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A NONLOCAL p-LAPLACIAN EVOLUTION EQUATION WITH NONHOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS*

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Abstract. In this paper we study the nonlocal p-Laplacian-type diffusion equation $u_t(t,x) = \int_{\mathbb{R}^N} J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))\,dy$, $(t,x) \in]0,T[\times\Omega$, with $u(t,x) = \psi(x)$ for $(t,x) \in]0,T[\times(\mathbb{R}^N \setminus \Omega)$. If $p > 1$, this is the nonlocal analogous problem to the well-known local p-Laplacian evolution equation $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with Dirichlet boundary condition $u(t,x) = \psi(x)$ on $(t,x) \in]0,T[\times\partial\Omega$. If $p = 1$, this is the nonlocal analogous to the total variation flow. When $p = +\infty$ (this has to be interpreted as the limit as $p \to +\infty$ in the previous model) we find an evolution problem that can be seen as a nonlocal model for the formation of sandpiles (here $u(t,x)$ stands for the height of the sandpile) with prescribed height of sand outside of Ω. We prove, as main results, existence, uniqueness, a contraction property that gives well posedness of the problem, and the convergence of the solutions to solutions of the local analogous problem when a rescaling parameter goes to zero.

Key words. nonlocal diffusion, p-Laplacian, nonhomogeneous Dirichlet boundary conditions, total variation flow, sandpiles

AMS subject classifications. 35B40, 45A07, 45G10

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1. Introduction.

In this paper we study the nonlocal diffusion equation

$$u_t(t,x) = \int_{\mathbb{R}^N} J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))\,dy \qquad (t,x) \in]0,T[\times\Omega,$$

where Ω is a bounded domain and u is prescribed in $\mathbb{R}^N \setminus \Omega$ as $u(t,x) = \psi(x)$ for $(t,x) \in]0,T[\times(\mathbb{R}^N \setminus \Omega)$. We consider $1 < p < +\infty$ as well as the extreme cases $p = 1$ and the limit $p \nearrow +\infty$. Throughout the paper, we assume that $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative, radial, continuous function, strictly positive in $B(0,1)$, vanishing in $\mathbb{R}^N \setminus B(0,1)$ and such that $\int_{\mathbb{R}^N} J(z)\,dz = 1$.

First, let us briefly introduce the prototype of nonlocal problem that will be considered along this work. Nonlocal evolution equations of the form

$$u_t(t,x) = (J * u - u)(t,x) = \int_{\mathbb{R}^N} J(x-y)u(t,y)\,dy - u(t,x), \tag{1.1}$$

and variations of it, have been recently widely used to model diffusion processes. More precisely, as stated in [31], if $u(t,x)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to

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location x, then $\int_{\mathbb{R}^N} J(y-x)u(t,y)\,dy = (J*u)(t,x)$ is the rate at which individuals are arriving at position x from all other places and $-u(t,x) = -\int_{\mathbb{R}^N} J(y-x)u(t,x)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies (1.1). For recent references on nonlocal diffusion, see [4], [5], [6], [9], [11], [12], [20], [21], [22], [23], [24], [25], [26], [27], [31], [33], [36] and references therein.

The first goal of this paper is to study the following nonlocal nonlinear diffusion problem:

$$P_p^J(u_0,\psi)\begin{cases} u_t(t,x) = \displaystyle\int_\Omega J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))\,dy \\ \qquad\qquad + \displaystyle\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)|\psi(y)-u(t,x)|^{p-2}(\psi(y)-u(t,x))\,dy, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in]0,T[\times\Omega, \\ u(0,x) = u_0(x), \quad x\in\Omega. \end{cases}$$

Here $\Omega_J = \Omega + \mathrm{supp}(J)$ and ψ is a given function $\psi : \Omega_J\setminus\overline{\Omega} \to \mathbb{R}$.

Observe that we can rewrite $P_p^J(u_0,\psi)$, setting $u(t,x)=\psi(x)$ in $\Omega_J\setminus\overline{\Omega}$, as

$$\begin{cases} u_t(t,x) = \displaystyle\int_{\Omega_J} J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))\,dy, \quad (t,x)\in]0,T[\times\Omega, \\ u(t,x) = \psi(x), \quad (t,x)\in]0,T[\times\left(\Omega_J\setminus\overline{\Omega}\right), \\ u(0,x) = u_0(x), \quad x\in\Omega, \end{cases}$$

and we call it the *nonlocal p-Laplacian problem with Dirichlet boundary condition.* Note that we are prescribing the values of u outside the domain Ω and not only on its boundary. This is due to the nonlocal character of the problem.

Let us state the precise definition of solution. Solutions to $P_p^J(u_0,\psi)$ will be understood in the following sense.

DEFINITION 1.1. *Let $1<p<+\infty$. A solution of $P_p^J(u_0,\psi)$ in $[0,T]$ is a function*

$$u\in C\left([0,T];L^1(\Omega)\right)\cap W^{1,1}\left(]0,T[;L^1(\Omega)\right),$$

which satisfies $u(0,x)=u_0(x)$ a.e. $x\in\Omega$ and

$$u_t(t,x) = \int_\Omega J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))\,dy$$
$$+\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)|\psi(y)-u(t,x)|^{p-2}(\psi(y)-u(t,x))\,dy,$$

for a.e. $t\in]0,T[$ and a.e. $x\in\Omega$.

Our first result shows existence and uniqueness of a global solution for this problem. Moreover, a contraction principle holds.

THEOREM 1.2. *Assume $p>1$ and let $u_0\in L^p(\Omega)$, $\psi\in L^p(\Omega_J\setminus\overline{\Omega})$. Then, there exists a unique solution to $P_p^J(u_0,\psi)$ in the sense of Definition* 1.1. *Moreover, if $u_{i0}\in L^1(\Omega)$ and u_i is a solution in $[0,T]$ of $P_p^J(u_{i0},\psi)$, $i=1,2$, respectively. Then*

$$\int_\Omega (u_1(t)-u_2(t))^+ \le \int_\Omega (u_{10}-u_{20})^+ \qquad \textit{for every } t\in[0,T].$$

If $u_{i0} \in L^p(\Omega)$, $i = 1, 2$, *then*

$$\|u_1(t) - u_2(t)\|_{L^p(\Omega)} \le \|u_{10} - u_{20}\|_{L^p(\Omega)} \qquad \textit{for every } t \in [0, T].$$

Our next step is to rescale the kernel J appropriately and take the limit as the scaling parameter goes to zero. To be more precise, for $p > 1$, we consider the local p-Laplace evolution equation with Dirichlet boundary condition

$$D_p(u_0, \tilde{\psi}) \quad \begin{cases} u_t = \Delta_p u & \text{in }]0, T[\times\Omega, \\ u = \tilde{\psi} & \text{on }]0, T[\times\partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where the boundary datum $\tilde{\psi}$ is assumed to be the trace of a function defined in a larger domain and the operator in the equation, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, is the usual local p-Laplacian.

We prove that the solutions of this local problem can be approximated by solutions of a sequence of nonlocal *p-Laplacian* problems of the form P_p^J. Indeed, for given $p \ge 1$ and J we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right), \qquad \text{where} \qquad C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p\, dz \tag{1.2}$$

is a normalizing constant in order to obtain the p-Laplacian in the limit instead of a multiple of it, and we obtain the following result.

THEOREM 1.3. *Let* Ω *be a smooth bounded domain in* $\mathbb{R}^N$ *and* $\tilde{\psi} \in W^{1/p',p}(\partial\Omega) \cap L^\infty(\partial\Omega)$. *Let* $\psi \in W^{1,p}(\Omega_J) \cap L^\infty(\Omega_J)$ *such that the trace* $\psi|_{\partial\Omega} = \tilde{\psi}$. *Assume* $J(x) \ge J(y)$ *if* $|x| \le |y|$. *Let* $T > 0$ *and* $u_0 \in L^p(\Omega)$. *Let* u_ε *be the unique solution of* $P_p^{J_{p,\varepsilon}}(u_0, \psi)$ *and* u *the unique solution of* $D_p(u_0, \tilde{\psi})$ *(see section* 2.2*). Then*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_\varepsilon(t, .) - u(t, .)\|_{L^p(\Omega)} = 0. \tag{1.3}$$

Note that the above result says that P_p^J is a nonlocal problem analogous to the p-Laplacian with Dirichlet boundary condition.

The second goal of this paper is to study the Dirichlet problem for $p = 1$, called the nonlocal total variation flow, which can be written formally as

$$P_1^J(u_0, \psi) \begin{cases} u_t(t, x) = \displaystyle\int_\Omega J(x - y)\frac{u(t, y) - u(t, x)}{|u(t, y) - u(t, x)|}\, dy \\ \qquad\qquad + \displaystyle\int_{\Omega_J\setminus\overline{\Omega}} J(x - y)\frac{\psi(t, y) - u(t, x)}{|\psi(t, y) - u(t, x)|}\, dy, \quad (t, x) \in]0, T[\times\Omega, \\ u(0, x) = u_0(x), \quad x \in \Omega. \end{cases}$$

We give the following definition of what we understand by a solution of $P_1^J(u_0, \psi)$.

DEFINITION 1.4. *A solution of* $P_1^J(u_0, \psi)$ *in* $[0, T]$ *is a function*

$$u \in C\left([0, T]; L^1(\Omega)\right) \cap W^{1,1}\left(]0, T[; L^1(\Omega)\right),$$

which satisfies $u(0, x) = u_0(x)$ *a.e.* $x \in \Omega$ *and*

$$u_t(t, x) = \int_{\Omega_J} J(x - y)g(t, x, y)\, dy \qquad \textit{a.e. in }]0, T[\times\Omega,$$

for some $g \in L^\infty(0,T;L^\infty(\Omega_J \times \Omega))$ *with* $\|g\|_\infty \le 1$ *such that for almost every* $t \in]0,T[$, $g(t,x,y) = -g(t,y,x)$ *and*

$$J(x-y)g(t,x,y) \in J(x-y)\mathrm{sign}(u(t,y)-u(t,x)), \qquad (x,y) \in \Omega \times \Omega,$$

$$J(x-y)g(t,x,y) \in J(x-y)\mathrm{sign}(\psi(y)-u(t,x)), \qquad (x,y) \in \Omega \times \left(\Omega_J \setminus \overline{\Omega}\right).$$

Here, sign is the multivalued function defined by

$$\mathrm{sign}(r) := \begin{cases} 1 & \text{if } \ r > 0 \\ [-1,1] & \text{if } \ r = 0 \\ -1 & \text{if } \ r < 0. \end{cases}$$

We use sign_0 to denote the univalued function

$$\mathrm{sign}_0(r) := \begin{cases} 1 & \text{if } \ r > 0 \\ 0 & \text{if } \ r = 0 \\ -1 & \text{if } \ r < 0. \end{cases}$$

To get the existence and uniqueness of these kinds of solutions, the idea is to take the limit as $p \searrow 1$ of solutions to P_p^J with $p > 1$.

THEOREM 1.5. *Let* $u_0 \in L^1(\Omega)$ *and* $\psi \in L^1(\Omega_J \setminus \overline{\Omega})$. *Then, there exists a unique solution to* $P_1^J(u_0)$ *in the sense of Definition* 1.4. *Moreover, if* $u_{i0} \in L^1(\Omega)$ *and* u_i *are solutions in* $[0,T]$ *of* $P_1^J(u_{i0})$, $i = 1,2$. *Then*

$$\int_\Omega (u_1(t) - u_2(t))^+ \le \int_\Omega (u_{10} - u_{20})^+ \quad \text{for every } t \in [0,T].$$

In this case we can rescale the kernel as in (1.2) in order to obtain convergence of the solutions of the corresponding rescaled problem to the solution of the Dirichlet problem for the total variational flow, that is,

$$D_1(u_0,\tilde{\psi}) \qquad \begin{cases} u_t = \mathrm{div}\left(\frac{Du}{|Du|}\right) & \text{in }]0,T[\times\Omega, \\ u = \tilde{\psi} & \text{on }]0,T[\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

THEOREM 1.6. *Let* Ω *be a smooth bounded domain in* $\mathbb{R}^N$. *Assume* $J(x) \ge J(y)$ *if* $|x| \le |y|$. *Let* $T > 0$, $u_0 \in L^1(\Omega)$, $\tilde{\psi} \in L^\infty(\partial\Omega)$, *and* $\psi \in W^{1,1}(\Omega_J \setminus \overline{\Omega}) \cap L^\infty(\Omega_J \setminus \overline{\Omega})$ *such that the trace* $\psi|_{\partial\Omega} = \tilde{\psi}$. *Let* u_ε *be the unique solution of* $P_1^{J_{1,\varepsilon}}(u_0,\psi)$. *Then, if* u *is the unique solution of* $D_1(u_0,\tilde{\psi})$ *(see section* 3.2*)*,

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon(t,.) - u(t,.)\|_{L^1(\Omega)} = 0.$$

Finally, the third goal of this paper is to study the limit case $p = +\infty$, which has to be understood as the limit of our nonlocal evolution problems as $p \to +\infty$ (see section 4). In this case we recover a nonlocal model for the evolution of sandpiles which is the nonlocal version of the Prigozhin model [35]. Then, the nonlocal limit problem with source for $p = +\infty$ can be written as

$$P_\infty^J(u_0,\psi,f) \qquad \begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial G_{\infty,\psi}^J(u(t)), & \text{a.e. } \ t \in]0,T[, \\ u(0,x) = u_0(x), \end{cases}$$

where $G^J_{\infty,\psi}$ is the functional

$$G^J_{\infty,\psi}(u) = \begin{cases} 0 & \text{if } \ |u(x)-u(y)| \le 1, \text{ for } x,y \in \Omega \\ & \text{and } |\psi(y)-u(x)| \le 1, \quad \text{for } x \in \Omega, y \in \Omega_J \setminus \overline{\Omega}, \\ & \text{with } x-y \in \operatorname{supp}(J) \\ +\infty & \text{in the other case,} \end{cases}$$

that is, $G^J_{\infty,\psi} = I_{K^J_{\infty,\psi}}$, the indicator function of the set

$$K^J_{\infty,\psi} := \left\{ u \in L^2(\Omega) \ : \ \begin{array}{l} |u(x)-u(y)| \le 1, x,y \in \Omega \\ \text{and } |\psi(y)-u(x)| \le 1, \quad \text{for } x \in \Omega, y \in \Omega_J \setminus \overline{\Omega}, \\ \text{with } x-y \in \operatorname{supp}(J) \end{array} \right\}.$$

More precisely, we obtain the following result.

THEOREM 1.7. *Let $\psi \in L^\infty(\Omega_J \setminus \overline{\Omega})$ such that $K^J_{\infty,\psi} \neq \emptyset$. Let $T > 0$, $f \in L^2(0,T;\cap_{q\ge 2} L^q(\Omega))$, $u_0 \in \cap_{q\ge 2} L^q(\Omega)$ such that $u_0 \in K^J_{\infty,\psi}$, and u_p, $p \ge 2$, the unique solution of the nonlocal p-Laplacian with a source term f, $P^J_p(u_0,\psi,f)$ (see section 4). Then, if u_∞ is the unique solution to $P^J_\infty(u_0,\psi,f)$,*

$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(t,\cdot) - u_\infty(t,\cdot)\|_{L^2(\Omega)} = 0.$$

Our next step is to rescale the kernel J appropriately and take the limit as the scaling parameter goes to zero. We will suppose that Ω is convex and ψ verifies $\|\nabla\psi\|_\infty \le 1$. For $\varepsilon > 0$, we rescale the functional $G^J_{\infty,\psi}$ as follows:

$$G^\varepsilon_{\infty,\psi}(u) = \begin{cases} 0 & \text{if } |u(x)-u(y)| \le \varepsilon, \text{ for } x,y \in \Omega \\ & \text{and } |\psi(y)-u(x)| \le \varepsilon, \quad \text{for } x \in \Omega, y \in \Omega_J \setminus \overline{\Omega}, \\ & \text{with } |x-y| \le \varepsilon \\ +\infty & \text{in the other case,} \end{cases}$$

that is, $G^\varepsilon_{\infty,\psi} = I_{K^\varepsilon_{\infty,\psi}}$, where

$$K^\varepsilon_{\infty,\psi} := \left\{ u \in L^2(\Omega) \ : \ \begin{array}{l} |u(x)-u(y)| \le \varepsilon, x,y \in \Omega \\ \text{and } |\psi(y)-u(x)| \le \varepsilon, \quad \text{for } x \in \Omega, y \in \Omega_J \setminus \overline{\Omega}, \\ \text{with } |x-y| \le \varepsilon \end{array} \right\}.$$

Consider the gradient flow associated to the functional $G^\varepsilon_{\infty,\psi}$

$$P^\varepsilon_\infty(u_0,\psi,f) \quad \begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial I_{K^\varepsilon_{\infty,\psi}}(u(t)), & \text{a.e. } t \in]0,T[, \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases}$$

and the limit problem

$$P_\infty(u_0,\psi,f) \quad \begin{cases} f(t,\cdot) - u_{\infty,t} \in \partial I_{K_\psi}(u_\infty), & \text{a.e. } t \in]0,T[, \\ u_\infty(0,x) = u_0(x), & \text{in } \Omega, \end{cases}$$

where

$$K_\psi := \left\{ u \in W^{1,\infty}(\Omega) \ : \ \|\nabla u\|_\infty \le 1, \ u|_{\partial\Omega} = \psi|_{\partial\Omega} \right\}.$$

Now we state our result concerning the limit as $\varepsilon \to 0$ for the sandpile model ($p = +\infty$).

THEOREM 1.8. *Assume Ω is a convex bounded domain in $\mathbb{R}^N$. Let $T > 0$, $f \in L^2(0,T;L^2(\Omega))$, $\psi \in W^{1,\infty}(\Omega_J \setminus \overline{\Omega})$ such that $\|\nabla\psi\|_\infty \leq 1$, $u_0 \in W^{1,\infty}(\Omega)$ such that $\|\nabla u_0\|_\infty \leq 1$ and $u_0|_{\partial\Omega} = \psi|_{\partial\Omega}$ (this means $u_0 \in K_\psi$), and consider $u_{\infty,\varepsilon}$ the unique solution of $P_\infty^\varepsilon(u_0,\psi,f)$. Then, if v_∞ is the unique solution of $P_\infty(u_0,\psi,f)$, we have*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_{\infty,\varepsilon}(t,\cdot) - v_\infty(t,\cdot)\|_{L^2(\Omega)} = 0.$$

Closely related to the present work are [5] and [6] where the homogeneous Neumann problem and its limit as p goes to infinity or to one are considered. The difference here is that we are now considering Dirichlet boundary conditions, not only the homogeneous case, but also the nonhomogeneous case, and this introduces new difficulties specially when one tries to recover the local models when $\varepsilon \to 0$. Remark that in our nonlocal formulation we are not imposing any continuity between the values of u inside Ω and outside it, ψ. However, when dealing with local problems usually the boundary datum is taken in the sense of traces, that is, $u|_{\partial\Omega} = \psi$. Recovering this condition as $\varepsilon \to 0$ is one of the main contributions of the present work.

Note that, as it happens for the local p-Laplacian, the Dirichlet problem can be written as a Neumann problem with a particular flux that depends on the solution itself. Indeed, the problem $P_p^J(u_0,\psi)$ can be written as

$$\begin{cases} u_t(t,x) = \displaystyle\int_\Omega J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))\,dy + \varphi(x,u(x)) \\ \hfill (t,x) \in]0,T[\times\Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

where

$$\varphi(x,u(x)) = \int_{\Omega_J\setminus\overline{\Omega}} J(x-y)|\psi(t,y)-u(t,x)|^{p-2}(\psi(t,y)-u(t,x))\,dy.$$

In the homogeneous case, $\psi \equiv 0$,

$$\varphi(x,u(x)) = -\left(\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)\,dy\right) |u(t,x)|^{p-2}u(t,x).$$

This problem is a nonhomogeneous Neumann problem (see [5]) with a prescribed flux given by φ.

Let us finish the introduction by collecting some notations and results that will be used in the sequel. Following [7] (see also [2]), let

$$X(\Omega) = \left\{ z \in L^\infty(\Omega,\mathbb{R}^n) \ : \ \operatorname{div}(z) \in L^1(\Omega) \right\}. \tag{1.4}$$

If $z \in X(\Omega)$ and $w \in BV(\Omega)\cap L^\infty(\Omega)$, we define the functional $(z,Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (z,Dw),\varphi\rangle = -\int_\Omega w\,\varphi\operatorname{div}(z)\,dx - \int_\Omega w\,z\cdot\nabla\varphi\,dx. \tag{1.5}$$

Then (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w \, dx \tag{1.6}$$

for all $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and

$$\left| \int_B (z, Dw) \right| \leq \int_B |(z, Dw)| \leq \|z\|_\infty \int_B \|Dw\| \tag{1.7}$$

for any Borel set $B \subseteq \Omega$.

In [7], a weak trace on $\partial\Omega$ of the normal component of $z \in X(\Omega)$ is defined. Concretely, it is proved that there exists a linear operator $\gamma : X(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(z)\|_\infty \leq \|z\|_\infty,$$

and

$$\gamma(z)(x) = z(x) \cdot \nu(x) \quad \text{ for all } x \in \partial\Omega \text{ if } z \in C^1\left(\overline{\Omega}, \mathbb{R}^N\right).$$

We shall denote $\gamma(z)(x)$ by $[z, \nu](x)$. Moreover, the following *Green's formula*, relating the function $[z, \nu]$ and the measure (z, Dw), for $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$, is established:

$$\int_\Omega w \, \operatorname{div}(z) \, dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z, \nu] w \, d\mathcal{H}^{N-1}. \tag{1.8}$$

Organization of the paper. The rest of the paper is organized as follows. In the second section we prove the existence and uniqueness of strong solutions for the nonlocal p-Laplacian problem with Dirichlet boundary conditions for $p > 1$ and we show that our model approaches local p-Laplacian evolution equation with Dirichlet boundary condition. In section 3 we study the Dirichlet problem for the nonlocal total variation flow, proving convergence to the local model when the problem is rescaled appropriately as well. Finally, in section 4 we study the case $p = \infty$, obtaining a model for sandpiles with Dirichlet boundary conditions.

2. The case $p > 1$.

2.1. Existence of solutions for the nonlocal problems. We first study $P_p^J(u_0, \psi)$ from the point of view of nonlinear semigroup theory ([15], [28]). For that we introduce in $L^1(\Omega)$ the following operator associated with our problem.

DEFINITION 2.1. For $1 < p < +\infty$ and $\psi : \Omega_J \setminus \overline{\Omega} \to \mathbb{R}$, such that $|\psi|^{p-1} \in L^1(\Omega_J \setminus \overline{\Omega})$, we define in $L^1(\Omega)$ the operator $B_{p,\psi}^J$ by

$$\begin{aligned} B_{p,\psi}^J(u)(x) \quad &= -\int_\Omega J(x-y)|u(y) - u(x)|^{p-2}(u(y) - u(x)) \, dy \\ &\quad - \int_{\Omega_J \setminus \overline{\Omega}} J(x-y)|\psi(y) - u(x)|^{p-2}(\psi(y) - u(x)) \, dy, \qquad x \in \Omega. \end{aligned}$$

Remark 2.2. (i). We will set overall the section,

$$u_\psi(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ \psi(x) & \text{if } x \in \Omega_J \setminus \overline{\Omega}, \\ 0 & \text{if } x \notin \Omega_J. \end{cases}$$

Therefore, we can rewrite

$$B^J_{p,\psi}(u)(x) = -\int_{\Omega_J} J(x-y)|u_\psi(y)-u(x)|^{p-2}(u_\psi(y)-u(x))\,dy, \qquad x \in \Omega.$$

(ii) If $\psi = 0$, then

$$B^J_{p,0}(u)(x) = -\int_{\Omega} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy + \left(\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)dy\right)\,|u(x)|^{p-2}u(x)\,, \qquad x \in \Omega.$$

Remark 2.3. It is easy to see that
(i) If $\psi = 0$, $B^J_{p,0}$ is positively homogeneous of degree $p-1$,
(ii) $L^{p-1}(\Omega) \subset \mathrm{Dom}(B^J_{p,\psi})$, if $p > 2$.
(iii) For $1 < p \le 2$, $\mathrm{Dom}(B^J_{p,\psi}) = L^1(\Omega)$ and $B^J_{p,\psi}$ is closed in $L^1(\Omega)\times L^1(\Omega)$.
We have the following monotonicity lemma, whose proof is straightforward.

LEMMA 2.4. *Let $1 < p < +\infty$, $\psi : \Omega_J \setminus \overline{\Omega} \to \mathbb{R}$, $|\psi|^{p-1} \in L^1(\Omega_J\setminus\overline{\Omega})$, and $T : \mathbb{R} \to \mathbb{R}$ a nondecreasing function. Then,*

(i) *for every u, $v \in L^p(\Omega)$ such that $T(u-v) \in L^p(\Omega)$, it holds*

$$\begin{aligned}
&\int_\Omega \left(B^J_{p,\psi}u(x) - B^J_{p,\psi}v(x)\right) T(u(x)-v(x))dx \\
&= \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)\left(T(u_\psi(y)-v_\psi(y)) - T(u_\psi(x)-v_\psi(x))\right) \\
&\qquad \times \big(|u_\psi(y)-u_\psi(x)|^{p-2}(u_\psi(y)-u_\psi(x)) \\
&\qquad\qquad - |v_\psi(y)-v_\psi(x)|^{p-2}(v_\psi(y)-v_\psi(x))\big)\,dydx.
\end{aligned} \tag{2.1}$$

(ii) *Moreover, if T is bounded,* (2.1) *holds for u, $v \in Dom(B^J_{p,\psi})$.*

We have the following Poincaré's type inequality.

PROPOSITION 2.5. *Given Ω a bounded domain in $\mathbb{R}^N$, $J : \mathbb{R}^N \to \mathbb{R}$ a nonnegative, radial, continuous function, such that $\int_{\mathbb{R}^N} J(z)\,dz > 0$, $p \ge 1$ and $\psi \in L^p(\Omega_J\setminus\overline{\Omega})$, there exists $\lambda = \lambda(J,\Omega,p) > 0$ such that*

$$\lambda\int_\Omega |u(x)|^p\,dx \le \int_\Omega\int_{\Omega_J} J(x-y)|u_\psi(y)-u(x)|^p\,dy\,dx + \int_{\Omega_J\setminus\overline{\Omega}} |\psi(y)|^p\,dy \tag{2.2}$$

for all $u \in L^p(\Omega)$.

Proof. First, let us assume that there exist $r,\alpha > 0$ such that $J(x) \ge \alpha$ in $B(0,r)$. Let

$$B_0 = \{x \in \Omega_J\setminus\overline{\Omega}\ :\ d(x,\Omega) \le r/2\},$$
$$B_1 = \{x \in \Omega\ :\ d(x,B_0) \le r/2\}\,,$$
$$B_j = \left\{x \in \Omega\setminus\cup_{k=1}^{j-1}B_k\ :\ d(x,B_{j-1}) \le r/2\right\}, \quad j = 2,3,\ldots$$

Observe that we can cover Ω by a finite number of nonnull sets $\{B_j\}_{j=1}^{l_r}$. Now

$$\int_\Omega\int_{\Omega_J} J(x-y)|u_\psi(y)-u(x)|^p\,dy\,dx \ge \int_{B_j}\int_{B_{j-1}} J(x-y)|u_\psi(y)-u(x)|^p\,dy\,dx,$$

$j = 1, \ldots, l_r$, and

$$\begin{aligned}
&\int_{B_j}\int_{B_{j-1}} J(x-y)|u_\psi(y)-u(x)|^p\,dy\,dx \\
&\geq \frac{1}{2^p}\int_{B_j}\int_{B_{j-1}} J(x-y)|u(x)|^p\,dy\,dx - \int_{B_j}\int_{B_{j-1}} J(x-y)|u_\psi(y)|^p\,dy\,dx \\
&= \frac{1}{2^p}\int_{B_j}\left(\int_{B_{j-1}} J(x-y)\,dy\right)|u(x)|^p\,dx - \int_{B_{j-1}}\left(\int_{B_j} J(x-y)\,dx\right)|u_\psi(y)|^p\,dy \\
&\geq \frac{1}{2^p}\min_{x\in\overline{B_j}}\int_{B_{j-1}} J(x-y)dy\int_{B_j}|u(x)|^p\,dx - \beta\int_{B_{j-1}}|u_\psi(y)|^p\,dy,
\end{aligned}$$

where $\beta = \int_{\mathbb{R}^N} J(x)dx$. Hence

$$\int_\Omega\int_{\Omega_J} J(x-y)|u_\psi(y)-u(x)|^p\,dy\,dx \geq \alpha_j\int_{B_j}|u(x)|^p\,dx - \beta\int_{B_{j-1}}|u_\psi(y)|^p\,dy,$$

where

$$\alpha_j = \frac{1}{2^p}\min_{x\in\overline{B_j}}\int_{B_{j-1}} J(x-y)dy > 0.$$

Therefore, since $u_\psi(y) = \psi(y)$ if $y \in B_0$, $u_\psi(y) = u(y)$ if $y \in B_j$, $j = 1, \ldots, l_r$, $B_j \cap B_i = \emptyset$, for all $i \neq j$ and $|\Omega \setminus \cup_{j=1}^{j_r} B_j| = 0$, it is easy to see that there exists $\hat{\lambda} = \hat{\lambda}(J, \Omega, p) > 0$ such that

$$\int_\Omega |u|^p \leq \hat{\lambda}\int_\Omega\int_{\Omega_J} J(x-y)|u_\psi(y)-u(x)|^p\,dy\,dx + \hat{\lambda}\int_{B_0}|\psi|^p.$$

The proof is finished by taking $\lambda = \hat{\lambda}^{-1}$.

In the general case we have that there exist $\mathtt{a} \geq 0$ and $r, \alpha > 0$ such that

$$J(x) \geq \alpha \text{ in the annulus } A(0, \mathtt{a}, r). \tag{2.3}$$

In this case we proceed as before with the same choice of the sets B_j for $j \geq 0$ and

$$B_{-j} = \left\{x \in \Omega_J \setminus \left(\Omega \cup \cup_{k=0}^{j-1} B_{-k}\right) \ :\ d(x, B_{-j+1}) \leq r/2\right\}, \quad j = 1, 2, 3, \ldots$$

Observe that for each B_j, $j \geq 1$, there exists B_{j^e} with $j^e < j$ and such that

$$|(x + A(0, \mathtt{a}, r)) \cap B_{j^e}| > 0 \quad \forall x \in \overline{B}_j. \tag{2.4}$$

With this choice of B_j and taking into account (2.3) and (2.4), as before, we obtain

$$\begin{aligned}
\int_\Omega\int_{\Omega_J} J(x-y)|u_\psi(y)-u(x)|^p\,dy\,dx &\geq \int_{B_j}\int_{B_{j^e}} J(x-y)|u_\psi(y)-u(x)|^p\,dy\,dx \\
&\geq \alpha_j\int_{B_j}|u(x)|^p\,dx - \beta\int_{B_{j^e}}|u_\psi(y)|^p\,dy,
\end{aligned}$$

$j = 1, \ldots, l_r$, where

$$\alpha_j = \frac{1}{2^p}\min_{x\in\overline{B_j}}\int_{B_{j^e}} J(x-y)dy > 0$$

and $\beta = \int_{\mathbb{R}^N} J(x)dx$. And we conclude as before. $\square$

Remark 2.6. Note that in [5] it is proved a Poincare's type inequality for Neumann boundary conditions, but assuming that $J(0) > 0$ (otherwise there is a counterexample). Surprisingly, for the Dirichlet problem we do not need positivity at the origin for J. This is due to the fact that for the Dirichlet problem the outside values influence the inside values.

In the next result we prove that $B^J_{p,\psi}$ is a completely accretive operator (see [14]) and verifies a range condition. In short, this means that for any $\phi \in L^p(\Omega)$ there is a unique solution of the problem $u + B^J_{p,\psi} u = \phi$ and the resolvent $(I + B^J_{p,\psi})^{-1}$ is a contraction in $L^q(\Omega)$ for all $1 \le q \le +\infty$.

THEOREM 2.7. *Let $1 < p < +\infty$. For $\psi \in L^p(\Omega_J \setminus \overline{\Omega})$, the operator $B^J_{p,\psi}$ is completely accretive and verifies the range condition*

$$L^p(\Omega) \subset Ran\left(I + B^J_{p,\psi}\right). \tag{2.5}$$

Proof. Given $u_i \in \mathrm{Dom}(B^J_{p,\psi})$, $i = 1, 2$, by the monotonicity Lemma 2.4, for any $q \in C^\infty(\mathbb{R})$, $0 \le q' \le 1$, supp(q') compact, $0 \notin$ supp(q), we have that

$$\int_\Omega \left(B^J_{p,\psi} u_1(x) - B^J_{p,\psi} u_2(x)\right) q(u_1(x) - u_2(x))\, dx \ge 0,$$

from where it follows that $B^J_{p,\psi}$ is a completely accretive operator (see [14]).

To show that $B^J_{p,\psi}$ satisfies the range condition we have to prove that for any $\phi \in L^p(\Omega)$ there exists $u \in \mathrm{Dom}(B^J_{p,\psi})$ such that $\phi = u + B^J_{p,\psi} u$.

Assume first $p \ge 2$. Let $\phi \in L^p(\Omega)$ and set

$$K = \left\{ w \in L^p(\Omega_J) \ : \ w = \psi \text{ in } \Omega_J \setminus \overline{\Omega} \right\}.$$

We consider the continuous monotone operator $A : K \to L^{p'}(\Omega_J)$ defined by

$$A(w)(x) := w(x) - \int_{\Omega_J} J(x-y)|w(y) - w(x)|^{p-2}(w(y) - w(x))\, dy.$$

A is coercive in $L^p(\Omega_J)$. In fact, by Proposition 2.5, for any $w \in K$,

$$\begin{aligned}
&\int_{\Omega_J} A(w)w = \int_{\Omega_J} w^2 - \int_{\Omega_J}\int_{\Omega_J} J(x-y)|w(y) - w(x)|^{p-2}(w(y) - w(x))\, dy w(x) dx \\
&\ge \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|w(y) - w(x)|^p\, dydx \\
&\ge \frac{1}{2}\int_{\Omega}\int_{\Omega_J} J(x-y)|w_\psi(y) - w(x)|^p\, dydx \ge \frac{\lambda}{2}\|w\|^p_{L^p(\Omega)} - \frac{1}{2}\int_{\Omega_J\setminus\overline{\Omega}} |\psi|^p.
\end{aligned}$$

Therefore,

$$\lim_{\substack{\|w\|_{L^p(\Omega_J)} \to +\infty \\ w \in K}} \frac{\displaystyle\int_{\Omega_J} A(w)w}{\|w\|_{L^p(\Omega_J)}} = +\infty.$$

Now, since $p \ge 2$, we have the function $\phi_\psi \in L^{p'}(\Omega_J)$. Then, applying [32, Corollary III.1.8] to the operator $B(w) := A(w) - \phi_\psi$, we get there exists $w \in K$, such that

$$w(x) - \int_{\Omega_J} J(x-y)|w(y) - w(x)|^{p-2}(w(y) - w(x))\, dy = \phi_\psi(x) \qquad \text{for all } \ x \in \Omega_J.$$

Hence, $u := w_{|\Omega}$ satisfies

$$u(x) - \int_{\Omega_J} J(x-y)|u_\psi(y) - u(x)|^{p-2}(u_\psi(y) - u(x))\, dy = \phi(x) \qquad \text{for all} \quad x \in \Omega,$$

and, consequently, $\phi = u + B^J_{p,\psi} u$.

Suppose now $1 < p < 2$. By the results in [5], we know that the operator

$$B^J_p u(x) = -\int_\Omega J(x-y)|u(y) - u(x)|^{p-2}(u(y) - u(x))\, dy$$

is m-accretive in $L^1(\Omega)$ and satisfies what is called property (M_0); that is, for any $q \in C^\infty(\mathbb{R})$, $0 \le q' \le 1$, $\operatorname{supp}(q')$ compact, $0 \notin \operatorname{supp}(q)$, and $(u, v) \in B^J_p$,

$$\int_\Omega q(u)v \ge 0.$$

On the other hand,

$$\varphi(x, r) = -\int_{\Omega_J \setminus \overline{\Omega}} J(x-y)|\psi(y) - r|^{p-2}(\psi(y) - r)\, dy$$

is continuous and nondecreasing in r for almost every $x \in \Omega$, and an $L^1(\Omega)$ function for all r. Therefore, by [3, Theorem 3.1], $B^J_{p,\psi} u(x) = B^J_p u(x) + \varphi(x, u(x))$ is m-accretive in $L^1(\Omega)$. $\square$

Remark 2.8. If $\mathcal{B}^J_{p,\psi}$ denotes the closure of $B^J_{p,\psi}$ in $L^1(\Omega)$, by Theorem 2.7, we have $\mathcal{B}^J_{p,\psi}$ is m-completely accretive in $L^1(\Omega)$ (see [14]). Therefore, by the nonlinear semigroup theory (see [15] and [14]), there exists an unique mild-solution of the abstract Cauchy problem

$$\begin{cases} u'(t) + B^J_{p,\psi} u(t) = 0, & t \in (0, T), \\ u(0) = u_0, \end{cases} \tag{2.6}$$

given by the Crandall–Liggett exponential formula

$$\mathrm{e}^{-t\mathcal{B}^J_{p,\psi}} u_0 = \lim_n \left(I + \frac{t}{n} \mathcal{B}^J_{p,\psi} \right)^{-n} u_0.$$

Now, due to regularity results for mild solutions, under certain hypothesis, this mild solution is a strong solution of the abstract Cauchy problem (2.6) (see [14]) which means, for our problem $P^J_p(u_0, \psi)$, a solution in the sense of Definition 1.1.

The following result states the existence and uniqueness results for $P^J_p(u_0, \psi)$. From it, Theorem 1.2 can be derived.

THEOREM 2.9. *Assume $p > 1$. Let $T > 0$, $\psi \in L^p(\Omega_J \setminus \overline{\Omega})$, and $u_0 \in L^1(\Omega)$. Then, there exists a unique mild-solution u of* (2.6)*. Moreover,*

(1) *if $u_0 \in L^p(\Omega)$, the unique mild solution u of* (2.6) *is a solution of $P^J_p(u_0, \psi)$ in the sense of Definition* 1.1*. If $1 < p \le 2$, this is true for any $u_0 \in L^1(\Omega)$ and any ψ such that $|\psi|^{p-1} \in L^1(\Omega_J \setminus \overline{\Omega})$.*

(2) *Let $u_{i0} \in L^1(\Omega)$ and u_i a solution in $[0, T]$ of $P^J_p(u_{i0})$, $i = 1, 2$. Then*

$$\int_\Omega (u_1(t) - u_2(t))^+ \le \int_\Omega (u_{10} - u_{20})^+ \quad \textit{for every } t \in]0, T[.$$

Moreover, for $q \in [1, +\infty]$, *if* $u_{i0} \in L^q(\Omega)$, $i = 1, 2$, *then*

$$\|u_1(t) - u_2(t)\|_{L^q(\Omega)} \leq \|u_{10} - u_{20}\|_{L^q(\Omega)} \quad \textit{for every } t \in]0, T[.$$

Proof. As a consequence of Theorem 2.7 we get the existence of mild solution of (2.6) (see Remark 2.8). Now, due to the complete accretivity of $B^J_{p,\psi}$ and the range condition (2.5), by regularity results for mild solutions (see [14]), $u(t)$ is a strong solution, that is, a solution of $P^J_p(u_0, \psi)$ in the sense of Definition 1.1. Moreover, in the case $1 < p \leq 2$, since $\mathrm{Dom}(B^J_{p,\psi}) = L^1(\Omega)$ and $B^J_{p,\psi}$ is closed in $L^1(\Omega) \times L^1(\Omega)$, the result holds for L^1-data. Finally, the contraction principle is a consequence of the general nonlinear semigroup theory ([15], [28]). $\square$

2.2. Convergence to the p-Laplacian. Our main goal in this section is to show that the solution to the Dirichlet problem for the p-Laplacian equation $D_p(u_0, \tilde{\psi})$ can be approximated by solutions to suitable nonlocal Dirichlet problems $P^J_p(u_0, \psi)$.

Let us first recall the following result from [5]. For a function g defined in a set D, we define

$$\overline{g}(x) = \begin{cases} g(x) & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases}$$

and we denote by χ_D the characteristic function of D.

PROPOSITION 2.10 ([5]). *Let* $1 \leq q < +\infty$, D *a bounded domain in* $\mathbb{R}^N$, $\rho : \mathbb{R}^N \to \mathbb{R}$ *a nonnegative continuous radial function with compact support, nonidentically zero, and* $\rho_n(x) := n^N \rho(nx)$. *Let* $\{f_n\}$ *be a sequence of functions in* $L^q(D)$ *such that*

$$\int_D \int_D |f_n(y) - f_n(x)|^q \rho_n(y - x)\, dx\, dy \leq M \frac{1}{n^q}. \tag{2.7}$$

1. *If* $\{f_n\}$ *is weakly convergent in* $L^q(D)$ *to* f, *then*
   (i) *if* $q > 1$, $f \in W^{1,q}(D)$ *and moreover*

$$(\rho(z))^{1/q} \chi_D\left(x + \frac{1}{n}z\right) \frac{\overline{f}_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \rightharpoonup (\rho(z))^{1/q}\, z \cdot \nabla f(x)$$

weakly in $L^q(D) \times L^q(\mathbb{R}^N)$;
   (ii) *if* $q = 1$, $f \in BV(D)$ *and moreover*

$$\rho(z)\chi_D\left(\cdot + \frac{1}{n}z\right) \frac{\overline{f}_n\left(\cdot + \frac{1}{n}z\right) - f_n(\cdot)}{1/n} \rightharpoonup \rho(z) z \cdot Df$$

weakly as measures.
2. *Assume* D *is a smooth bounded domain in* $\mathbb{R}^N$ *and* $\rho(x) \geq \rho(y)$ *if* $|x| \leq |y|$. *Then* $\{f_n\}$ *is relatively compact in* $L^q(D)$ *and, consequently, there exists a subsequence* $\{f_{n_k}\}$ *such that*
   (i) *if* $q > 1$, $f_{n_k} \to f$ *in* $L^q(D)$ *with* $f \in W^{1,q}(D)$;
   (ii) *If* $q = 1$, $f_{n_k} \to f$ *in* $L^1(D)$ *with* $f \in BV(D)$.

Let us now recall some results about the p-Laplacian equation

$$D_p(u_0, \tilde{\psi}) \quad \begin{cases} u_t = \Delta_p u & \text{in }]0, T[\times\Omega, \\ u = \tilde{\psi} & \text{on }]0, T[\times\partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

In the case $\tilde\psi \in W^{1/p',p}(\partial\Omega)$, associated to the p-Laplacian with nonhomogeneous Dirichlet boundary condition, in [2] it is defined the operator $A_{p,\psi} \subset L^1(\Omega)\times L^1(\Omega)$ as $(u,\hat u)\in A_{p,\tilde\psi}$ if and only if $\hat u \in L^1(\Omega)$, $u \in W^{1,p}_{\tilde\psi}(\Omega) := \{u \in W^{1,p}(\Omega) : u|_{\partial\Omega} = \tilde\psi \ \mathcal{H}^{N-1} - a.e. \text{ on } \partial\Omega\}$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(u-v) \le \int_\Omega \hat u(u-v) \quad \text{for every } \ v \in W^{1,p}_{\tilde\psi}(\Omega)\cap L^\infty(\Omega).$$

This inequality is equivalent to

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla w = \int_\Omega \hat u w \quad \text{for every } \ w \in W^{1,p}_0(\Omega)\cap L^\infty(\Omega).$$

Moreover, for $\tilde\psi \in W^{1/p',p}(\partial\Omega)\cap L^\infty(\partial\Omega)$, $A_{p,\tilde\psi}$ is proved to be a completely accretive operator in $L^1(\Omega)$, satisfying the range condition $L^\infty(\Omega)\subset \mathrm{Ran}(I + A_{p,\tilde\psi})$, and it is easy to see that $\overline{\mathrm{D}(A_{p,\tilde\psi})}^{L^1(\Omega)} = L^1(\Omega)$. Therefore, its closure $\mathcal{A}_{p,\tilde\psi}$ in $L^1(\Omega)\times L^1(\Omega)$ is an m-completely accretive operator in $L^1(\Omega)$. Consequently, for any $u_0 \in L^1(\Omega)$ there exists a unique mild solution $u(t) = e^{-t\mathcal{A}_{p,\tilde\psi}}u_0$ of the abstract Cauchy problem associated to $D_p(u_0,\tilde\psi)$, given by Crandall–Liggett's exponential formula. Due to the complete accretivity of the operator $\mathcal{A}_{p,\tilde\psi}$, in the case $u_0 \in D(\mathcal{A}_{p,\tilde\psi})$ this mild solution is the unique strong solution of problem $D_p(u_0,\tilde\psi)$.

In the homogeneous case $\tilde\psi = 0$, due to the results in [13], we can say that for any $u_0\in L^1(\Omega)$, the mild solution $u(t) = e^{-t\mathcal{A}_{p,0}}u_0$ is the unique entropy solution of problem $D_p(u_0,0)$.

For given $p>1$ and J, we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right), \qquad \text{where} \qquad C^{-1}_{J,p} := \frac12\int_{\mathbb{R}^N} J(z)|z_N|^p\,dz$$

is a normalizing constant in order to obtain the p-Laplacian in the limit instead of a multiple of it.

Proposition 2.11. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let $\tilde\psi \in W^{1/p',p}(\partial\Omega)\cap L^\infty(\partial\Omega)$. Let $\psi \in W^{1,p}(\Omega_J)\cap L^\infty(\Omega_J)$ such that $\psi|_{\partial\Omega} = \tilde\psi$. Assume $J(x)\ge J(y)$ if $|x|\le|y|$. Then, for any $\phi\in L^\infty(\Omega)$,*

$$\left(I + B^{J_{p,\varepsilon}}_{p,\psi}\right)^{-1}\phi \to \left(I + A_{p,\tilde\psi}\right)^{-1}\phi \quad \textit{in } L^p(\Omega) \textit{ as } \varepsilon\to 0. \tag{2.8}$$

Proof. We denote

$$\Omega_\varepsilon := \Omega_{J_{p,\varepsilon}} = \Omega + \mathrm{supp}(J_{p,\varepsilon}).$$

For $\varepsilon>0$ small, let $u_\varepsilon = \left(I + B^{J_{p,\varepsilon}}_{p,\psi}\right)^{-1}\phi$. Then,

$$\begin{aligned}\int_\Omega u_\varepsilon v - \frac{C_{J,p}}{\varepsilon^{p+N}} \quad &\int_\Omega\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)|(u_\varepsilon)_\psi(y) - u_\varepsilon(x)|^{p-2}\\ &\times ((u_\varepsilon)_\psi(y) - u_\varepsilon(x))\,dy\,v(x)\,dx = \int_\Omega \phi v\end{aligned} \tag{2.9}$$

for every $v\in L^\infty(\Omega)$.

Let $M := \max\{\|\phi\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega_J)}\}$. Taking $v = (u_\varepsilon - M)^+$ in (2.9), we get

$$\begin{aligned}\int_\Omega u_\varepsilon(x)(u_\varepsilon(x) - M)^+ dx - \frac{C_{J,p}}{\varepsilon^{p+N}} \int_\Omega \int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) |(u_\varepsilon)_\psi(y) - u_\varepsilon(x)|^{p-2} \\ \times ((u_\varepsilon)_\psi(y) - u_\varepsilon(x))\, dy (u_\varepsilon(x) - M)^+\, dx \\ = \int_\Omega \phi(x)(u_\varepsilon(x) - M)^+\, dx.\end{aligned}$$

Now,

$$\begin{aligned}&-\frac{C_{J,p}}{\varepsilon^{p+N}} \int_\Omega \int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) |(u_\varepsilon)_\psi(y) - u_\varepsilon(x)|^{p-2}((u_\varepsilon)_\psi(y) - u_\varepsilon(x))\, dy \\ &\qquad \times (u_\varepsilon(x) - M)^+ dx \\ &= -\frac{C_{J,p}}{\varepsilon^{p+N}} \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) |(u_\varepsilon)_\psi(y) - (u_\varepsilon)_\psi(x)|^{p-2}((u_\varepsilon)_\psi(y) - (u_\varepsilon)_\psi(x))\, dy \\ &\qquad \times ((u_\varepsilon)_\psi(x) - M)^+ dx \\ &= \frac{C_{J,p}}{2\varepsilon^{p+N}} \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) |(u_\varepsilon)_\psi(y) - (u_\varepsilon)_\psi(x)|^{p-2}((u_\varepsilon)_\psi(y) - (u_\varepsilon)_\psi(x)) \\ &\qquad \times (((u_\varepsilon)_\psi(y) - M)^+ - ((u_\varepsilon)_\psi(x) - M)^+)\, dy\, dx \\ &\geq 0.\end{aligned}$$

Therefore,

$$\int_\Omega u_\varepsilon(x)(u_\varepsilon(x) - M)^+ dx \leq \int_\Omega \phi(x)(u_\varepsilon(x) - M)^+\, dx.$$

Consequently, we have

$$\int_\Omega (u_\varepsilon(x) - M)(u_\varepsilon(x) - M)^+ dx \leq \int_\Omega (\phi(x) - M)(u_\varepsilon(x) - M)^+\, dx \leq 0,$$

and $u_\varepsilon(x) \leq M$ for almost all $x \in \Omega$. Analogously, we can obtain $-M \leq u_\varepsilon(x)$ for almost all $x \in \Omega$. Thus

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq M \qquad \text{for all } \epsilon > 0, \tag{2.10}$$

and, therefore, there exists a sequence $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in } L^1(\Omega).$$

Taking $v = u_\varepsilon - \psi$ in (2.9) we get

$$\begin{aligned}\int_\Omega u_\varepsilon(u_\varepsilon - \psi) - \frac{C_{J,p}}{\varepsilon^{p+N}} \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) |(u_\varepsilon)_\psi(y) - (u_\varepsilon)_\psi(x)|^{p-2} \\ \times ((u_\varepsilon)_\psi(y) - (u_\varepsilon)_\psi(x))\, dy\, ((u_\varepsilon)_\psi(x) - \psi(x))\, dx = \int_\Omega \phi(u_\varepsilon - \psi).\end{aligned} \tag{2.11}$$

Now, by (2.11) and (2.10),

$$\begin{aligned}&\frac{C_{J,p}}{2\varepsilon^N} \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) \frac{|(u_\varepsilon)_\psi(y) - (u_\varepsilon)_\psi(x)|^p}{\varepsilon^p}\, dy\, dx \\ &\leq \frac{C_{J,p}}{2\varepsilon^N} \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) \frac{|(u_\varepsilon)_\psi(y) - (u_\varepsilon)_\psi(x)|^{p-1}}{\varepsilon^{p-1}} \frac{|\psi(y) - \psi(x)|}{\varepsilon} dy\, dx + M_1.\end{aligned}$$

Since $\psi \in W^{1,p}(\Omega_J)$, using Young's inequality, we obtain

$$\frac{1}{\varepsilon^N}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\frac{|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)|^p}{\varepsilon^p}\,dy\,dx \le M_2.$$

Moreover,

$$\begin{aligned}
&\int_{\Omega_J}\int_{\Omega_J}\frac{1}{\varepsilon^N}J\left(\frac{x-y}{\varepsilon}\right)\left|\frac{(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)}{\varepsilon}\right|^p dx\,dy\\
&=\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon}\frac{1}{\varepsilon^N}J\left(\frac{x-y}{\varepsilon}\right)\left|\frac{(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)}{\varepsilon}\right|^p dx\,dy\\
&\quad+2\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\int_{\Omega_\varepsilon}\frac{1}{\varepsilon^N}J\left(\frac{x-y}{\varepsilon}\right)\left|\frac{\psi(y)-(u_\varepsilon)_\psi(x)}{\varepsilon}\right|^p dx\,dy\\
&\quad+\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\frac{1}{\varepsilon^N}J\left(\frac{x-y}{\varepsilon}\right)\left|\frac{\psi(y)-\psi(x)}{\varepsilon}\right|^p dx\,dy\\
&=\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon}\frac{1}{\varepsilon^N}J\left(\frac{x-y}{\varepsilon}\right)\left|\frac{(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)}{\varepsilon}\right|^p dx\,dy\\
&\quad+2\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\int_{\Omega_\varepsilon\setminus\overline{\Omega}}\frac{1}{\varepsilon^N}J\left(\frac{x-y}{\varepsilon}\right)\left|\frac{\psi(y)-\psi(x)}{\varepsilon}\right|^p dx\,dy\\
&\quad+\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\frac{1}{\varepsilon^N}J\left(\frac{x-y}{\varepsilon}\right)\left|\frac{\psi(y)-\psi(x)}{\varepsilon}\right|^p dx\,dy \le M_3.
\end{aligned}\tag{2.12}$$

Therefore, by Proposition 2.10, there exists a subsequence, denoted as above, and $w \in W^{1,p}(\Omega_J)$ such that

$$(u_{\varepsilon_n})_\psi \to w \quad \text{strongly in } L^p(\Omega_J).$$

Hence, $w = u$ in Ω and, by [18, Proposition IX.18] and the properties of the trace, $u \in W^{1,p}_{\tilde\psi}(\Omega)$. Moreover, by Proposition 2.10,

$$\left(\frac{C_{J,p}}{2}J(z)\right)^{1/p}\chi_\Omega(x+\varepsilon_n z)\frac{(u_{\varepsilon_n})_\psi(x+\varepsilon_n z)-(u_{\varepsilon_n})_\psi(x)}{\varepsilon_n} \rightharpoonup \left(\frac{C_{J,p}}{2}J(z)\right)^{1/p} z\cdot\nabla u(x)\tag{2.13}$$

weakly in $L^p(\Omega)\times L^p(\mathbb{R}^N)$ (observe that $\chi_\Omega(x+\varepsilon_n z)(u_{\varepsilon_n})_\psi(x+\varepsilon_n z) = \chi_\Omega(x+\varepsilon_n z)\overline{u}_{\varepsilon_n}(x+\varepsilon_n z)$). We can also assume that

$$\begin{aligned}
(J(z))^{1/p'}&\left|\frac{(u_{\varepsilon_n})_\psi(x+\varepsilon_n z)-(u_{\varepsilon_n})_\psi(x)}{\varepsilon_n}\right|^{p-2}\chi_{\Omega_{\varepsilon_n}}(x+\varepsilon_n z)\\
&\times\frac{(u_{\varepsilon_n})_\psi(x+\varepsilon_n z)-(u_{\varepsilon_n})_\psi(x)}{\varepsilon_n} \rightharpoonup (J(z))^{1/p'}\chi(x,z)
\end{aligned}$$

weakly in $L^{p'}(\Omega_J)\times L^{p'}(\mathbb{R}^N)$, for some function $\chi \in L^{p'}(\Omega_J)\times L^{p'}(\mathbb{R}^N)$.

Passing to the limit in (2.9) for $\varepsilon = \varepsilon_n$, we get

$$\int_\Omega uv + \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla v(x)\,dx\,dz = \int_\Omega \phi v\tag{2.14}$$

for every v smooth with support in Ω and by approximation for every $v \in W_0^{1,p}(\Omega)$.

Finally, working as in Proposition 3.3. of [5], we can prove

$$\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi(x,z)z\cdot\nabla v(x)\,dx\,dz = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v \tag{2.15}$$

and the proof is finished. ∎

From the above Proposition, by the standard results of the nonlinear semigroup theory (see [19] or [15]), we obtain Theorem 1.3.

3. The nonlocal total variation flow. The case $p = 1$.

3.1. Existence of solutions for the nonlocal problem. This section deals with the existence and uniqueness of solutions for the nonlocal 1-Laplacian problem with Dirichlet boundary condition,

$$P_1^J(u_0,\psi)\quad \begin{cases} u_t(t,x) = \displaystyle\int_\Omega J(x-y)\frac{u(t,y)-u(t,x)}{|u(t,y)-u(t,x)|}\,dy. \\ \qquad\qquad + \displaystyle\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)\frac{\psi(y)-u(t,x)}{|\psi(y)-u(t,x)|}\,dy, \qquad x\in\Omega. \\ u(0,x) = u_0(x). \end{cases}$$

As in the case $p > 1$, to prove existence and uniqueness of solutions of $P_1^J(u_0,\psi)$ we use the Nonlinear Semigroup Theory, so we start by introducing the following operator in $L^1(\Omega)$.

DEFINITION 3.1. *Given $\psi\in L^1(\Omega_J\setminus\overline{\Omega})$, we define the operator $B_{1,\psi}^J$ in $L^1(\Omega)\times L^1(\Omega)$ by $\hat{u}\in B_{1,\psi}^J u$ if and only if $u,\hat{u}\in L^1(\Omega)$, there exists $g\in L^\infty(\Omega_J\times\Omega_J)$, $g(x,y) = -g(y,x)$ for almost all $(x,y)\in\Omega_J\times\Omega_J$, $\|g\|_\infty\le 1$,*

$$\hat{u}(x) = -\int_{\Omega_J} J(x-y)g(x,y)\,dy \quad \text{a.e. } x\in\Omega, \tag{3.1}$$

and

$$J(x-y)g(x,y)\in J(x-y)\operatorname{sign}(u(y)-u(x)) \quad \text{a.e. } (x,y)\in\Omega\times\Omega, \tag{3.2}$$

$$J(x-y)g(x,y)\in J(x-y)\operatorname{sign}(\psi(y)-u(x)) \quad \text{a.e. } (x,y)\in\Omega\times(\Omega_J\setminus\overline{\Omega}). \tag{3.3}$$

Remark 3.2. Observe that

(i) we can rewrite (3.2) + (3.3) as

$$J(x-y)g(x,y)\in J(x-y)\operatorname{sign}(u_\psi(y)-u(x)) \quad \text{a.e. } (x,y)\in\Omega\times\Omega_J, \tag{3.4}$$

where we set as above, and overall the section,

$$u_\psi(x) := \begin{cases} u(x) & \text{if } x\in\Omega, \\ \psi(x) & \text{if } x\in\Omega_J\setminus\overline{\Omega}, \\ 0 & \text{if } x\notin\Omega_J. \end{cases}$$

(ii) It holds $L^1(\Omega) = \mathrm{Dom}(B_{1,\psi}^J)$ and $B_{1,\psi}^J$ is closed in $L^1(\Omega)\times L^1(\Omega)$.

(iii) It is not difficult to see that, if $g\in L^\infty(\Omega_J\times\Omega_J)$, $g(x,y) = -g(y,x)$ for almost all $(x,y)\in\Omega_J\times\Omega_J$, $\|g\|_\infty\le 1$,

$$J(x-y)g(x,y)\in J(x-y)\ \operatorname{sign}(z(y)-z(x)) \quad \text{a.e. } (x,y)\in\Omega_J\times\Omega_J$$

is equivalent to

$$-\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)\,dy\,z(x)\,dx = \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|z(y)-z(x)|\,dy\,dx.$$

THEOREM 3.3. *Let $\psi \in L^1(\Omega_J \setminus \overline{\Omega})$. The operator $B^J_{1,\psi}$ is completely accretive and satisfies the range condition*

$$L^\infty(\Omega) \subset Ran\left(I + B^J_{1,\psi}\right).$$

Proof. Let $\hat{u}_i \in B^J_{1,\psi}u_i$, $i = 1,2$, and set $u_i(y) = \psi(y)$ in $\Omega_J \setminus \overline{\Omega}$. Then, there exist $g_i \in L^\infty(\Omega_J \times \Omega_J)$, $\|g_i\|_\infty \le 1$, $g_i(x,y) = -g_i(y,x)$, $J(x-y)g_i(x,y) \in J(x-y)\text{sign}(u_i(y)-u_i(x))$ for almost all $(x,y) \in \Omega \times \Omega_J$, such that

$$\hat{u}_i(x) = -\int_{\Omega_J} J(x-y)g_i(x,y)\,dy \quad a.e.\ x \in \Omega$$

for $i = 1,2$. Given $q \in C^\infty(\mathbb{R})$, $0 \le q' \le 1$, $\text{supp}(q')$ compact, $0 \notin \text{supp}(q)$, we have

$$\begin{aligned}
&\int_\Omega (\hat{u}_1(x)-\hat{u}_2(x))q(u_1(x)-u_2(x))\,dx\\
&= \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(g_1(x,y)-g_2(x,y))\left(q(u_1(y)-u_2(y))-q(u_1(x)-u_2(x))\right)\,dxdy\\
&\quad -\int_\Omega\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)(g_1(x,y)-g_2(x,y)\left(q(u_1(x)-u_2(x))\right)\,dx\,dy\\
&\ge \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(g_1(x,y)-g_2(x,y))\left(q(u_1(y)-u_2(y))-q(u_1(x)-u_2(x))\right)\,dxdy.
\end{aligned}$$

Now, by the mean value theorem

$$\begin{aligned}
&J(x-y)(g_1(x,y)-g_2(x,y))\left[q(u_1(y)-u_2(y))-q(u_1(x)-u_2(x))\right]\\
&= J(x-y)(g_1(x,y)-g_2(x,y))q'(\xi)\left[(u_1(y)-u_2(y))-(u_1(x)-u_2(x))\right]\\
&= J(x-y)q'(\xi)\left[g_1(x,y)(u_1(y)-u_1(x))-g_1(x,y)(u_2(y)-u_2(x))\right]\\
&\qquad -J(x-y)q'(\xi)\left[g_2(x,y)(u_1(y)-u_1(x))-g_1(x,y)(u_2(y)-u_2(x))\right] \ge 0,
\end{aligned}$$

since

$$J(x-y)g_i(x,y)(u_i(y)-u_i(x)) = J(x-y)|u_i(y)-u_i(x)|, \quad i=1,2,$$

and

$$-J(x-y)g_i(x,y)(u_j(y)-u_j(x)) \ge -J(x-y)|u_j(y)-u_j(x)|, \quad i \ne j.$$

Hence

$$\int_\Omega (\hat{u}_1(x)-\hat{u}_2(x))q(u_1(x)-u_2(x))\,dx \ge 0,$$

from which it follows that $B^J_{1,\psi}$ is a completely accretive operator.

To show that $B^J_{1,\psi}$ satisfies the range condition, let us see that for any $\phi \in L^\infty(\Omega)$,

$$\lim_{p\to 1+}\left(I + B^J_{p,\psi}\right)^{-1}\phi = \left(I + B^J_{1,\psi}\right)^{-1}\phi \quad \text{weakly in } L^1(\Omega).$$

We prove this in several steps.

Step 1. Let us first suppose that $\psi \in L^\infty(\Omega_J \setminus \overline{\Omega})$. For $1 < p < +\infty$, by Theorem 2.7, there is u_p such that $u_p = (I + B^J_{p,\psi})^{-1}\phi$, that is,

$$u_p(x) - \int_{\Omega_J} J\,(x-y)\,|(u_p)_\psi(y) - u_p(x)|^{p-2}((u_p)_\psi(y) - u_p(x))\,dy = \phi(x), \tag{3.5}$$

a.e. $x \in \Omega$. It is easy to see that $\|u_p\|_\infty \leq \sup\{\|\phi\|_\infty, \|\psi\|_\infty\}$. Therefore, there exists a sequence $p_n \to 1$ such that

$$u_{p_n} \rightharpoonup u \quad \text{weakly in } L^2(\Omega).$$

On the other hand, we also have

$$\frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)\,|(u_{p_n})_\psi(y) - (u_{p_n})_\psi(x)|^{p_n}\;dy\,dx \leq M_2, \qquad \forall\, n \in \mathbb{N}.$$

Consequently, for any measurable subset $E \subset \Omega_J \times \Omega_J$, we have

$$\begin{aligned}&\left|\int\int_E J(x-y)|(u_{p_n})_\psi(y) - (u_{p_n})_\psi(x)|^{p_n-2}\left((u_{p_n}(y))_\psi - (u_{p_n})_\psi(x)\right)\right|\\ &\qquad \leq \int\int_E J(x-y)|(u_{p_n})_\psi(y) - (u_{p_n})_\psi(x)|^{p_n-1} \leq M_2|E|^{\frac{1}{p_n}}.\end{aligned}$$

Hence, by the Dunford–Pettis theorem we may assume that there exists $g(x,y)$ such that

$$J(x-y)|(u_{p_n})_\psi(y) - (u_{p_n})_\psi(x)|^{p_n-2}\left((u_{p_n})_\psi(y) - (u_{p_n})_\psi(x)\right) \rightharpoonup J(x-y)g(x,y),$$

weakly in $L^1(\Omega_J \times \Omega_J)$, $g(x,y) = -g(y,x)$ for almost all $(x,y) \in \Omega_J \times \Omega_J$, and $\|g\|_\infty \leq 1$.

Therefore, by (3.5),

$$u(x) - \int_{\Omega_J} J(x-y)g(x,y)\,dy = \phi(x) \qquad \text{a.e. } x \in \Omega. \tag{3.6}$$

Then, to finish the proof it is enough to show that

$$\begin{aligned}&-\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)\,dy\,u_\psi(x)\,dx\\ &\qquad = \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|u_\psi(y) - u_\psi(x)|\,dy\,dx.\end{aligned} \tag{3.7}$$

In fact, by (3.5) and (3.6),

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)\left|(u_{p_n})_\psi(y)-(u_{p_n})_\psi(x)\right|^{p_n}\,dy\,dx = \int_\Omega \phi u_{p_n} - \int_\Omega u_{p_n}u_{p_n}\\
&-\int_{\Omega_J\setminus\overline{\Omega}}\int_{\Omega_J} J(x-y)\left|\psi(y)-(u_{p_n})_\psi(x)\right|^{p_n-2}\left(\psi(y)-(u_{p_n})_\psi(x)\right)\,dy\,\psi(x)\,dx\\
&=\int_\Omega \phi u - \int_\Omega uu - \int_\Omega \phi(u-u_{p_n}) + \int_\Omega 2u(u-u_{p_n}) - \int_\Omega (u-u_{p_n})(u-u_{p_n})\\
&\qquad -\int_{\Omega_J\setminus\overline{\Omega}}\int_{\Omega_J} J(x-y)\left|\psi(y)-(u_{p_n})_\psi(x)\right|^{p_n-2}\left(\psi(y)-(u_{p_n})_\psi(x)\right)\,dy\,\psi(x)\,dx\\
&\le -\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)\,dy\,u(x)\,dx - \int_\Omega \phi(u-u_{p_n}) + \int_\Omega 2u(u-u_{p_n})\\
&\qquad +\int_{\Omega_J\setminus\overline{\Omega}}\int_{\Omega_J} J(x-y)g(x,y)\,dy\,\psi(x)\,dx\\
&\qquad -\int_{\Omega_J\setminus\overline{\Omega}}\int_{\Omega_J} J(x-y)\left|\psi(y)-(u_{p_n})_\psi(x)\right|^{p_n-2}\left(\psi(y)-(u_{p_n})_\psi(x)\right)\,dy\,\psi(x)\,dx,
\end{aligned}$$

and so,

$$\begin{aligned}
\limsup_{n\to+\infty}\frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)\left|u_{p_n}(y)-u_{p_n}(x)\right|^{p_n}\,dy\,dx&\\
\le -\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,u(x)\,dx.&
\end{aligned}$$

Now, by the monotonicity Lemma 2.4, for all $\rho\in L^\infty(\Omega)$,

$$\begin{aligned}
&-\int_{\Omega_J}\int_{\Omega_J} J(x-y)|\rho(y)-\rho(x)|^{p_n-2}(\rho(y)-\rho(x))\,dy\,(u_{p_n}(x)-\rho(x))\,dx\\
&\le -\int_{\Omega_J}\int_{\Omega_J} J(x-y)|u_{p_n}(y)-u_{p_n}(x)|^{p_n-2}(u_{p_n}(y)-u_{p_n}(x))dy(u_{p_n}(x)-\rho(x))dx.
\end{aligned}$$

Taking limits,

$$\begin{aligned}
-\int_{\Omega_J}\int_{\Omega_J} J(x-y)\,\mathrm{sign}_0(\rho(y)-\rho(x))\,dy\,(u(x)-\rho(x))\,dx&\\
\le -\int_{\Omega_J}\int_{\Omega_J} J(x-y)g(x,y)\,dy\,(u(x)-\rho(x))\,dx.&
\end{aligned}$$

Taking now, $\rho = u\pm\lambda u$, $\lambda>0$, and letting $\lambda\to 0$, we get (3.7), and the proof is finished for this class of data.

Step 2. Let us now suppose that ψ^- is bounded. Let $\psi_n = T_n(\psi)$, n large enough such that $\psi_n^- = \psi^-$. Then, $\{\psi_n\}$ is a nondecreasing sequence that converges in L^1 to ψ. By *Step* 1, there exists $u_n = (I+B^J_{1,\psi_n})^{-1}\phi$, that is, there exists $g_n\in L^\infty(\Omega_J\times\Omega_J)$, $g_n(x,y) = -g_n(y,x)$ for almost all $(x,y)\in\Omega_J\times\Omega_J$, $\|g_n\|_\infty\le 1$,

$$u_n(x) - \int_{\Omega_J} J(x-y)g_n(x,y)\,dy = \phi(x) \qquad \text{a.e. } x\in\Omega \tag{3.8}$$

and

$$\begin{aligned}
&-\int_{\Omega_J}\int_{\Omega_J} J(x-y)g_n(x,y)\,dy\,(u_n)_{\psi_n}(x)\,dx\\
&= \frac{1}{2}\int_{\Omega_J}\int_{\Omega_J} J(x-y)|(u_n)_{\psi_n}(y)-(u_n)_{\psi_n}(x)|\,dy\,dx.
\end{aligned}\tag{3.9}$$

Therefore, by monotonicity,

$$\int_{\Omega_J}\int_{\Omega_J} \left((u_n)_{\psi_n} - (u_{n+1})_{\psi_{n+1}}\right)\left((u_n)_{\psi_n} - (u_{n+1})_{\psi_{n+1}}\right)^+ \le 0,$$

which implies $u_n \le u_{n+1}$. Since $\{u_n\}$ is bounded in L^∞ we have $\{u_n\}$ converges to a function u in L^2. On the other hand, we can suppose that $J(x-y)g_n(x,y)$ converges weakly in L^2 to $J(x-y)g(x,y)$, $g(x,y) = -g(y,x)$ for almost all $(x,y) \in \Omega_J \times \Omega_J$, and $\|g\|_\infty \le 1$. Hence, passing to the limit in (3.8) and (3.9) we obtain $u = (I + B^J_{1,\psi})^{-1}\phi$.

Step 3. For a general $\psi \in L^1(\Omega_J \setminus \overline{\Omega})$, apply *Step* 2 to $\psi_n = \sup\{\psi, -n\}$ and use monotonicity in a similar way to finish the proof. □

Proof of Theorem 1.5. As a consequence of the above results, we have that the abstract Cauchy problem

$$\begin{cases} u'(t) + B^J_{1,\psi} u(t) \ni 0, & t \in (0,T), \\ u(0) = u_0 \end{cases} \tag{3.10}$$

has a unique mild solution u for every initial datum $u_0 \in L^1(\Omega)$ and $T > 0$ (see [15]). Moreover, due to the complete accretivity of the operator $B^J_{1,\psi}$, the mild solution of (3.10) is a strong solution ([14]). Consequently, the proof is concluded. □

3.2. Convergence to the total variation flow. Let us start recalling some results from [1] (see also [2]) about the Dirichlet problem for the total variational flow, that is,

$$D_1(u_0, \tilde\psi) \quad \begin{cases} u_t = \mathrm{div}\left(\frac{Du}{|Du|}\right) & \text{in }]0,T[\times\Omega, \\ u = \tilde\psi & \text{on }]0,T[\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

with $\tilde\psi \in L^1(\partial\Omega)$.

THEOREM 3.4 ([1]). *Let $T > 0$ and $\tilde\psi \in L^1(\partial\Omega)$. For any $u_0 \in L^1(\Omega)$ ($L^2(\Omega)$) there exists a unique entropy (strong) solution $u(t)$ of $D_1(u_0, \tilde\psi)$.*

Associated to $-\mathrm{div}(\frac{Du}{|Du|})$ with Dirichlet boundary conditions, in [1] it is defined the operator $\mathcal{A}_{\tilde\psi} \subset L^1(\Omega) \times L^1(\Omega)$ as follows: $(u,v) \in \mathcal{A}_{\tilde\psi}$ if and only if $u, v \in L^1(\Omega)$, $q(u) \in BV(\Omega)$ for all $q \in \mathcal{P} := \{q \in W^{1,\infty}(\mathbb{R}) : q' \ge 0, \mathrm{supp}(q') \text{ is compact}\}$, and there exists $\zeta \in X(\Omega)$ (where $X(\Omega)$ is defined by (1.4)), with $\|\zeta\|_\infty \le 1$, $v = -\mathrm{div}(\zeta)$ in $\mathcal{D}'(\Omega)$ such that

$$\int_\Omega (w - q(u))v \le \int_\Omega (\zeta, Dw) - |Dq(u)| + \int_{\partial\Omega} |w - q\left(\tilde\psi\right)| - \int_{\partial\Omega} |q(u) - q\left(\tilde\psi\right)| \tag{3.11}$$

for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ and every $q \in \mathcal{P}$. Also in [1] it is proved that the following assertions are equivalent:

(a) $(u,v) \in \mathcal{A}_{\tilde\psi}$,

(b) $u, v \in L^1(\Omega)$, $q(u) \in BV(\Omega)$ for all $q \in \mathcal{P}$, and there exists $\zeta \in X(\Omega)$, with $\|\zeta\|_\infty \le 1$, $v = -\mathrm{div}(\zeta)$ in $\mathcal{D}'(\Omega)$ such that

$$\int_\Omega (\zeta, Dq(u)) = |Dq(u)| \qquad \forall\, q \in \mathcal{P}, \tag{3.12}$$

$$[\zeta, \nu] \in \mathrm{sign}\left(q\left(\tilde\psi\right) - q(u)\right) \qquad \mathcal{H}^{N-1} - a.e. \text{ on } \partial\Omega, \ \ \forall\, q \in \mathcal{P}. \tag{3.13}$$

Moreover, it is shown that $\mathcal{A}_{\tilde{\psi}}$ is an m-completely accretive operator in $L^1(\Omega)$ with dense domain and that for any $u_0 \in L^1(\Omega)$, the unique entropy solution $u(t)$ of problem $D_1(u_0, \tilde{\psi})$ coincides with the unique mild solution $e^{-t\mathcal{A}_{\tilde{\psi}}} u_0$ given by Crandall–Liggett's exponential formula.

Now, given J, we consider the rescaled kernels

$$J_{1,\varepsilon}(x) := \frac{C_{J,1}}{\varepsilon^{1+N}} J\left(\frac{x}{\varepsilon}\right), \qquad \text{with} \quad C_{J,1}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|\, dz,$$

that is, a normalizing constant in order to obtain the 1-Laplacian in the limit instead of a multiple of it.

Proposition 3.5. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $\tilde{\psi} \in L^\infty(\partial\Omega)$. Let $\psi \in W^{1,1}(\Omega_J \setminus \overline{\Omega}) \cap L^\infty(\Omega_J \setminus \overline{\Omega})$ such that $\psi|_{\partial\Omega} = \tilde{\psi}$. Assume $J(x) \geq J(y)$ if $|x| \leq |y|$. Then, for any $\phi \in L^\infty(\Omega)$,*

$$\left(I + B_{1,\psi}^{J_{1,\varepsilon}}\right)^{-1} \phi \to \left(I + \mathcal{A}_{\tilde{\psi}}\right)^{-1} \phi \quad \textit{strongly in } L^1(\Omega) \textit{ as } \varepsilon \to 0. \tag{3.14}$$

Proof. Given $\varepsilon > 0$ small, we set $u_\varepsilon = (I + B_{1,\psi}^{J_{1,\varepsilon}})^{-1}\phi$ and denote

$$\Omega_\varepsilon := \Omega_{J_{1,\varepsilon}} = \Omega + \operatorname{supp}(J_{1,\varepsilon}).$$

Then, there exists $g_\epsilon \in L^\infty(\Omega_\varepsilon \times \Omega_\varepsilon)$, $g_\varepsilon(x,y) = -g_\varepsilon(y,x)$ for almost all $(x,y) \in \Omega_\varepsilon \times \Omega_\varepsilon$, $\|g_\varepsilon\|_\infty \leq 1$, such that

$$J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y) \in J\left(\frac{x-y}{\varepsilon}\right) \operatorname{sign}(u_\varepsilon(y) - u_\varepsilon(x)) \qquad \text{a.e. } (x,y) \in \Omega \times \Omega,$$
$$J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y) \in J\left(\frac{x-y}{\varepsilon}\right) \operatorname{sign}(\tilde{\psi}(y) - u_\varepsilon(x)) \qquad \text{a.e. } (x,y) \in \Omega \times (\Omega_\epsilon \setminus \overline{\Omega})$$

and

$$u_\varepsilon(x) - \frac{C_{J,1}}{\varepsilon^{1+N}} \int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)\, dy = \phi(x) \qquad a.e.\ x \in \Omega. \tag{3.15}$$

Therefore, for $v \in L^\infty(\Omega_J)$, we can write

$$\begin{aligned}\int_\Omega u_\varepsilon(x)v(x)\, dx - \frac{C_{J,1}}{\varepsilon^{1+N}} \int_\Omega \int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)v(x)\, dy\, dx \\ = \int_\Omega \phi(x)v(x)\, dx.\end{aligned} \tag{3.16}$$

Observe that we can extend g_ε to a function in $L^\infty(\Omega_J \times \Omega_J)$, $g_\varepsilon(x,y) = -g_\varepsilon(y,x)$ for almost all $(x,y) \in \Omega_J \times \Omega_J$, $\|g_\varepsilon\|_{L^\infty(\Omega_J)} \leq 1$, such that

$$J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y) \in J\left(\frac{x-y}{\varepsilon}\right) \operatorname{sign}((u_\varepsilon)_\psi(y) - (u_\varepsilon)_\psi(x)) \quad \text{a.e. } (x,y) \in \Omega_J \times \Omega_J.$$

Let $M := \max\{\|\phi\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega_J \setminus \overline{\Omega})}\}$. Taking $v = (u_\varepsilon - M)^+$ in (3.16), we get

$$\begin{aligned}&\int_\Omega u_\varepsilon(x)(u_\varepsilon(x) - M)^+ dx - \frac{C_{J,1}}{\varepsilon^{1+N}} \int_\Omega \int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)(u_\varepsilon(x) - M)^+\, dy\, dx \\ &= \int_\Omega \phi(x)(u_\varepsilon(x) - M)^+\, dx.\end{aligned}$$

Now

$$
\begin{aligned}
&-\frac{C_{J,1}}{\varepsilon^{1+N}}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)((u_\varepsilon)_\psi(x)-M)^+\,dy\,dx\\
&=\frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)(((u_\varepsilon)_\psi(y)-M)^+-\left((u_\varepsilon)_\psi(x)-M)^+\right)\,dy\,dx\\
&\geq 0.
\end{aligned}
$$

Hence, we get

$$
\int_\Omega u_\varepsilon(x)(u_\varepsilon(x)-M)^+dx \leq \int_\Omega \phi(x)(u_\varepsilon(x)-M)^+dx.
$$

Consequently,

$$
0\leq \int_\Omega (u_\varepsilon(x)-M)(u_\varepsilon(x)-M)^+dx \leq \int_\Omega (\phi(x)-M)(u_\varepsilon(x)-M)^+dx \leq 0,
$$

and we deduce $u_\varepsilon(x)\leq M$ for almost all $x\in\Omega$. Analogously, we can obtain $-M\leq u_\varepsilon(x)$ for almost all $x\in\Omega$. Thus

$$
\|u_\varepsilon\|_{L^\infty(\Omega)}\leq M \qquad \text{for all } \epsilon>0; \tag{3.17}
$$

from here, we can assume there exists a sequence $\varepsilon_n\to 0$ such that

$$
u_{\varepsilon_n}\rightharpoonup u \quad \text{weakly in } L^1(\Omega).
$$

Taking $v=u_\varepsilon$ in (3.16), we have

$$
\int_\Omega u_\varepsilon(x)u_\varepsilon(x)\,dx-\frac{C_{J,1}}{\varepsilon^{1+N}}\int_\Omega\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)\,dy u_\varepsilon(x)\,dx=\int_\Omega \phi(x)u_\varepsilon(x)dx. \tag{3.18}
$$

Observe that

$$
\begin{aligned}
&-\frac{C_{J,1}}{\varepsilon^{1+N}}\int_\Omega\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)dy u_\varepsilon(x)\,dx\\
&=-\frac{C_{J,1}}{\varepsilon^{1+N}}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)dy(u_\varepsilon)_\psi(x)\,dx\\
&\quad+\frac{C_{J,1}}{\varepsilon^{1+N}}\int_{\Omega_\varepsilon\setminus\overline{\Omega}}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)dy\psi(x)\,dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left|\frac{C_{J,1}}{\varepsilon^{1+N}}\int_{\Omega_\varepsilon\setminus\overline{\Omega}}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)dy\psi(x)dx\right|\\
&\leq \frac{C_{J,1}}{\varepsilon^{1+N}}\int_{\Omega_\varepsilon\setminus\overline{\Omega}}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) dy|\psi(x)|dx\\
&\leq \frac{C_{J,1}}{\varepsilon}M\int_{\Omega_\varepsilon\setminus\overline{\Omega}}\left(\frac{1}{\varepsilon^N}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)dy\right)dx\\
&\leq \frac{C_{J,1}}{\varepsilon}M|\Omega_\varepsilon\setminus\Omega|\leq M_1.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&-\frac{C_{J,1}}{\varepsilon^{1+N}}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)dy(u_\varepsilon)_\psi(x)dx \\
&\quad = \frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx.
\end{aligned}
$$

Consequently, from (3.17) and (3.18), it follows that

$$
\frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx \le M_2. \tag{3.19}
$$

Let us compute,

$$
\begin{aligned}
&\frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_J}\int_{\Omega_J} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx \\
&= \frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx \\
&\quad + \frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_\varepsilon}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx \\
&\quad + \frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx \\
&\quad + \frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx.
\end{aligned}
$$

Now, since $\psi \in W^{1,1}(\Omega_J \setminus \overline{\Omega})$, we get

$$
\begin{aligned}
&\frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx \\
&\quad = \frac{C_{J,1}}{2\varepsilon^{N}}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\frac{|\psi(y)-\psi(x)|}{\varepsilon} dydx \le M_3.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx \\
&\quad = \frac{C_{J,1}}{2\varepsilon^{N}}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\int_{\Omega_\varepsilon\setminus\overline{\Omega}} J\left(\frac{x-y}{\varepsilon}\right)\frac{|\psi(y)-\psi(x)|}{\varepsilon}\, dy\, dx \\
&\quad \le M_4\frac{C_{J,1}}{2}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon}\left(\frac{1}{\varepsilon^N}\int_{\Omega_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right) dy\right) dx \le M_5.
\end{aligned}
$$

With similar arguments we obtain

$$
\frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_\varepsilon}\int_{\Omega_J\setminus\overline{\Omega}_\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx \le M_6.
$$

Therefore,

$$
\frac{C_{J,1}}{2\varepsilon^{1+N}}\int_{\Omega_J}\int_{\Omega_J} J\left(\frac{x-y}{\varepsilon}\right)\left|(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)\right| dy\, dx \le M_7. \tag{3.20}
$$

In particular, we get

$$\int_{\Omega_J}\int_{\Omega_J} \frac{1}{2}\frac{C_{J,1}}{\varepsilon^N} J\left(\frac{x-y}{\varepsilon}\right)\left|\frac{(u_\varepsilon)_\psi(y)-(u_\varepsilon)_\psi(x)}{\varepsilon}\right| dx\,dy \le M_7 \qquad \forall\, n\in\mathbb{N}.$$

By Proposition 2.10, there exists a subsequence, denote equal, and $w\in BV(\Omega_J)$ such that

$$(u_{\varepsilon_n})_\psi \to w \qquad \text{strongly in } L^1(\Omega_J)$$

and

$$\frac{C_{J,1}}{2}J(z)\chi_\Omega(\cdot+\varepsilon_n z)\frac{(u_{\varepsilon_n})_\psi(\cdot+\varepsilon_n z)-(u_{\varepsilon_n})_\psi(\cdot)}{\varepsilon_n} \rightharpoonup \frac{C_{J,1}}{2}J(z)z\cdot Dw \tag{3.21}$$

weakly as measures. Hence, it is easy to obtain that

$$w(x)=u_\psi(x)=\begin{cases} u(x) & \text{in } \ x\in\Omega,\\ \psi(x) & \text{in } \ x\in\Omega_J\setminus\overline{\Omega},\end{cases}$$

and $u\in BV(\Omega)$.

Moreover, we can also assume that

$$J(z)\chi_{\Omega_J}(x+\varepsilon_n z)\overline{g}_{\varepsilon_n}(x,x+\varepsilon_n z) \rightharpoonup \Lambda(x,z) \tag{3.22}$$

weakly* in $L^\infty(\Omega_J)\times L^\infty(\mathbb{R}^N)$ for some function $\Lambda\in L^\infty(\Omega_J)\times L^\infty(\mathbb{R}^N)$, $\Lambda(x,z)\le J(z)$ almost everywhere in $\Omega_J\times\mathbb{R}^N$. Taking in (3.16) $v\in\mathcal{D}(\Omega)$, we get for $\varepsilon=\varepsilon_n$ small enough

$$\begin{aligned}\int_\Omega u_{\varepsilon_n}(x)v(x)dx - \frac{C_{J,1}}{{\varepsilon_n}^{1+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right) g_{\varepsilon_n}(x,y)v(x)\,dy\,dx \\ = \int_\Omega \phi(x)v(x)\,dx.\end{aligned} \tag{3.23}$$

Changing variables and taking into account (3.23), we can write

$$\begin{aligned}&\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\int_\Omega J(z)\chi_\Omega(x+\varepsilon_n z)\overline{g}_{\varepsilon_n}(x,x+\varepsilon_n z)\,dz\frac{\overline{v}(x+\varepsilon_n z)-v(x)}{\varepsilon_n}\,dx\\ &= -\frac{C_{J,1}}{\varepsilon_n}\int_{\mathbb{R}^N}\int_\Omega J(z)\chi_\Omega(x+\varepsilon_n z)\overline{g}_{\varepsilon_n}(x,x+\varepsilon_n z)\,dz\,v(x)\,dx\\ &= \int_\Omega(\phi(x)-u_{\varepsilon_n}(x))v(x)\,dx.\end{aligned} \tag{3.24}$$

By (3.22), passing to the limit in (3.24), we get

$$\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\int_\Omega \Lambda(x,z)z\cdot\nabla v(x)\,dx\,dz = \int_\Omega(\phi(x)-u(x))v(x)\,dx \tag{3.25}$$

for all $v\in\mathcal{D}(\Omega)$. We set $\zeta=(\zeta_1,\ldots,\zeta_N)$, the vector field defined by

$$\zeta_i(x) := \frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,z)z_i\,dz, \qquad i=1,\ldots,N.$$

Then, $\zeta \in L^\infty(\Omega_J, \mathbb{R}^N)$, and from (3.25),

$$-\mathrm{div}(\zeta) = \phi - u \quad \text{ in } \mathcal{D}'(\Omega).$$

Let us see that

$$\|\zeta\|_{L^\infty(\Omega_J)} \le 1.$$

Given $\xi \in \mathbb{R}^N \setminus \{0\}$, let R_ξ be the rotation such that $R_\xi^t(\xi) = \mathbf{e}_1|\xi|$. If we make the change of variables $z = R_\xi(y)$, we obtain

$$\begin{aligned}\zeta(x)\cdot\xi = \frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,z) z\cdot\xi\,dz &= \frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,R_\xi(y))R_\xi(y)\cdot\xi\,dy\\ &= \frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,R_\xi(y))y_1|\xi|\,dy.\end{aligned}$$

On the other hand, since J is a radial function and $\Lambda(x,z) \le J(z)$ almost everywhere,

$$C_{J,1}{}^{-1} = \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_1|\,dz$$

and

$$|\zeta(x)\cdot\xi| \le \frac{C_{J,1}}{2}\int_{\mathbb{R}^N} J(y)|y_1|\,dy|\xi| = |\xi| \quad \text{ a.e. } x \in \Omega_J.$$

Therefore, $\|\zeta\|_{L^\infty(\Omega_J)} \le 1$.

To finish the proof, that is, to show that $u = (I + \mathcal{A}_{\tilde\psi})^{-1}\phi$, since $u \in L^\infty(\Omega)$ and $\tilde\psi \in L^\infty(\partial\Omega)$, we need only to prove that

$$(\zeta, Du) = |Du| \quad \text{ as measures in } \Omega \tag{3.26}$$

and

$$[\zeta,\nu] \in \mathrm{sign}\left(\tilde\psi - u\right) \quad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega. \tag{3.27}$$

Given $0 \le \varphi \in \mathcal{D}(\Omega)$, taking $\varepsilon = \varepsilon_n$ and $v = \varphi u_{\varepsilon_n}$ in (3.16), we get

$$\begin{aligned}&-\frac{C_{J,1}}{\varepsilon_n{}^{1+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right) g_{\varepsilon_n}(x,y)u_{\varepsilon_n}(x)\varphi(x)\,dy\,dx\\ &\quad= \frac{C_{J,1}}{2\varepsilon_n{}^{1+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right) g_{\varepsilon_n}(x,y)(u_{\varepsilon_n}(y)\varphi(y) - u_{\varepsilon_n}(x)\varphi(x))\,dy\,dx\\ &\quad= \int_\Omega(\phi(x) - u_{\varepsilon_n}(x))u_{\varepsilon_n}(x)\varphi(x)\,dx.\end{aligned} \tag{3.28}$$

Now, we decompose the double integral as follows,

$$I_n := \frac{C_{J,1}}{2\varepsilon_n{}^{1+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right) g_{\varepsilon_n}(x,y)(u_{\varepsilon_n}(y)\varphi(y) - u_{\varepsilon_n}(x)\varphi(x))\,dy\,dx = I_n^1 + I_n^2,$$

where

$$\begin{aligned}I_n^1 &:= \frac{C_{J,1}}{2\varepsilon_n{}^{1+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right)|u_{\varepsilon_n}(y) - u_{\varepsilon_n}(x)|\varphi(y)\,dy\,dx\\ &= \frac{C_{J,1}}{2}\int_\Omega\int_\Omega J(z)\,\chi_\Omega(x+\varepsilon_n z)\frac{|\overline{u}_{\varepsilon_n}(x+\varepsilon_n z) - u_{\varepsilon_n}(x)|}{\varepsilon_n}\overline{\varphi}(x+\varepsilon_n z)\,dz\,dx\end{aligned}$$

and

$$I_n^2 := \frac{C_{J,1}}{2{\varepsilon_n}^{1+N}} \int_\Omega \int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right) g_{\varepsilon_n}(x,y) u_{\varepsilon_n}(x)(\varphi(y)-\varphi(x))\,dy\,dx$$
$$= \frac{C_{J,1}}{2} \int_\Omega \int_\Omega J(z)\chi_\Omega(x+\varepsilon_n z)\overline{g}_{\varepsilon_n}(x,x+\varepsilon_n z) u_{\varepsilon_n}(x) \frac{\overline{\varphi}(x+\varepsilon_n z)-\varphi(x)}{\varepsilon_n}\,dz\,dx.$$

Having in mind (3.21), it follows that

$$\lim_{n\to\infty} I_n^1 \ge \frac{C_{J,1}}{2} \int_\Omega \int_\Omega J(z)\,\varphi(x)|z\cdot Du| = \int_\Omega \varphi\,|Du|.$$

On the other hand, since

$$u_{\varepsilon_n} \to u \qquad \text{strongly in } L^1(\Omega),$$

by (3.22), we get

$$\lim_{n\to\infty} I_n^2 = \frac{C_{J,1}}{2} \int_\Omega \int_{\mathbb{R}^N} u(x)\Lambda(x,z) z\cdot\nabla\varphi(x)\,dz\,dx = \int_\Omega u(x)\zeta(x)\cdot\nabla\varphi(x)\,dx.$$

Therefore, taking $n\to+\infty$ in (3.28), we obtain

$$\int_\Omega \varphi\,|Du| + \int_\Omega u(x)\zeta(x)\cdot\nabla\varphi(x)\,dx \le \int_\Omega (\phi(x)-u(x))u(x)\varphi(x)\,dx. \tag{3.29}$$

By Green's formula,

$$\int_\Omega (\phi(x)-u(x))u(x)\varphi(x)dx = -\int_\Omega \operatorname{div}(\zeta)u\varphi\,dx = \int_\Omega (\zeta, D(\varphi u))$$
$$= \int_\Omega \varphi(\zeta,Du) + \int_\Omega u(x)\zeta(x)\cdot\nabla\varphi(x)\,dx.$$

Since $|(\zeta,Du)|\le|Du|$, the last identity and (3.29) give (3.26).

Finally, we show that (3.27) holds. We take $w_m \in W^{1,1}(\Omega)\cap C(\Omega)$ such that $w_m = \tilde{\psi}\ \mathcal{H}^{N-1}$-a.e. on $\partial\Omega$, and $w_m \to u$ in $L^1(\Omega)$. Taking $v = v_{m,n} := (u_{\varepsilon_n})_\psi - (w_m)_\psi$ in (3.16), we get

$$\begin{aligned} &\int_\Omega (\phi(x)-u_{\varepsilon_n}(x))(u_{\varepsilon_n}(x)-w_m(x))\,dx \\ &= -\frac{C_{J,1}}{{\varepsilon_n}^{1+N}} \int_{\Omega_J}\int_{\Omega_J} J\left(\frac{x-y}{\varepsilon_n}\right) g_{\varepsilon_n}(x,y) v_{m,n}(x)\,dy\,dx \\ &= \frac{C_{J,1}}{2{\varepsilon_n}^{1+N}} \int_{\Omega_J}\int_{\Omega_J} J\left(\frac{x-y}{\varepsilon_n}\right) g_{\varepsilon_n}(x,y)(v_{m,n}(x)-v_{m,n}(x))\,dy\,dx \\ &= H_n^1 + H_{m,n}^1, \end{aligned} \tag{3.30}$$

where

$$H_n^1 = \frac{C_{J,1}}{2} \int_{\Omega_J}\int_{\mathbb{R}^N} J(z)\chi_{\Omega_J}(x+\varepsilon_n z) \left|\frac{(u_{\varepsilon_n})_\psi(x+\varepsilon_n z)-(u_{\varepsilon_n})_\psi(x)}{\varepsilon_n}\right|\,dz\,dx$$

and

$$H_{m,n}^2 = -\frac{C_{J,1}}{2} \int_{\Omega_J}\int_{\mathbb{R}^N} J(z)\chi_{\Omega_J}(x+\varepsilon_n z)\overline{g}_{\varepsilon_n}(x,x+\varepsilon_n z)$$
$$\times \frac{(w_m)_\psi(x+\varepsilon_n z)-(w_m)_\psi(x)}{\varepsilon_n}\,dz\,dx.$$

Arguing as before,

$$\lim_{n\to\infty} H_n^1 \geq \int_{\Omega_J} |Du_\psi| = \int_\Omega |Du| + \int_{\partial\Omega} \left|u - \tilde{\psi}\right| d\mathcal{H}^{N-1} + \int_{\Omega_J\setminus\overline{\Omega}} |\nabla\psi|.$$

On the other hand, since $(w_m)_\psi \in W^{1,1}(\Omega_J)$, by (3.22),

$$\lim_{n\to\infty} H_{m,n}^2 = -\frac{C_{J,1}}{2}\int_{\Omega_J}\int_{\mathbb{R}^N} \Lambda(x,z)z\cdot\nabla(w_m)_\psi(x)\,dz\,dx = -\int_{\Omega_J} \zeta(x)\cdot\nabla(w_m)_\psi(x)\,dx.$$

Consequently, taking $n\to\infty$ in (3.30), we get

$$\begin{aligned}(3.31)\qquad &\int_\Omega (\phi(x)-u(x))(u(x)-w_m(x))\,dx\\ &\geq \int_\Omega |Du| + \int_{\partial\Omega}\left|u-\tilde{\psi}\right| d\mathcal{H}^{N-1} + \int_{\Omega_J\setminus\overline{\Omega}}|\nabla\psi| - \int_{\Omega_J}\zeta(x)\cdot\nabla(w_m)_\psi(x)\,dx.\end{aligned}$$

Now,

$$\begin{aligned}-\int_{\Omega_J}\zeta(x)\cdot\nabla(w_m)_\psi(x)\,dx &= -\int_\Omega \zeta(x)\cdot\nabla w_m(x)\,dx - \int_{\Omega_J\setminus\overline{\Omega}}\zeta(x)\cdot\nabla\psi(x)\,dx\\ &= \int_\Omega \mathrm{div}\zeta(x) w_m(x)\,dx - \int_{\partial\Omega}[\zeta,\nu]\tilde{\psi}\,d\mathcal{H}^{N-1} - \int_{\Omega_J\setminus\overline{\Omega}}\zeta(x)\cdot\nabla\psi(x)\,dx.\end{aligned}$$

Since

$$\int_{\Omega_J\setminus\overline{\Omega}}|\nabla\psi| - \int_{\Omega_J\setminus\overline{\Omega}}\zeta(x)\cdot\nabla\psi(x)\,dx \geq 0,$$

from (3.31), we have

$$\begin{aligned}&\int_\Omega (\phi(x)-u(x))(u(x)-w_m(x))\,dx\\ &\qquad\geq \int_\Omega |Du| + \int_{\partial\Omega}\left|u-\tilde{\psi}\right| d\mathcal{H}^{N-1} + \int_\Omega \mathrm{div}\zeta(x)w_m(x)\,dx - \int_{\partial\Omega}[\zeta,\nu]\tilde{\psi}\,d\mathcal{H}^{N-1}.\end{aligned}$$

Letting $m\to\infty$, and using Green's formula, we deduce

$$\begin{aligned}0 &\geq \int_\Omega |Du| + \int_{\partial\Omega}\left|u-\tilde{\psi}\right| d\mathcal{H}^{N-1} + \int_\Omega \mathrm{div}\zeta(x)u(x)\,dx - \int_{\partial\Omega}[\zeta,\nu]\tilde{\psi}\,d\mathcal{H}^{N-1}\\ &= \int_\Omega |Du| + \int_{\partial\Omega}\left|u-\tilde{\psi}\right| d\mathcal{H}^{N-1} - \int_\Omega (\zeta, Du) + \int_{\partial\Omega}[\zeta,\nu]u\,d\mathcal{H}^{N-1}\\ &\qquad - \int_{\partial\Omega}[\zeta,\nu]\tilde{\psi}\,d\mathcal{H}^{N-1}.\end{aligned}$$

By (3.26), we obtain

$$\int_{\partial\Omega}\left|u-\tilde{\psi}\right| d\mathcal{H}^{N-1} \leq \int_{\partial\Omega}[\zeta,\nu]\left(\tilde{\psi}-u\right) d\mathcal{H}^{N-1} \leq \int_{\partial\Omega}\left|u-\tilde{\psi}\right| d\mathcal{H}^{N-1}.$$

Therefore,

$$[\zeta,\nu]\in \mathrm{sign}\left(\tilde{\psi}-u\right) \quad \mathcal{H}^{N-1}-\text{a.e. on } \partial\Omega,$$

and the proof is finished. □

From the above Proposition, by standard results of the Nonlinear Semigroup Theory (see, [19] or [15]), we obtain Theorem 1.6.

4. Limit as $p \to +\infty$. A model for sandpiles.

4.1. A model for sandpiles. Let sand be poured out onto a rigid surface, $y = u_0(x)$, given in a bounded open subset Ω of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. If the support boundary is open and we assume that the angle of stability is equal to $\frac{\pi}{4}$, a model for pile surface evolution was proposed by Prigozhin [35] as

$$\partial_t u + \operatorname{div} \mathbf{q} = f, \qquad u|_{t=0} = u_0, \qquad u|_{\partial\Omega} = u_0|_{\partial\Omega}, \tag{4.1}$$

where $u(t,x)$ is the unknown pile surface, $f(t,x) \geq 0$ is the given source density, and $\mathbf{q}(t,x)$ is the unknown horizontal projection of the flux of sand pouring down the pile surface. If the support has no slopes steeper than the sand angle of repose, $\|\nabla u_0\|_\infty \leq 1$, Prigozhin ([35], see also [10], [29], and the references therein) proposed to take $\mathbf{q} = -m\nabla u$, where $m \geq 0$ is the Lagrange multiplier related to the constraint $\|\nabla u\|_\infty \leq 1$ and satisfies $m(\|\nabla u\|^2 - 1) = 0$ and reformulated this model as the following variational inequality:

$$\begin{cases} f(t,\cdot) - u_t(t) \in \partial I_{K(u_0)}(u(t)), & \text{a.e. } t \in]0,T[, \\ u(0,x) = u_0(x), \end{cases} \tag{4.2}$$

where

$$K(u_0) := \left\{ v \in W^{1,\infty}(\Omega) \ : \ \|\nabla v\|_\infty \leq 1, \ v|_{\partial\Omega} = u_0|_{\partial\Omega} \right\}.$$

Our aim is to approximate the Prigozhin model for the sandpile by a nonlocal model (Theorem 1.8) obtained as the limit as $p \to +\infty$ of the nonlocal p-Laplacian problem with Dirichlet boundary condition (Theorem 1.7).

To identify the limit as $p \to +\infty$ of the solutions u_p of problem $P_p^J(u_0,\psi)$ we will use the methods of convex analysis, and so we first recall some terminology (see [30], [17], and [8]). If H is a real Hilbert space with inner product $(\ ,\ )$ and $\Psi : H \to (-\infty, +\infty]$ is convex, then the subdifferential of Ψ is defined as the multivalued operator $\partial\Psi$ given by

$$v \in \partial\Psi(u) \iff \Psi(w) - \Psi(u) \geq (v, w-u) \quad \forall\, w \in H.$$

The epigraph of Ψ is defined by $\operatorname{Epi}(\Psi) = \{(u,\lambda) \in H \times \mathbb{R} \ : \ \lambda \geq \Psi(u)\}$.

Given K a closed convex subset of H, the indicator function of K is defined by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Then it is easy to see that the subdifferential is characterized as follows:

$$v \in \partial I_K(u) \iff u \in K \text{ and } (v, w-u) \leq 0 \quad \forall\, w \in K.$$

In case the convex functional $\Psi : H \to (-\infty, +\infty]$ is proper, lower-semicontinuous, and $\min \Psi = 0$, it is well known (see [17]) that the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni f(t), & \text{a.e. } t \in]0,T[, \\ u(0) = u_0, \end{cases}$$

has a unique strong solution for any $f \in L^2(0,T;H)$ and $u_0 \in \overline{D(\partial\Psi)}$.

The following convergence was studied by Mosco in [34] (see [8]). Suppose X is a metric space and $A_n \subset X$. We define

$$\liminf_{n\to\infty} A_n = \{x \in X \ : \ \exists x_n \in A_n,\ x_n \to x\}$$

and

$$\limsup_{n\to\infty} A_n = \{x \in X \ : \ \exists x_{n_k} \in A_{n_k},\ x_{n_k} \to x\}.$$

In the case X is a normed space, we note by $s - \lim$ and $w - \lim$ the above limits associated, respectively, to the strong and to the weak topology of X.

Given a sequence $\Psi_n, \Psi : H \to (-\infty, +\infty]$ of convex lower-semicontinuous functionals, we say that Ψ_n converges to Ψ in the sense of Mosco if

$$w - \limsup_{n\to\infty} \mathrm{Epi}(\Psi_n) \subset \mathrm{Epi}(\Psi) \subset s - \liminf_{n\to\infty} \mathrm{Epi}(\Psi_n). \tag{4.3}$$

It is easy to see that (4.3) is equivalent to the two following conditions:

$$\forall\, u \in D(\Psi) \ \ \exists u_n \in D(\Psi_n) \ : \ u_n \to u \ \text{ and } \ \Psi(u) \geq \limsup_{n\to\infty} \Psi_n(u_n); \tag{4.4}$$

$$\text{for every subsequence } n_k,\ \text{ when } u_k \rightharpoonup u,\ \text{ it holds } \Psi(u) \leq \liminf_k \Psi_{n_k}(u_k). \tag{4.5}$$

As a consequence of the results in [19] and [8] we can write the following result.

THEOREM 4.1. *Let $\Psi_n, \Psi : H \to (-\infty, +\infty]$ convex lower-semicontinuous functionals. Then the following statements are equivalent.*

(i) *Ψ_n converges to Ψ in the sense of Mosco.*

(ii) *$(I + \lambda\partial\Psi_n)^{-1}u \to (I + \lambda\partial\Psi)^{-1}u, \quad \forall\, \lambda > 0,\ u \in H$.*

Moreover, any of these two conditions (i) *or* (ii) *imply that*

(iii) *for every $u_0 \in \overline{D(\partial\Psi)}$ and $u_{0,n} \in \overline{D(\partial\Psi_n)}$ such that $u_{0,n} \to u_0$, and every $f_n, f \in L^2(0,T;H)$ with $f_n \to f$, if $u_n(t)$, $u(t)$ are the strong solutions of the abstract Cauchy problems*

$$\begin{cases} u_n'(t) + \partial\Psi_n(u_n(t)) \ni f_n, & a.e. \ \ t \in]0,T[, \\ u_n(0) = u_{0,n}, \end{cases}$$

and

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni f, & a.e. \ \ t \in]0,T[, \\ u(0) = u_0, \end{cases}$$

respectively, then

$$u_n \to u \qquad in\ C([0,T] : H).$$

4.2. Limit as $p \to +\infty$. Let us consider the *nonlocal p-Laplacian evolution problem* with source

$$P_p^J(u_0,\psi,f) \begin{cases} u_t(t,x) = \displaystyle\int_\Omega J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy + f(t,x), \\ \hfill (t,x) \in]0,T[\times\Omega, \\ u(t,x) = \psi(x), \quad (t,x) \in]0,T[\times(\Omega_J \setminus \overline{\Omega}), \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases}$$

This problem is associated to the energy functional

$$G^J_{p,\psi}(u) = \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^p\,dy\,dx + \frac{1}{p}\int_\Omega\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)|\psi(y)-u(x)|^p\,dy\,dx.$$

With a formal calculation, taking limit as $p\to+\infty$, we arrive to the functional

$$G^J_{\infty,\psi}(u) = \begin{cases} 0 & \text{if } |u(x)-u(y)|\le 1, \text{ for } x,y\in\overline{\Omega} \\ & \text{and } |\psi(y)-u(x)|\le 1, \quad \text{for } x\in\Omega, y\in\Omega_J\setminus\overline{\Omega}, \\ & \text{with } x-y\in\operatorname{supp}(J) \\ +\infty & \text{in the other case.} \end{cases}$$

Hence, if we define

$$K^J_{\infty,\psi} := \left\{ u\in L^2(\Omega) \ : \begin{array}{l} |u(x)-u(y)|\le 1, x,y\in\Omega \\ \text{and } |\psi(y)-u(x)|\le 1, \quad \text{for } x\in\Omega, y\in\Omega_J\setminus\overline{\Omega}, \\ \text{with } x-y\in\operatorname{supp}(J) \end{array} \right\},$$

we have that the functional $G^J_{\infty,\psi}$ is given by the indicator function of $K^J_{\infty,\psi}$; that is, $G^J_{\infty,\psi} = I_{K^J_{\infty,\psi}}$. Then, the *nonlocal limit problem* can be written as

$$P^J_\infty(u_0,\psi,f) \quad \begin{cases} f(t,\cdot)-u_t(t)\in\partial I_{K^J_{\infty,\psi}}(u(t)), & \text{a.e. } t\in]0,T[, \\ u(0,x)=u_0(x). \end{cases}$$

Proof of Theorem 1.7. Let $T>0$. By Theorem 4.1, to prove the result it is enough to show that the functionals

$$G^J_{p,\psi}(u) = \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^p\,dy\,dx + \frac{1}{p}\int_\Omega\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)|\psi(y)-u(x)|^p\,dy\,dx$$

converge to

$$G^J_{\infty,\psi}(u) = \begin{cases} 0 & \text{if } |u(x)-u(y)|\le 1, \text{ for } x,y\in\Omega \\ & \text{and } |\psi(y)-u(x)|\le 1, \quad \text{for } x\in\Omega, y\in\Omega_J\setminus\overline{\Omega}, \\ & \text{with } x-y\in\operatorname{supp}(J) \\ +\infty & \text{in the other case} \end{cases}$$

as $p\to+\infty$, in the sense of Mosco. First, let us check that

$$\operatorname{Epi}\left(G^J_{\infty,\psi}\right) \subset s-\liminf_{p\to+\infty}\operatorname{Epi}\left(G^J_{p,\psi}\right). \tag{4.6}$$

To this end let $(u,\lambda)\in\operatorname{Epi}(G^J_{\infty,\psi})$. We can assume that $u\in K^J_{\infty,\psi}$ and $\lambda\ge 0$ (as $G^J_{\infty,\psi}(u)=0$). Now take

$$v_p = u \qquad \text{and} \qquad \lambda_p = G^J_{p,\psi}(u)+\lambda. \tag{4.7}$$

Then, as $\lambda \geq 0$ we have $(v_p, \lambda_p) \in \mathrm{Epi}(G^J_{p,\psi})$. Obviously, $v_p = u \to u$ in $L^2(\Omega)$, and as $u \in K^J_{\infty,\psi}$,

$$\begin{aligned} G^J_{p,\psi}(u) &= \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^p\,dydx \\ &\quad + \frac{1}{p}\int_\Omega\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)|\psi(y)-u(x)|^p\,dy\,dx \\ &\leq \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)\,dydx + \frac{1}{p}\int_\Omega\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)\,dy\,dx \to 0 \end{aligned}$$

as $p \to +\infty$. Therefore, we get (4.6). Finally, let us prove that

$$w - \limsup_{p\to+\infty} \mathrm{Epi}\left(G^J_{p,\psi}\right) \subset \mathrm{Epi}\left(G^J_{\infty,\psi}\right). \tag{4.8}$$

To this end, let us consider a sequence $(u_{p_j}, \lambda_{p_j}) \in \mathrm{Epi}(G^J_{p_j,\psi})$; that is, $G^J_{p_j,\psi}(u_{p_j}) \leq \lambda_{p_j}$, with

$$u_{p_j} \rightharpoonup u, \qquad \text{and} \qquad \lambda_{p_j} \to \lambda.$$

Since, $0 \leq G^J_{p_j,\psi}(u_{p_j}) \leq \lambda_{p_j} \to \lambda$, $0 \leq \lambda$. On the other hand, we have that there exists a constant $C > 0$ such that

$$\begin{aligned} (p_jC)^{1/p_j} \geq \left(p_j G^J_{p,\psi}(u_{p_j})\right)^{1/p_j} &= \Bigg(\frac{1}{2}\int_\Omega\int_\Omega J(x-y)|u_{p_j}(y)-u_{p_j}(x)|^{p_j}\,dy\,dx \\ &\quad + \int_\Omega\int_{\Omega_J\setminus\overline{\Omega}} J(x-y)|\psi(y)-u_{p_j}(x)|^{p_j}\,dy\,dx\Bigg)^{1/p_j}. \end{aligned}$$

Then, by the above inequality,

$$\begin{aligned} &\left(\int_\Omega\int_\Omega J(x-y)\left|u_{p_j}(y)-u_{p_j}(x)\right|^q\,dy\,dx\right)^{1/q} \\ &\quad\leq \left(\int_\Omega\int_\Omega J(x-y)\,dy\,dx\right)^{(p_j-q)/p_jq} \\ &\qquad\times \left(\int_\Omega\int_\Omega J(x-y)\left|u_{p_j}(y)-u_{p_j}(x)\right|^{p_j}\,dy\,dx\right)^{1/p_j} \\ &\quad\leq \left(\int_\Omega\int_\Omega J(x-y)\,dy\,dx\right)^{(p_j-q)/p_jq} (Cp_j)^{1/p_j}. \end{aligned}$$

Hence, we can extract a subsequence (if necessary) and let $p_j \to +\infty$ to obtain

$$\left(\int_\Omega\int_\Omega J(x-y)\,|u(y)-u(x)|^q\,dy\,dx\right)^{1/q} \leq \left(\int_\Omega\int_\Omega J(x-y)\,dy\,dx\right)^{1/q}.$$

Now, just taking $q \to +\infty$, we get

$$|u(x)-u(y)| \leq 1 \qquad \text{a.e. } (x,y) \in \Omega\times\Omega,\ x-y \in \mathrm{supp}(J).$$

With a similar argument we obtain

$$|u(x)-\psi(y)| \leq 1 \qquad \text{a.e. } x \in \Omega,\ y \in \Omega_J\setminus\overline{\Omega},\ \text{with}\ \ x-y \in \mathrm{supp}(J).$$

Hence, we conclude that $u \in K^J_{\infty,\psi}$. This ends the proof. $\square$

4.3. Rescaling. We will assume now that Ω is convex and ψ verifies $\|\nabla\psi\|_\infty \le 1$. For $\varepsilon > 0$, we rescale the functional $G^J_{\infty,\psi}$ as follows:

$$G^{\varepsilon}_{\infty,\psi}(u) = \begin{cases} 0 & \text{if } |u(x)-u(y)| \le \varepsilon, \text{ for } x, y \in \Omega \\ & \text{and } |\psi(y)-u(x)| \le \varepsilon, \quad \text{for } x \in \Omega, y \in \Omega_J \setminus \overline{\Omega}, \\ & \text{with } |x-y| \le \varepsilon \\ +\infty & \text{in the other case.} \end{cases}$$

In other words, $G^{\varepsilon}_{\infty,\psi} = I_{K^{\varepsilon}_{\infty,\psi}}$, where

$$K^{\varepsilon}_{\infty,\psi} := \left\{ u \in L^2(\Omega) \ : \ \begin{array}{l} |u(x)-u(y)| \le \varepsilon, x, y \in \Omega \\ \text{and } |\psi(y)-u(x)| \le \varepsilon, \quad \text{for } x \in \Omega, y \in \Omega_J \setminus \overline{\Omega}, \\ \text{with } |x-y| \le \varepsilon \end{array} \right\}.$$

Consider the gradient flow associated to the functional $G^{\varepsilon}_{\infty,\psi}$

$$P^{\varepsilon}_{\infty}(u_0,\psi,f) \quad \begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial I_{K^{\varepsilon}_{\infty,\psi}}(u(t)), & \text{a.e. } t \in]0,T[, \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases}$$

and the problem

$$P_{\infty}(u_0,\psi,f) \quad \begin{cases} f(t,\cdot) - u_{\infty,t} \in \partial I_{K_\psi}(u_\infty), & \text{a.e. } t \in]0,T[, \\ u_\infty(0,x) = u_0(x), & \text{in } \Omega, \end{cases}$$

where

$$K_\psi := \left\{ u \in W^{1,\infty}(\Omega) \ : \ \|\nabla u\|_\infty \le 1, \ u|_{\partial\Omega} = \psi|_{\partial\Omega} \right\}.$$

Observe that if $u \in K_\psi$, $\|\nabla u\|_\infty \le 1$. Then, since $\|\nabla\psi\|_\infty \le 1$ and Ω is convex, we have $|u(x)-u(y)| \le |x-y|$ and $|u(x)-\psi(y)| \le |x-y|$, from where it follows that $u \in K^{\varepsilon}_{\infty,\psi}$, that is, $K_\psi \subset K^{\varepsilon}_{\infty,\psi}$.

With all these definitions and notations, we can proceed with the limit as $\varepsilon \to 0$ for the sandpile model ($p = +\infty$).

Proof of Theorem 1.8. Since $u_0 \in K_\psi$, $u_0 \in K^{\varepsilon}_{\infty,\psi}$ for all $\varepsilon > 0$. Again we are using that $\|\nabla\psi\|_\infty \le 1$. Consequently, there exists $u_{\infty,\varepsilon}$ the unique solution of $P^{\varepsilon}_{\infty}(u_0,\psi,f)$.

By Theorem 4.1, to prove the result it is enough to show that $I_{K^{\varepsilon}_{\infty,\psi}}$ converges to I_{K_ψ} in the sense of Mosco. Using that $\|\nabla\psi\|_\infty \le 1$ it is easy to obtain that

$$K^{\varepsilon_1}_{\infty,\psi} \subset K^{\varepsilon_2}_{\infty,\psi}, \quad \text{if } \ \varepsilon_1 \le \varepsilon_2. \tag{4.9}$$

Since $K_\psi \subset K^{\varepsilon}_{\infty,\psi}$ for all $\varepsilon > 0$, we have

$$K_\psi \subset \bigcap_{\varepsilon > 0} K^{\varepsilon}_{\infty,\psi}.$$

On the other hand, if

$$u \in \bigcap_{\varepsilon > 0} K^{\varepsilon}_{\infty,\psi},$$

we have

$$|u(y) - u(x)| \leq |y - x|, \quad \text{a.e. } x, y \in \Omega,$$

and moreover

$$|u(y) - \psi(x)| \leq |y - x|, \quad \text{a.e. } x \in \Omega_J \setminus \overline{\Omega},\ y \in \Omega,$$

from where it follows that $u \in K_\psi$. Therefore, we have

$$K_\psi = \bigcap_{\varepsilon > 0} K^\varepsilon_{\infty,\psi}. \tag{4.10}$$

Note that

$$\operatorname{Epi}(I_{K_\psi}) = K_\psi \times [0, \infty[, \quad \operatorname{Epi}\left(I_{K^\varepsilon_{\infty,\psi}}\right) = K^\varepsilon_{\infty,\psi} \times [0, \infty[\ \ \forall\, \varepsilon > 0. \tag{4.11}$$

By (4.10) and (4.11),

$$\operatorname{Epi}\left(I_{K_\psi}\right) \subset s - \liminf_{\varepsilon \to 0} \operatorname{Epi}\left(I_{K^\varepsilon_{\infty,\psi}}\right). \tag{4.12}$$

On the other hand, given $(u, \lambda) \in w - \limsup_{\varepsilon \to 0} \operatorname{Epi}(I_{K^\varepsilon_{\infty,\psi}})$ there exists $(u_{\varepsilon_k}, \lambda_k) \in K_{\varepsilon_k,\psi} \times [0, \infty[$, such that $\varepsilon_k \to 0$ and

$$u_{\varepsilon_k} \rightharpoonup u \quad \text{in } L^2(\Omega), \quad \lambda_k \to \lambda \ \text{ in } \mathbb{R}.$$

By (4.9), given $\varepsilon > 0$, there exists k_0, such that $u_{\varepsilon_k} \in K^\varepsilon_{\infty,\psi}$ for all $k \geq k_0$. Then, since $K^\varepsilon_{\infty,\psi}$ is a closed convex set, we get $u \in K^\varepsilon_{\infty,\psi}$, and, by (4.10), we obtain that $u \in K_0$. Consequently,

$$w - \limsup_{n \to \infty} \operatorname{Epi}\left(I_{K^\varepsilon_{\infty,\psi}}\right) \subset \operatorname{Epi}\left(I_{K_\psi}\right). \tag{4.13}$$

Finally, by (4.12), (4.13), and having in mind (4.3), we obtain that $I_{K^\varepsilon_{\infty,\psi}}$ converges to I_{K_ψ} in the sense of Mosco. ∎

4.4. Explicit solutions. Our goal now is to show some explicit examples that illustrate the behavior of the solutions when $p = +\infty$.

Remark 4.2. There is a natural upper bound (and of course also a natural lower bound) for the solutions with boundary datum ψ outside Ω (regardless the source term f). Indeed, given a bounded domain $\Omega \subset \mathbb{R}^N$ let us define inductively

$$\Omega_1 = \left\{x \in \Omega \ : \ |x - y| < 1 \text{ for some } y \in \Omega_J \setminus \overline{\Omega}\right\}$$

and, for $j \geq 2$,

$$\Omega_j = \left\{x \in \Omega \setminus \cup_{i=1}^{j-1} \Omega_i \ : \ |x - y| < 1 \text{ for some } y \in \Omega_{j-1}\right\}.$$

Then, since $u(t) \in K^J_{\infty,\psi}$ we must have

$$u(t,x) \leq \psi(y) + 1 \qquad \text{if } \ |x - y| \leq 1,\ x \in \Omega_1,\ y \in \Omega_J \setminus \overline{\Omega},$$

and for any $j \geq 2$

$$u(t,x) \leq u(t,y) + 1 \qquad \text{if } \ |x - y| \leq 1,\ x \in \Omega_j,\ y \in \Omega_{j-1} \setminus \Omega_j.$$

Therefore we have an upper bound for $u(t,x)$ in the whole Ω,

$$u(t,x) \le \Psi_1(x),$$

where Ψ_1 is defined by the inductive formula,

$$\Psi_1(x) = \max\left\{\psi(y)+1 \ : \ y \in \Omega_J \setminus \overline{\Omega},\ |x-y| \le 1\right\}, \text{ for } x \in \Omega_1,$$

and

$$\Psi_1(x) = \max\left\{\Psi_1(y)+1 \ : \ y \in \Omega_{j-1},\ |x-y| \le 1\right\}, \text{ for } x \in \Omega_j, \text{ if } j \ge 2.$$

Analogously, we can obtain a lower bound for $u(t,x)$,

$$u(t,x) \ge \Phi_1(x),$$

where Φ_1 is defined by the inductive formula,

$$\Phi_1(x) = \min\left\{\psi(y)-1 \ : \ y \in \Omega_J \setminus \overline{\Omega},\ |x-y| \le 1\right\}, \text{ for } x \in \Omega_1,$$

and

$$\Phi_1(x) = \min\left\{\Phi_1(y)-1 \ : \ y \in \Omega_{j-1},\ |x-y| \le 1\right\}, \text{ for } x \in \Omega_j, \quad \text{if } j \ge 2.$$

With this remark in mind we show some explicit examples of solutions to

$$P_\infty^J(u_0,\psi,f) \quad \begin{cases} f(t,x) - u_t(t,x) \in \partial G^J_{\infty,\psi}(u(t)), & \text{a.e. } t \in]0,T[, \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases}$$

where

$$G^J_{\infty,\psi}(u) = \begin{cases} 0 & \text{if } u \in L^2(\Omega),\ |u(x)-u(y)| \le 1, \text{ for } x,y \in \Omega,\ |x-y| \le 1, \\ & \text{and } |u(x)-\psi(y)| \le 1, \text{for } x \in \Omega,\ y \in \Omega_J \setminus \overline{\Omega},\ |x-y| \le 1, \\ +\infty & \text{in the other case.} \end{cases}$$

In order to verify that a function $u(t,x)$ is a solution to $P_\infty^J(u_0,\psi,f)$, we need to check that

$$G^J_{\infty,\psi}(v) \ge G^J_{\infty,\psi}(u) + \langle f-u_t,\ v-u\rangle, \qquad \text{for all } v \in L^2(\Omega). \tag{4.14}$$

To this end we can assume that $v \in K^J_{\infty,\psi}$ (otherwise $G^J_{\infty,\psi}(v) = +\infty$ and then (4.14) becomes trivial). Therefore, we need to check that

$$u(t,\cdot) \in K^J_{\infty,\psi} \tag{4.15}$$

and, by (4.14), that

$$\int_\Omega (f(t,x) - u_t(t,x))(v(x) - u(t,x))\,dx \le 0 \tag{4.16}$$

for every $v \in K^J_{\infty,\psi}$.

Example 1. Let us consider a nonnegative source f and as initial condition the upper bound defined in the previous remark, $u_0(x) = \Psi_1(x)$. Then the solution to $P_\infty^J(u_0,\psi,f)$ is given by

$$u(t,x) \equiv \Psi_1(x)$$

for every $t > 0$. Indeed, $\Psi_1(x) \in K^J_{\infty,\psi}$ and for every $v \in K^J_{\infty,\psi}$ we have that $v(x) \le \Psi_1(x)$, and therefore

$$\int_\Omega (f(t,x) - u_t(t,x))(v(x) - u(t,x))\, dx = \int_\Omega f(t,x)(v(x) - \Psi_1(x))\, dx \le 0,$$

as we have to show.

In general, given a nonnegative source f supported in $D \subset \Omega$, any initial condition $u_0 \in K^J_{\infty,\psi}$ that verifies $u_0(x) = \Psi_1(x)$ in D produces a stationary solution $u(t,x) \equiv u_0(x)$.

Analogously, it can be shown that $u(t,x) \equiv \Phi_1(x)$ when $u_0(x) = \Phi_1(x)$ and $f(t,x) \le 0$.

Example 2. Now, let us assume that we are in an interval $\Omega = (-L, L)$, $\psi = 0$, $\varepsilon = L/n$, $n \in \mathbf{N}$, $u_0 = 0$ which belongs to $K_{\varepsilon,0}$, and the source f is an approximation of a delta function,

$$f(t,x) = f_\eta(t,x) = \frac{1}{\eta}\chi_{[-\frac{\eta}{2},\frac{\eta}{2}]}(x), \quad 0 < \eta \le 2\varepsilon.$$

Using the same ideas of [6], it is easy to verify the following general formula that describes the solution of $P^\varepsilon_\infty(u_0,\psi,f)$ for every $t \ge 0$. For any given integer $l \ge 0$ we have

$$u(t,x) = \begin{cases} l\varepsilon + k_l(t-t_l), & x \in [-\frac{\eta}{2}, \frac{\eta}{2}], \\ (l-1)\varepsilon + k_l(t-t_l), & x \in [-\frac{\eta}{2}-\varepsilon, \frac{\eta}{2}+\varepsilon] \setminus [-\frac{\eta}{2}, \frac{\eta}{2}], \\ \dots & \\ k_l(t-t_l), & x \in [-\frac{\eta}{2}-l\varepsilon, \frac{\eta}{2}+l\varepsilon] \setminus [-\frac{\eta}{2}-(l-1)\varepsilon, \frac{\eta}{2}+(l-1)\varepsilon], \\ 0, & x \notin [-\frac{\eta}{2}-l\varepsilon, \frac{\eta}{2}+l\varepsilon], \end{cases}$$

for $t \in [t_l, t_{l+1})$, where

$$k_l = \frac{1}{2l\varepsilon + \eta} \quad \text{and} \quad t_{l+1} = t_l + \frac{\varepsilon}{k_l}, \quad t_0 = 0.$$

This general formula is valid until the time at which the solution verifies $u(t,x) = \Psi_\varepsilon(x)$ for $x \in [-\frac{\eta}{2}, \frac{\eta}{2}]$ (the support of f), that is, until $T = t_{l^*+1}$, where

$$l^* \ \text{ is the first } l \text{ such that } \ l\varepsilon + k_l(t_{l+1} - t_l) = \Psi_\varepsilon(0)$$

and

$$\Psi_\varepsilon \text{ is the natural upper bound defined in Remark 4.2}$$

for the corresponding rescaled kernel. Observe that for this l^*, $\frac{\eta}{2} + l^*\varepsilon \le L$. From that time on the solution is stationary, that is, $u(t,x) = u(T,x)$ for all $t > T$.

From the above formula, taking limits as $\eta \to 0$, we get that the expected solution to $P^\varepsilon_\infty(u_0,\psi,\delta_0)$ is given, for any given integer $l \ge 1$, by

$$u(t,x) = \begin{cases} (l-1)\varepsilon + k_l(t-t_l), & x \in [-\varepsilon, \varepsilon], \\ (l-2)\varepsilon + k_l(t-t_l), & x \in [-2\varepsilon, 2\varepsilon] \setminus [-\varepsilon, \varepsilon], \\ \dots & \\ k_l(t-t_l), & x \in [-l\varepsilon, l\varepsilon] \setminus [-(l-1)\varepsilon, (l-1)\varepsilon], \\ 0, & x \notin [-l\varepsilon, l\varepsilon], \end{cases} \tag{4.17}$$

for $t \in [t_l, t_{l+1})$, where $k_l = \frac{1}{2l\varepsilon}$, $t_{l+1} = t_l + \frac{\varepsilon}{k_l}$, $t_1 = 0$, until $T = t_{l^*+1}$, where

$$l^* \text{ is the first } l \text{ such that } l\varepsilon + k_l(t_{l+1} - t_l) = \Psi_\varepsilon(0).$$

And from that time on the solution is stationary, that is, $u(t,x) = u(T,x)$ for all $t > T$.

Remark that, since the space of functions $K^\varepsilon_{\infty,\psi}$ is not contained into $C(\mathbb{R})$, the formulation (4.16) with $f = \delta_0$ does not make sense. Hence the function $u(t,x)$ described by (4.17) is to be understood as a *generalized solution* to $P^\varepsilon_\infty(u_0, \psi, \delta_0)$ (it is obtained as a limit of solutions to approximating problems).

Note that the function $u(T,x)$ is a "regular and symmetric pyramid" composed by squares of side ε which is one step below the upper profile Ψ_ε.

Recovering the sandpile model as $\varepsilon \to 0$. Now, to recover the sandpile model, take the limit as $\varepsilon \to 0$ in the previous example to get that $u(t,x) \to v(t,x)$, where

$$v(t,x) = (l - |x|)^+ \qquad \text{for } t = l^2,$$

until the time at which $t = L^2$, and from that time the solution is stationary.

A similar argument shows that, for any $a \in (0, L)$, the *generalized solution* to $P^\varepsilon_\infty(0, 0, \delta_a)$ converges as $\varepsilon \to 0$ to $v(t,x)$, where

$$v(t,x) = (l - |x - a|)^+ \qquad \text{for } t = l^2,$$

until the time at which $t = (L-a)^2$, and from that time the solution is stationary.

These concrete examples illustrate the general convergence result in Theorem 1.8.

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