Tema 2

Edgeworth’s Exchange Theory
The exchange Theory of Edgeworth. A simple exchange model 2X2.

- 2 agents A y B and 2 goods: $x_1 y x_2$
- No production
- Initial endowments are given by:

$$w^A = (w_1^A, w_2^A) \quad y \quad w^B = (w_1^B, w_2^B) \quad \text{con}$$

$$w_1^A + w_1^B = w_1 \quad y \quad w_2^A + w_2^B = w_2$$

- Each agent has well-defined preferences over baskets of goods and can consume either her initial endowment or exchange it with the other agents.
The exchange Theory of Edgeworth. A simple exchange model 2X2.

- Let a consumption basket of A and B be:
  \[ x^A = (x_1^A, x_2^A) \quad \text{and} \quad x^B = (x_1^B, x_2^B) \]

- An allocation is a pair of consumption baskets:
  \[ x = (x^A, x^B) \]

- An allocation is feasible if:
  \[ x_1^A + x_1^B = w_1^A + w_1^B = w_1 \quad \text{and} \quad x_2^A + x_2^B = w_2^A + w_2^B = w_2 \]
A simple model of Pure exchange. The Edgeworth-Bowley box summarizes the set of all feasible allocations.
The Exchange Theory of Edgeworth. Pareto Rationality.

PARETO RATIONALITY: An allocation of goods is **Pareto efficient** if no one can be made better off without making someone else worse off. Formally:

Definition: A feasible allocation \( x \) is *Pareto optimal* (or Pareto Efficient) if there is no other feasible allocation \( y \) such that:

1) \( u_i(y_i) \geq u_i(x_i) \) for all \( i \), and
2) \( u_j(y_j) > u_j(x_j) \) for at least some \( j \)

- The **contract curve** shows all the efficient allocations of goods between two consumers.
- To calculate the contract curve, the utility of an agent is maximized subject to both the feasibility constraint and to the utility level of the other agent’s constraint: \( \text{Max } u^1(x^1) \), subject to \( u^2(x^2) \geq y^2 \) and subject to feasibility (the two FOC of the assoc. Lagrangian \( \rightarrow \text{RMS}^1=\text{RMS}^2 \) and feasibility).
The Exchange Theory of Edgeworth.

The core of an economy

- **INDIVIDUAL RATIONALITY**: An allocation $x^i$ satisfies individual rationality (IR) with respect to $w^i$ if: $u^i(x^i) \geq u^i(w^i)$

- The core of an exchange economy is the set of feasible allocations which cannot be improved upon (or blocked) by any coalition of agents.

- For 2 agent-exchange economies core allocations are those satisfying *individual rationality and Pareto efficiency*.

- For n agent-economies: We need to define "coalitions" of agents and how they can block a given allocation.
The Exchange Theory of Edgeworth.

The core of an economy

**Coalition**: A coalition $S$ is any subset of agents with mandatory agreements.

- Any coalition $S$ can block a proposed allocation $x$ whenever the agents in $S$ can reallocate their initial endowments among themselves and be better than under $x$.

- **Core**: RI, Pareto Rationality and rationality of all the remaining coalitions.

- Example: three agents $\{A,B,C\}$
  - Coalitions: $\{A\}, \{B\}, \{C\}; \{A,B,C\}; \{A,B\}, \{B,C\}$
  - $\{A,C\}$
Core of an exchange economy

- Let \( n \) be the number of agents of the economy,
- \( w = (w^1, w^2, \ldots, w^n) \) the vector of initial endowments,
- \( x = (x^1, x^2, \ldots, x^n) \) an allocation of the economy, and
- \( F(w) = \{ x : \sum_i x^i = \sum_i w^i \} \) the set of feasible allocations.

**Blocking coalition:** Let \( S \) be a coalition. \( S \) blocks allocation \( x \) in \( F(W) \), through \( y \) in \( F(W) \) if:
- 1) \( u^i(y^i) \geq u^i(x^i) \) for all \( i \) in \( S \), and
- 2) \( u^j(y^j) > u^j(x^j) \) for at least some \( j \) in \( S \)
- 3) \( \sum_{i \in S} y^i \leq \sum_{i \in S} w^i \) (feasibility in \( S \))

- The core of an exchange economy \( C(w) \):
  - \( C(w) = \{ x : \text{there is no } y \text{ satisfying 1), 2) } y \text{ 3) with } x \text{ and } y \text{ in } F(w) \} \)
Existence of the core of an exchange economy. Is the core empty? No, whenever there exists a WE, since it belongs to the core.

Definition (alternative): A pair allocation-price \((x^*, p^*)\) is a WE:

1) \(\sum_i x^*_i = \sum_i w^i\) (\(x^*\) is feasible), and
2) If \(u^i (x^i) > u^i (x^*_i)\), then \(p^* x^i > p^* w^i\) (\(x\) is not affordable).

**Proposition:** If \((x^*, p^*)\) is a WE for the initial endowment \(w\), then \(x^*\) belongs to \(C(w)\).

**Proof:** Suppose on the contrary that \(x^*\) does not belong to \(C(w)\). Then there is a coalition \(S\) and an allocation \(x\) such that for all \(i\) in \(S\), \(u^i (x^i) > u^i (x^*_i)\), and

\[\sum_{i \in S} x^i = \sum_{i \in S} w^i\] (\(x\) is feasible for \(S\)), \(\rightarrow p^* \sum_{i \in S} x^i = p^* \sum_{i \in S} w^i\) (1)

As \(x^*\) is a WE, then by definition, for all \(i\) in \(S\)

\(p^* x^i > p^* w^i\) and adding over all \(i\)'s in \(S\):

\[p^* \sum_{i \in S} x^i > p^* \sum_{i \in S} w^i,\] which contradicts (1) \((= \sum_{i \in S} x^i)\)

Then \(x^*\) belongs to \(C(w)\).
Contraction of the core and replica-economies

The core has ore allocations than the WE.

We show that if the economy increases its size, then new coalitions will appear and more opportunities to block or to improve upon:

*The core shrinks (contracts)*

We use a very simple type of growth.

**Definition:** 2 agents are of the same type if both their preferences and their initial endowment are identical.

**Definition:** An economy is a *replica of size* $r$ of another economy, if there are $r$-times as many agents of each type in the former economy as in the later.

Thus, if a large economy replicates a smaller economy, then it will just be as a “scale up” version of the small one.

We only consider 2 types of agents: type A and type B.
Equal Treatment in the Core

The $r$-core ($r$-C) of an economy is the core of its replica of size $r$.

**Lemma:** Equal treatment in the Core.
Suppose that agents’ preferences are strictly convex, continuous and strongly monotone. If $x$ belongs to the $r$-core of a given economy, then any two agents of the same type will receive the same bundle in $x$.

Proof: Let

\[ A_1, A_2, \ldots, A_r \quad \text{and} \quad B_1, B_2, \ldots, B_r \]

2 types of agents in the $r$-replica.
If all agents of the same type do not get the same allocation, there will be one agent of each type who is the most poorly treated.

Call these two agents: type A underdog (marginated): $A_M$ and type B underdog: $B_M$. 
Equal Treatment in the Core.

- (cont. proof.) Let the mean (average) allocations be:

\[ x_A = \frac{1}{r} \sum_{j=1}^{r} x_{A_j} \quad y \quad x_B = \frac{1}{r} \sum_{j=1}^{r} x_{B_j} \]

- and we have that: \( x_{A_M} < x_A, \quad x_{B_M} \leq x_B \)

- (Note that if all agents receive the same, then they will get the average allocations). By convexity of preferences \( A_M \) and \( B_M \) prefer the mean allocations to their allocation in \( x \):

\[ u_A(x_A) > u_A(x_{A_M}), \quad u_B(x_B) \geq u_B(x_{B_M}) \]

- Can \( A_M \) and \( B_M \) block core-allocation \( x \) through the average allocations?

- They could whenever average allocations are \textbf{feasible} for the coalition of them:

\[ x_A + x_B = w_A + w_B \]
Equal treatment in the Core.

- (cont. proof.) We check the feasibility of average allocations for the underdog-coalition: by feasibility of $x$ and given that all agents of the same type have the same initial endowment.

$$
\bar{x}_A + \bar{x}_B = \frac{1}{r} \sum_{j=1}^{r} x_{A_j} + \frac{1}{r'} \sum_{j=1}^{r'} x_{B_j} = x_{A_1} + x_{A_2} + \ldots + x_{A_r} + x_{B_1} + \ldots + x_{B_{r'}} =
$$

$$
\bar{w}_A + \bar{w}_A + \ldots + \bar{w}_{A_r} + \bar{w}_B + \ldots + \bar{w}_{B_{r'}} = \frac{1}{r} \sum_{j=1}^{r} w_{A_j} + \frac{1}{r'} \sum_{j=1}^{r'} w_{B_j} =
$$

$$
\frac{1}{r} \sum_{j=1}^{r} w_A + \frac{1}{r'} \sum_{j=1}^{r'} w_B = w_A + w_B.
$$

- Then, average allocations are feasible for the coalition of $A_M$ and $B_M$. 

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Chapter 16

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(Cont). The underdog $A_M$ strictly prefers its type average allocation to $x_{AM}$ and the underdog $B_M$ considers its type average allocation at least as good as $x_{BM}$.

Strong monotonicity allows $A_M$ to remove a little quantity from its average allocation $\overline{x_A} - \varepsilon$

and to bribe $B_M$ by offering him: $\overline{x_B} + \varepsilon$

thus forming a coalition that can improve upon allocation $x$.

Then agents cannot receive a different treatment in the \textit{r-core} of an exchange economy. In the core, all agents of the same type have to receive the same bundle.
Contraction of the Core

- **Lemma implications**: To simplify the analysis of the core in replica-economies: an allocation \( x \) in \( C \) tell us what each agent type A and type B obtain and then we can keep on representing core-allocations in two-dimensions (in the Edgeworth’s box).

- Any allocation \( x \) that is not a WE must eventually not be in the r-core of the economy. Hence, core allocations of large economies look like market equilibria.

- **Proposition**: Contraction of the core:
  - Suppose that preferences are strictly convex and strongly monotone and that there is a unique WE: \( x^* \), for initial endowments \( w \). Then if \( y \) is not a WE allocation, then, there will exit some \( r \)-replication of the economy, such that \( y \) is not in the \( r \)-core.
Contraction of the Core

- **Proof:** Observe the following drawing:
Contraction of the Core

Since \( y \) is not a EW, the line through \( w \) and \( y \) must cut at least one agent’s indifference, say \( u_{A1} \) through \( y \). Then, it is possible to choose a point such as \( g \) which \( A \) prefers to \( y \).

We look for a replica-economy and a coalition that blocks allocation \( y \).

By continuity of the preferences: \( g = \theta w + (1 - \theta)y \)

Let \( \theta = T/V < 1 \), con T y V integer numbers.

Then: \( g_A = (T/V)w_A + (1 - T/V)y_A \)

Take the V-replica of the economy.

Form the coalition: V agents of type A and V-T agents of type B.

And consider the allocation asignación \( z \) giving \( g_A \) to type A agents and \( y_B \) to those of type B:

\( Z: g_A \) to type A with \( u_A(z_A) > u_A(y_A) \)
\( y_B \) to type B, with \( u_B(z_B) \geq u_B(y_B) \).

Then, \( z \) is strictly preferred to \( y \) since agents’ type A can always bribe agents’ type B giving them some epsilon.
Contraction of the Core

- In order this coalition can block allocation $y$ through $z$, $z$ has to be feasible for the coalition. Let us check that this is the case:

- $Vz_A + (V-T)z_B = Vg_A + (V-T)y_B = V[(T/V)w_A + (1-T/V)y_A] + (V-T)y_B = Tw_A + (V-T)y_B = Tw_A + (V-T)(y_A + y_B) = (by feasbl y)$
- $Tw_A + (V-T)(w_A + w_B) = Vw_A + (V-T)w_B$, which is the initial endowment of the proposed coalition.

- Then, the proposed coalition can block allocation $y$ through $z$, in the $V$-replica of the economy.

- In this way, all allocations of the core not being WE will disappear in some replicas of the economy.
- All core-allocations in huge economies are WE.
Contraction of the Core. Example

- 2 agents: A y B

\[ g = \frac{1}{2} w + \frac{1}{2} y \]
Contraction of the Core. Example

- What **replica** of the economy and what **coalition** can block **y** through **z**, with **g** subst for type A and **y** subst for type B?
- **g**=1/2 **w**+1/2 **y** then, \( \theta=T/V=1/2 \),
- That is: \( V=2 \ y \ V-T=2-1=1 \)
- Replicate the economy at scale \( V=2 \) (duplicate the economy: 4 agents) and form the coalition:
- \( V=2 \) agents of type A and \( (V-T)=1 \) agent of type B (coalition of 3 agents).
- This coalition can block **y** through **z** whenever **z** is feasible for the coalition. Checking feasibility:
  - \( Vz_A+(V-T)z_B=Vg_A+ +(V-T)y_B=2(1/2 \ w_A+1/2 \ y_A)+y_B= \)
  - \( w_A+y_A+y_B=w_A+w_A+w_B=2w_A+w_B \),
- **which are the initial endowments of the coalition.** Then, **y** can be blocked in the duplication of the economy by the proposed coalition of three agents and through allocation **z**.
Contraction of the Core. Example

In general:
If the allocation is $g = \frac{1}{n} w + \frac{(1-1/n)}{} y$
Then $\theta = \frac{T}{V} = 1/n$, that is: $V = n \ y \ V - T = n - 1$
With $g_A = \frac{1}{n} w_A + \frac{(n-1)}{n} y_A$

- Take the replica of the economy at escale $V = n$
- and the coalition:
- $V = n$ agents type A and $V - T = n - 1$ agents type B