \in \gamma(u), & \text{in } \Omega \times (0,T) \\ \\ w_t + \mathbf{a}(x, Du) \cdot \eta = g, \ w \in \beta(u), & \text{on } \partial\Omega \times (0,T) \\ \\ v(0) = v_0 \ \text{ in } \Omega, \ w(0) = w_0 \ \text{ in } \partial\Omega. \end{cases}$$

In principle, it is not clear how these mild solutions have to be interpreted respect to the problem (DP). In [1] and [3] we characterize these mild solutions as weak solutions or as entropy solutions depending on the regularity of the data.

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