Lesson 3

Welfare Economics.

Welfare Properties of Walrasian Equilibrium: The two Fundamental Theorems of Welfare.

- The existence of a Walrasian Equilibrium is interesting as a positive result insofar as we believein the behavioral assumptions –primarily that agents are price takers- which underly the model. However, we may still be interested in WE for their *normative* content:
- ***Are WE optimal or efficient in some sense?***
- Recall Pareto efficiency:
- Pareto Efficiency: An allocation of goods is **Pareto efficient** if no one can be made better off without making someone else worse off. Formally:
- Definition: A feasible allocation ***x*** *is Pareto optimal* (or Pareto Efficient) if there is no other feasible allocation ***y*** such that:

  1) $u^i(y^i) \geq u^i(x^i)$ for all i, and

  2) $u^j(y^j) > u^j(x^j)$ for at least some j

Welfare Properties of Walrasian Equilibrium: The First Fundamental Welfare Theorem.

Recall the alternative definition of WE taking into account the Equilibrium allocation:

Definition (alternative): A pair allocation-price *(x*, p*)* is a WE if:

- 1) $\sum_i x^{*i} = \sum_i w^i$ (x^* is feasible), and
- 2) If $u^i(x^i) > u^i(x^{*i})$, then $p^* x^i > p^* w^i$ (x is not affordable).

Proposition FTW : If (x^*, p^*) is a WE for the initial endowment w, then x^* is Pareto efficient.

Proof: Suppose on the contrary that x^* is not Pareto efficient. Then there is *an allocation* x such that for all i, $u^i(x^i) > u^i(x^{*i})$, and
$\sum_I x^i = \sum_i w^i$ (x is feasible), $\rightarrow p^* \sum_i x^i = p^* \sum_i w^i$ *(1)*
As x^* is a WE, then by definition, for all i
$p^* x^i > p^* w^i$ and adding over all i's *:*

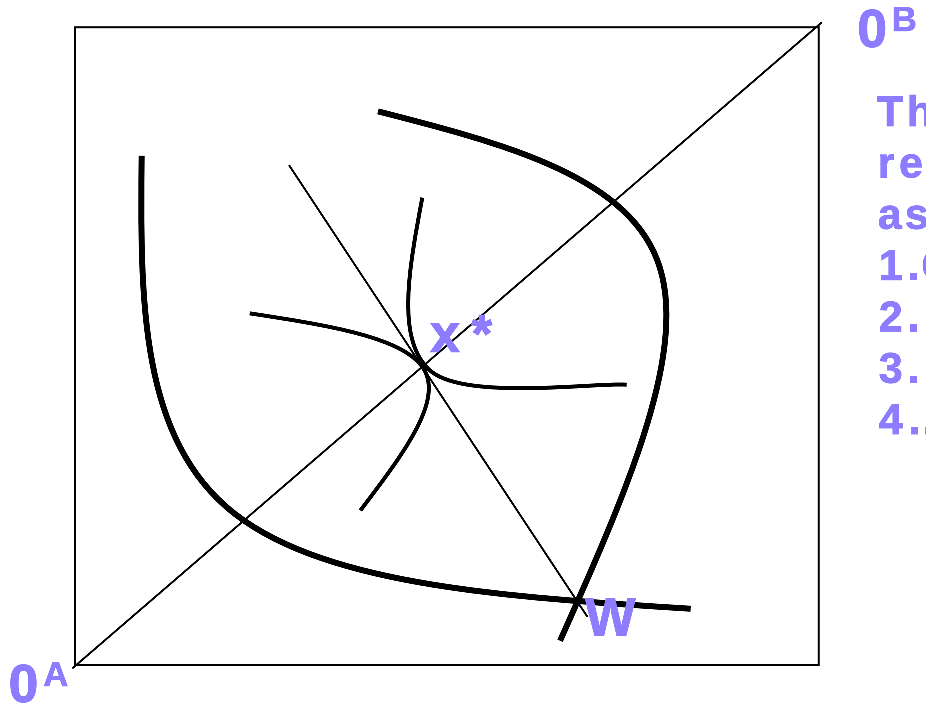
$$p^* \sum_i x^i > p^* \sum_i w^i \text{, which contradicts } (1) \ (= \sum_i x^i)$$

Then x* is Pareto efficient.

Welfare Properties of Walrasian Equilibrium: The First Fundamental Welfare Theorem.

Recall Edgewoth's box: It looks like that in the text book case:

Every WE is a PO allocation (1º TW) and every PO is a WE (2º TW)



The text book case relies on many assumptions:

1.Convex Preferec.

2. No satiation

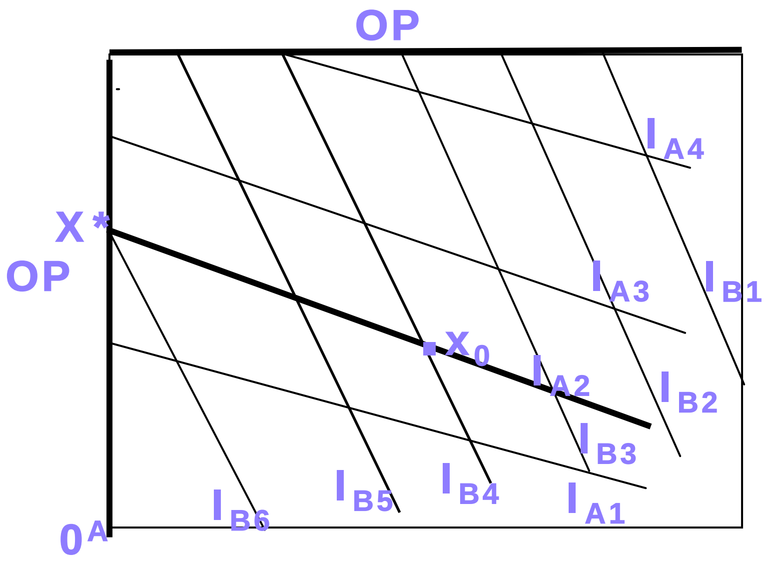
3. Perfect divisibility

4…

Welfare Properties of Walrasian Equilibrium: The First Fundamental Welfare Theorem.

Is every WE always a PO allocation**?** Under the assumptions of our model the answer is **YES**, but, in general, we can find problems when relaxing two axioms: ***1) Satiation***→allowing points of satiation inside the box and ***2) Good indivisibilities***.

Example 1**.** B has a point of maximal satiation inside the box:



Indifference curves are convex but not strictly convex.
Suppose first that there is no satiation point inside the box.
Pareto optimal allocations: North and west sides of the box

Suppose that the price budget line coincides with I_{A2} y $W=X_0$.

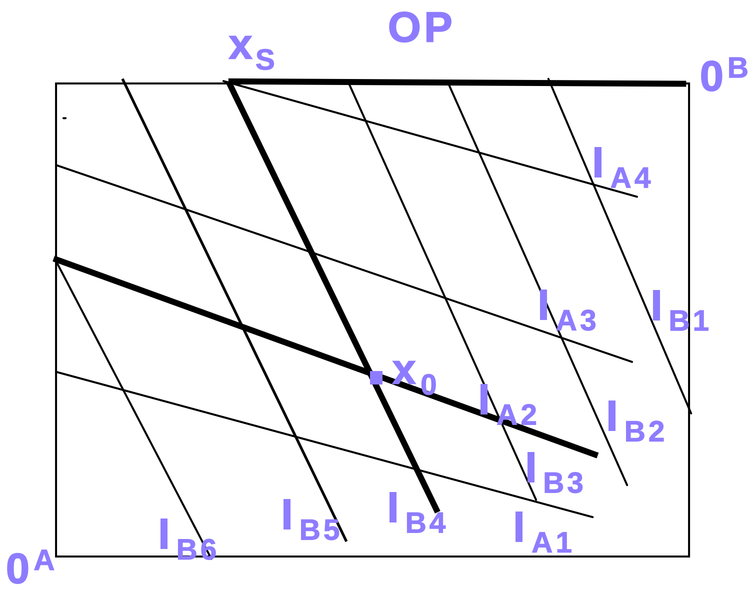
WE=X* and WE=PO

Welfare Properties of Walrasian Equilibrium:

The First Fundamental Welfare Theorem.

Example1 (cont.). Suppose now that B has a point of maximal satiation inside the box. Let $\boldsymbol{x_S}$ be such point a recall that $\boldsymbol{x_0}$ is the Initial endowment.

B is completely satiated over the straight line $\boldsymbol{x_S}$-$\boldsymbol{x_0}$



Pareto optimal allocations: from x_S to O_B in the north side of the box.

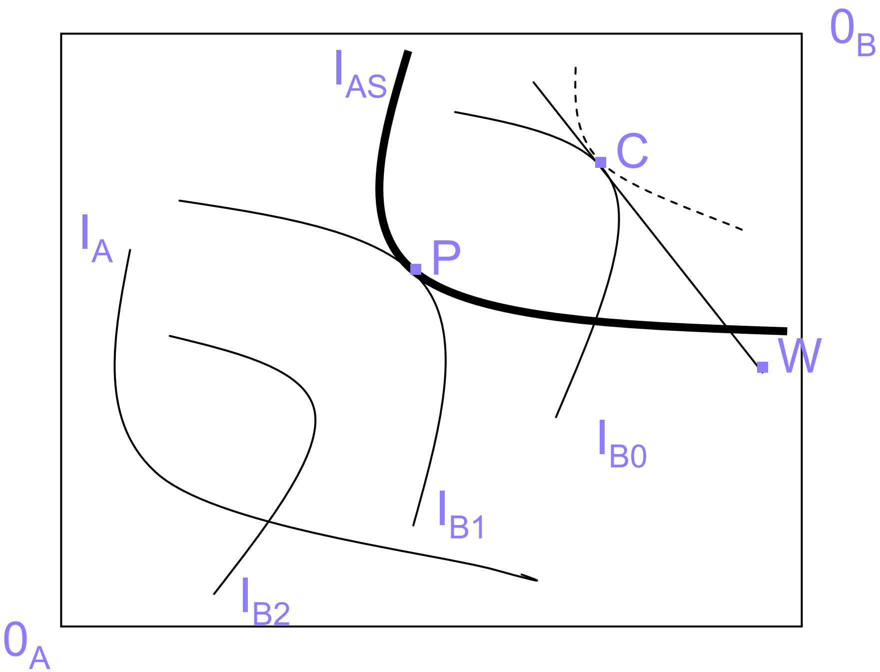
Suppose, as before, that the price budget line coincides with I_{A2} and that $W=X_0$.

By satiation:

WE=X_0 and WE is not PO, since A Is better off in x_S without B being worse off.

Welfare Properties of Walrasian Equilibrium: The First Fundamental Welfare Theorem.

- ***Example 2*:** A variation of the above argument. A feels satiated over and from I_{AS}. Each allocation in the area from I_{AS} gives him the same utility.

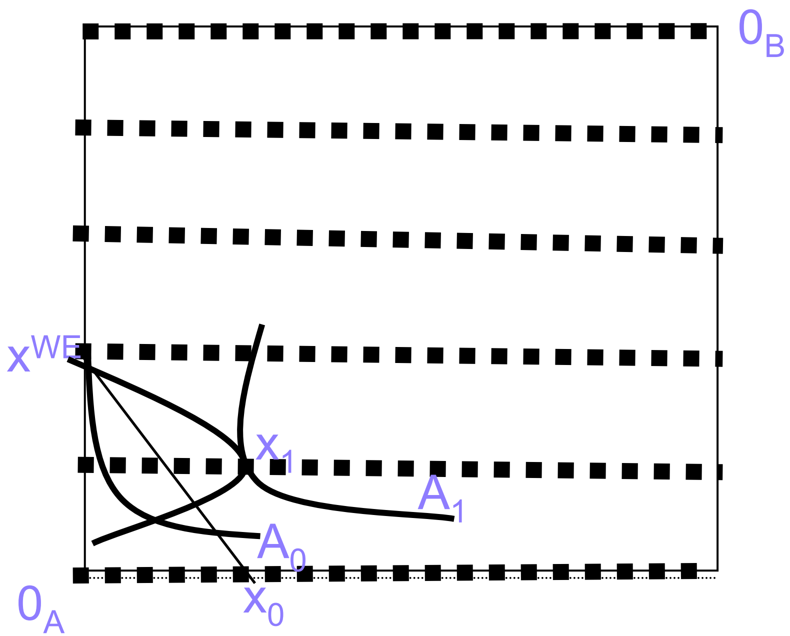


Let W be the initial endowment. With the budget constraint through W, the allocation C=WE.

But C is not a PO allocation, since B is better off in allocation P, without A being worse off.

Welfare Properties of Walrasian Equilibrium: The First Fundamental Welfare Theorem.

Example 3: Indivisible goods. Preferences are defined over bundles in $\mathbf{R^2}$, but the consumer can only choose points in the grid. Let x_0 be the initial endowment.



The WE is allocation x^{WE}, but this allocation is not PO, since A is better off in x_1 without B being worse off.

Welfare Properties of Walrasian Equilibrium: The Second Fundamental Welfare Theorem.

The normative content of WE comes from the 2º Theorem of Welfare:

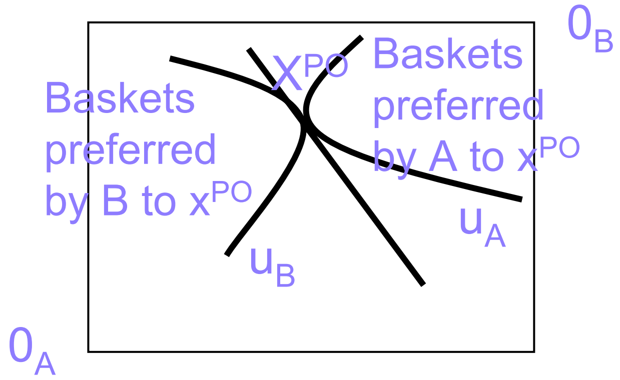
P0→WE

Notice that at each PO allocation: all the baskets preferred by A to this PO have an empty intersection with those preferred by B***,*** that is, the agents's preferred sets to a given PO are ***disjoint sets.***

Then, it is possible to draw a straight line (a hyperplane) passing between the two sets and "separating" them, and going through the PO allocation as well.

Then:

This PO could be "supported" by a descentralized price system.



Welfare Properties of Walrasian Equilibrium:

The Second Fundamental Welfare Theorem.

- There is a Theorem giving the *sufficient conditions* for the existence of a hyperplane "separating" sets: ***Theorem of the Separating Hyperplane.***
- ***Hyperplane*:** Let a be in $\boldsymbol{R}$, p in $\boldsymbol{R}^n$. A hyperplane $H(p,a)$ in $\boldsymbol{R}^n$ is a set such that: $H(p,a)=\{x \text{ in } \boldsymbol{R}^n : px=a\}$

H(p,a) is a n-1 dimensional set: in $\boldsymbol{R}^2$ ***is a straight line***, in $\boldsymbol{R}^3$ a plane.

Example: In a model with two goods, the budget line is the hyperplane $H(p,M)=\{x \text{ en } \boldsymbol{R}^2 : px=M\}$, where M is the agent's wealth, p are the prices and x the agent's consumptions.

***Separating Hyperplane*:** $H(p,a)=\{x \text{ in } \boldsymbol{R}^n : px=a\}$ separates (or strictly separates) the non-empty sets S_1 and S_2 in $\boldsymbol{R}^n$ if:

x_1 in S_1 implies that $p\,x_1 \geq a$ $(> a)$

x_2 in S_2 implies that $p\,x_2 \leq a$ $(< a)$

If H exists, then S_1 and S_2 are separable.

Welfare Properties of Walrasian Equilibrium:

The Second Fundamental Welfare Theorem.

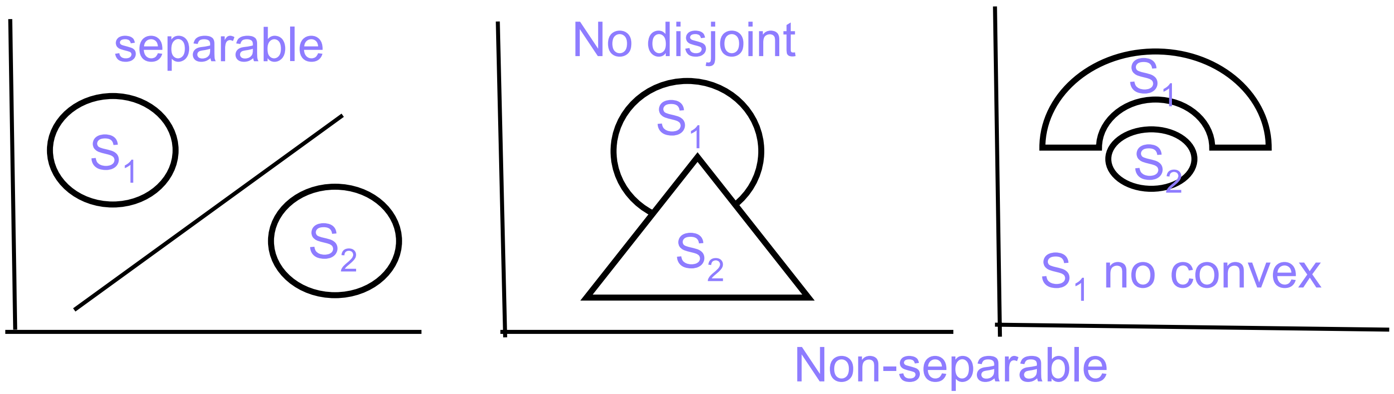
Theorem of the Separating Hyperplane : *(Minkowski)*

Let S_1 and S_2 *in* $\mathbf{R}^n$ *be convex, disjoint and non-empty sets, then there exists a hyperplane separating them, that is: there exists* $H(p,a)=\{x \text{ in } \mathbf{R}^n : px=a\}$ *, such that*

x_1 in S_1 implies that $p\,x_1 \geq a$

x_2 in S_2 implies that $p\,x_2 \leq a$ ⇒ $p\,x_1 \geq p\,x_2$

The theorem gives ***sufficient conditions*** for existence:



Welfare Properties of Walrasian Equilibrium:

The Second Fundamental Welfare Theorem..

The logic-foundations of the 2º TW is the Separation Theorem ➡convex preferences are needed in order the agents' preferred sets are convex and can be separated by a hyperplane.

2º Theorem of Welfare: *Suppose that x^* a PO allocation with $x^{*i} >> 0$, for all $i=1,2..,n$, and that the agents' preferences are convex, continuous and monotone. Then, x^* is a WE for the initial endowments $w^i=x^{*i}$, for all $i=1,2..,n$.*

Proof: Let $\boldsymbol{P}_i=\{x \text{ in } \boldsymbol{R}^k: u^i(x^i)> u^i(x^{*i})\}$, (the set of baskets preferred by i to x^{*i}) and let

$\boldsymbol{P}=\sum_i \boldsymbol{P}_i =\{z : z=\sum_i x^i, x^i \text{ in } \boldsymbol{P}_i\}$, (the set of all aggregate bundles that can be distributed among the n agents so as to make all of them better off).

Welfare Properties of Walrasian Equilibrium: The Second Fundamental Welfare Theorem..

Proof (cont.) Since the agents' preferences are convex ➡each $\boldsymbol{P}_i$ *is un a convex set.*

Since the sum of convex sets is convex➡ $\boldsymbol{P}$ *is a convex set.*

Let $w=\sum_i x^{*i}$ be the current aggregate bundle

Since x^* is a PO allocation, then there is no redistribution of x^* that makes everyone better off ➡

w does not belong to $\boldsymbol{P}$ *and* $w \cap \boldsymbol{P}=\varnothing$ (empty intersection).

We have then two non-empty, convex and disjoint sets $w=\sum_i x^{*i}$ and $\boldsymbol{P}$: then there exists a p such that

$$pz \geq p \sum_i x^{*i} = pw, \text{ for all } z \text{ in } \boldsymbol{P},$$

or rearranging,

$$p(z-\sum_i x^{*i}) \geq 0, \text{ for all } z \text{ in } \boldsymbol{P}.$$

Welfare Properties of Walrasian Equilibrium: The Second Fundamental Welfare Theorem.

- It has to be shown now that p is in fact an equilibrium price vector ➡ that the pair (x^*, p) is a WE.

- By the (alternative) definition of WE, it tranlates to showing that:
- 1) $\sum_i x^{*i} = \sum_i w^i$ (x is feasible), and
- 2) If $u^i(x^i) > u^i(x^{*i})$, then $p^* x^i > p^* w^i$ (*x is not affordable*).

1) Is trivially satisfied by hypothesis. Then, it remains to show 2).

The proof consists of three steps: (we do not prove them here):

a) p es non-negative

b) If $u^i(y^i) > u^i(x^{*i})$, then $p^* y^i \geq p^* w^i$, for all i

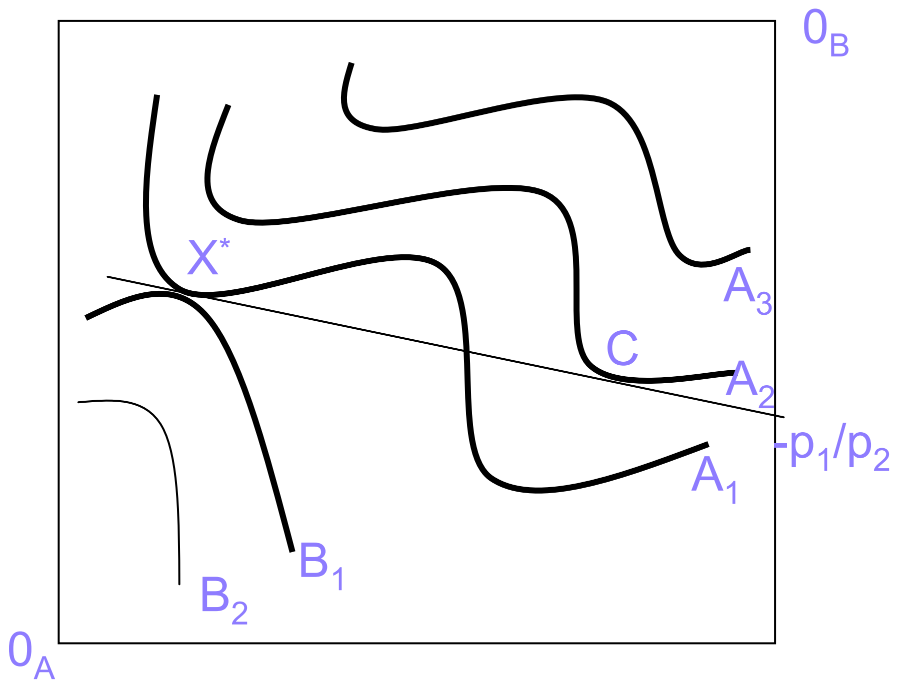
c) If $u^i(y^i) > u^i(x^{*i})$, *then* $p^* y^i > p^* w^i$, *for all i.*

Welfare Properties of Walrasian Equilibrium: The Second Fundamental Welfare Theorem.

- *Implications of the 2º TW* :
- Distribution problems can be *separated from* efficiency problems.
- The market enables to achieve any resource allocation: it is ***neutral*** from a distributive viewpoint.
- ***Normative content***: All PO allocations can be sustained by a price-system when there is a ***right*** redistribution of initial endowments➡ implications for Polítical Economy.

The Second Fundamental Welfare Theorem : Examples where PO allocations cannot be descentralized by a price system

- ***Example 1:*** No convexity of preferences: A's preferences are not convex.



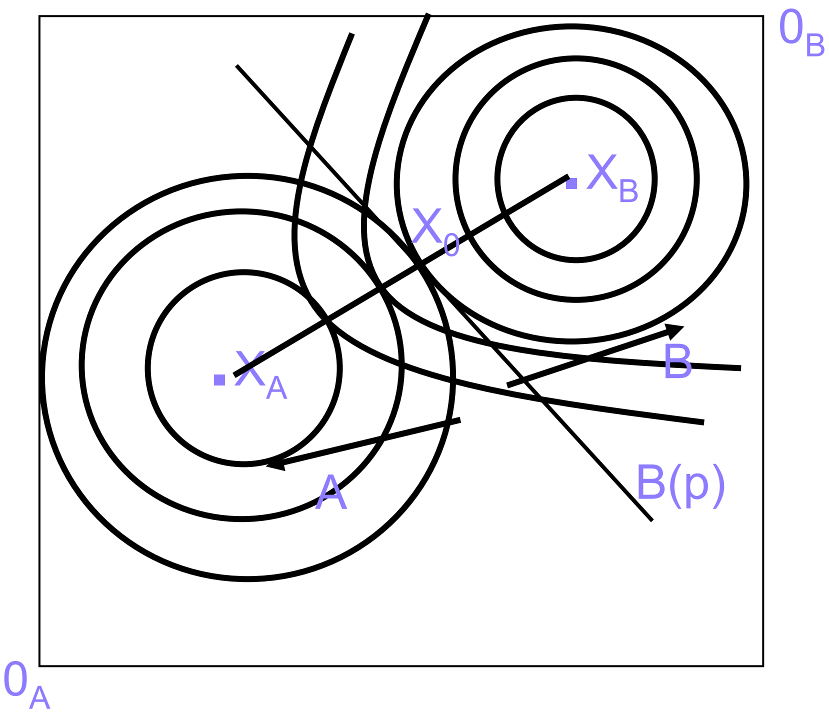
Let X* be PO, and let W= X* .

The price-vector $p=(p_1,p_2)$ cannot support X* as a WE, since A would prefer allocation C to X*:

X* is not a WE.

The Second Fundamental Welfare Theorem : Examples where PO allocations cannot be descentralized by a price system

- ***Example 2:*** Relaxation of non-satiation. The points of maxima satisfaction (bliss points) of A and B are in the Edgeworth' box. Let X_A and X_B be such points for A and B.

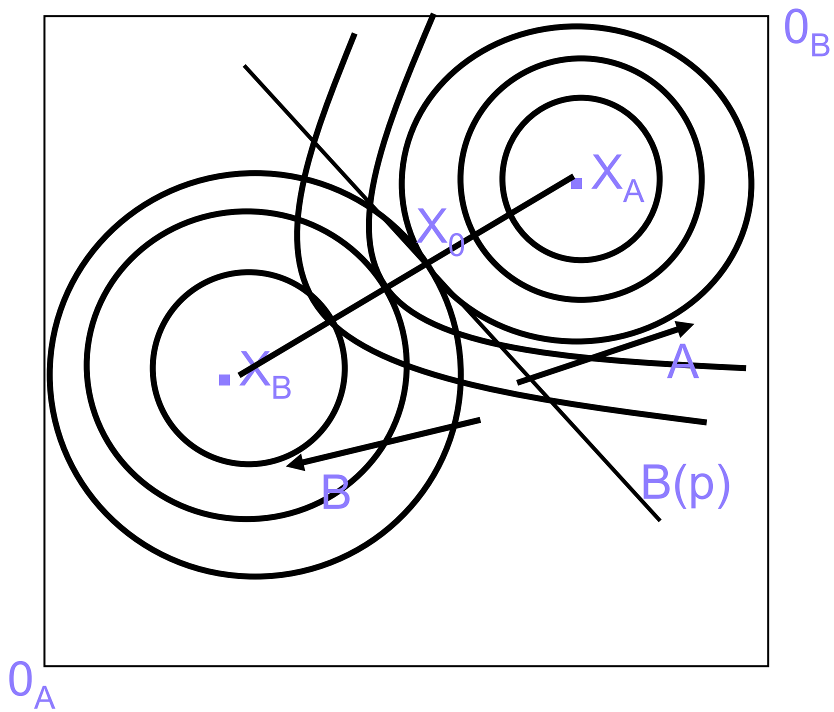


PO allocations= tangency points between X_A y $X_{B..}$
Allocation X_0=PO and let W=X_0 and B(p) the budget line.

Is X_0 a WE? NO if prices are positive, since both A and B are better off in X_A and X_B and are feasible for them.

The Second Fundamental Welfare Theorem : Examples where PO allocations cannot be descentralized by a price system

- ***Example 2 (**cont). Notice that if both X_A and X_B change of place instead, each agent in his bliss point is not feasible:



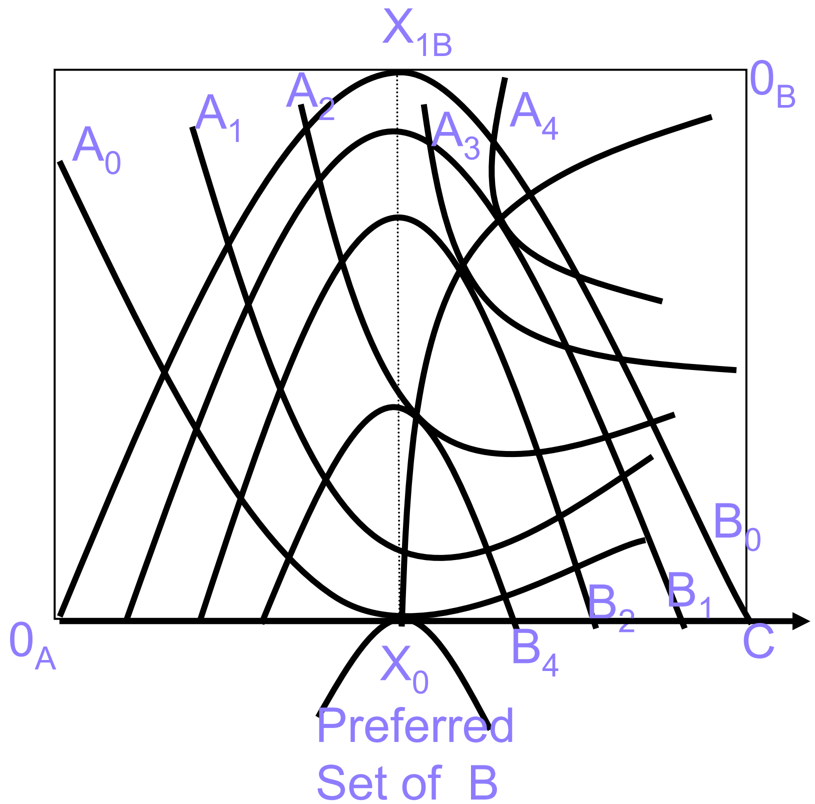
PO allocations= tangency points between X_A y X_B.

Allocation X_0=PO and let W=X_0 and B(p) the budget line.

Is X_0 a WE? YES, since now X_A andX_B are not feasible for the agents.

Examples where PO allocations cannot be decentralized by a price system: Arrow's exceptional case.

- ***Example 3.*** *Arrow's exceptional case.* Let us assume satiation: B is satiated of x_1 in x_{1B}. Let $W=X_0$



In W, A has nothing but good 1.

X_0=PO, Is X_0 a WE?

The unique price-vector tangent to X_0 is $p_1/p_2=0$, and then $p_1=0$.

But if $p_1=0$, A will maximize in C, since x_1 is a free good. .

Then X_0 is not a WE.

Arrow's exceptional case.

- Notice that in X_0, the value of A's consumption basket is zero and B's preferred set is outside the box (not feasible)
- Arrow's exceptional case can be summarized in any of the following statements:
- 1. There is no other state in the economy where the value of good 2 for A is lower than in X_0. Then, for p_1=0, the value of goods in X_0 for A, is the minimum one (=0).
- 2. In X_0, the MgU of x_1 for B is not positive (B is satiated)
- 3. In X_0, A has nothing desired by B (then it is not possible to trade).

- In order any PO is achievable as a WE the above 1), 2) y 3) statements have to be eliminated, that is, all the agents posses some units of a good that are desired by someone else ➡ nobody can be excluded from trading.

The maximization of Social Welfare.

- Up to now we have only considered individual decisions to discuss the existence and optimality of WE.
- However, when studing the 2º TW some problems of ***collective decision*** seem to appear: for instance, some criterion is needed to decide which Pareto optimum is going to be decentralized➡ how to distribute welfare in the society.
- ***Collective decision problems*** are related back with:
- 1. Bentham and Mills' studies on personal welfare.
- 2. Voting Theory of Condorcet and Borda.
- *Modern formulation*: start with Bergson (1938), who defined a ***social welfare function (swf)*** and has evolved with
- Arrow (1951, 1963), who changed the viewpoint ***(SWF).***
- Collective decision problems are known nowadays as the "***Theory of Social Choice"***, whose aim is to design evaluation rules gathering individual preferences: "to design aggregation criteria of individual preferences in order to obtain social preferences***"***. ➡he same criteria are the SWF's

Paretian judgement values

- 1. ***Independence of the process***: the process by which a particular allocation is achieved is not important.
- 2. ***Individualism***: Under the Paretian criterion the only important aspect of an allocation is its effect on the individuals of a asociety.
- 3. ***No paternalism***: Individuals are the best judges of their own welfare.
- *Doubts*: drugs, child pornography, etc.
- 4. ***Benevolence***: The Paretian criterion is benevolent with individuals since an increase (caeteris paribus) in the utility of any individual is considered a welfare improvement.
- *Doubts*: an increase in the utility of the richest person of a society is considered welfare improving regardless of some other people dying by starving.

Paretian criterion:

Set of utility possibilities:

The set of utility vectors assigned to the feasible allocations.

$U=\{(u_1, u_2, \ldots, u_n)$ in R^n: there is a feasible allocation x such that $u_i \leq u_i(x_i)$, for all $i=1,2,3,\ldots,n\}$.

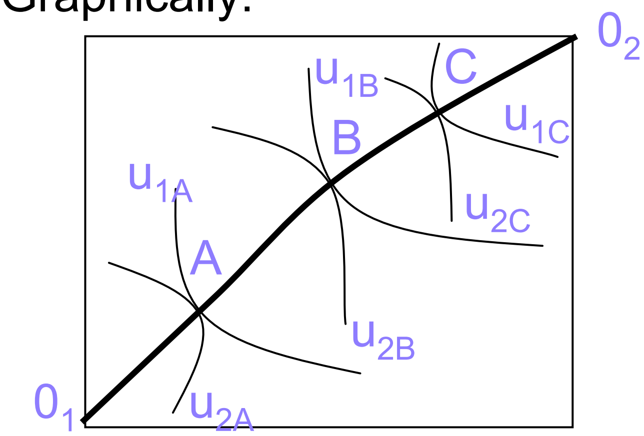
Utility Frontier or Pareto Frontier:

$UF=\{(u_1, u_2, \ldots, u_n)$ in R^n: there is no other vector $(u'_1, u'_2, \ldots, u'_n)$ in U such that $u'_j \geq u_j$ for all $j=1,2,..,n$, and $u'_i > u_i$ for some $i\}$.

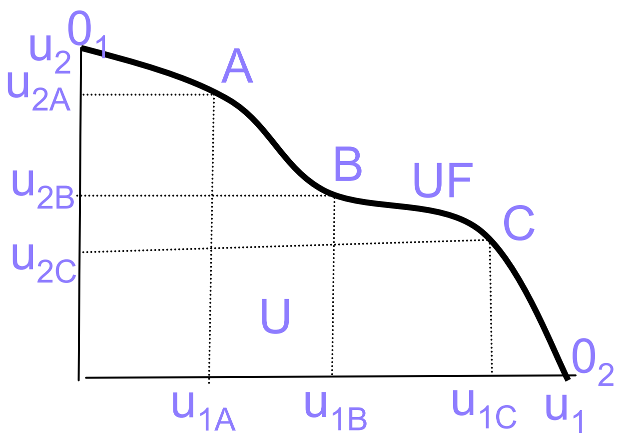
The utility frontier is all the utility vectors assigned to the Pareto-efficient allocations.

Utility Frontier:

Graphically:



Feasible allocations and Contract curve.



Set of utility possibilities, U and Utility frontier, UF (or Paretian frontier)

Utility Frontier:

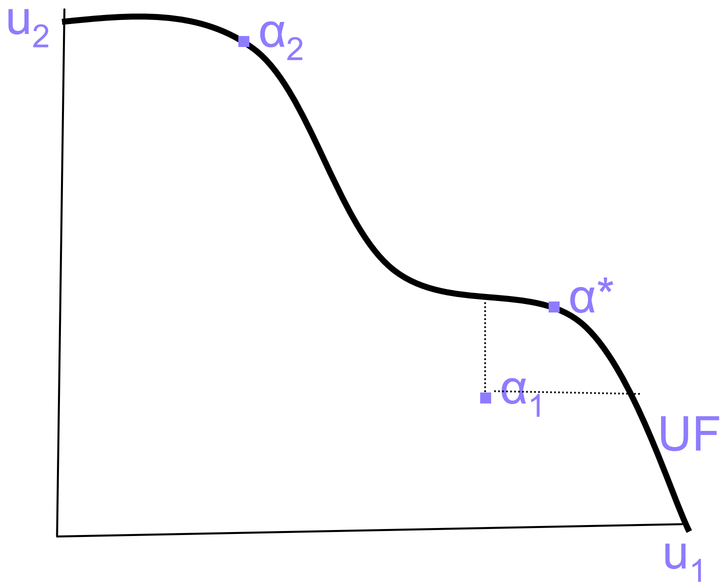
- The utility frontier as a function of utility levels can be calculated from the optimization problem characterizing the efficient allocations:
- Max $u_1(x_1)$
- s.t. $u_2(x_2) \geq u_2 = c$ (utility constraint)
- s.t. $x_{11}+x_{21}=w_1$ and $x_{12}+x_{22}=w_2$ (feasibility constraint)

- Solution: $x_1^*(u_2, w_1, w_2)$ and $x_2^*(u_2, w_1, w_2)$, and substituing into agent 1's utility function:
- $u_1(x_1^*(u_2, w_1, w_2)) = u_1(u_2, w_1, w_2)$,

- Or implicitely:
- $F(u_1, u_2) = 0$

Social Welfare functions and Social Optima.

- **Problem**: The Pareto efficiency criterion cannot generate a complete order over the feasible allocations. Even some pairs of allocationas cannot be compared.



α^* and α_2 cannot be compared: Neither α^* is Pareto superior to α_2, nor is α_2 Pareto superior to α^* .

α_1 is inefficient but α_2 is not Pareto superior to α_1.

Social Welfare functions and Social Optima.

- ***Bergson's social welfare function (swf)*** is a function assigning utility values to the feasible allocations of an economy, and generating a complete, transitive and reflexive ordering on the set of feasible allocations.
- $W: R^n \rightarrow R, \quad W(u_1(x_1), u_2(x_2), \ldots, u_n(x_n))$
- Any swf defined as a Bergsonian's swf has some underlying distributive principles.
- ***Paretian swf:*** swf with paretian judgement values:
- 1. Independende of the process
- 2. Individualism $\rightarrow W(x_1, x_2, \ldots, x_n)$
- 3. No paternalism$\rightarrow W(u_1(x_1), u_2(x_2), \ldots u_n(x_n)) = W(u_1, u_2, \ldots u_n)$
- 4. Benevolence (Monotonicity): W increasing in each u_j

$$\frac{\delta W(u_1, u_2, .., u_n)}{\delta u_j} = W_j > 0, \text{ for all } j = 1, 2, .., n$$

Social Welfare functions and Social Optima.

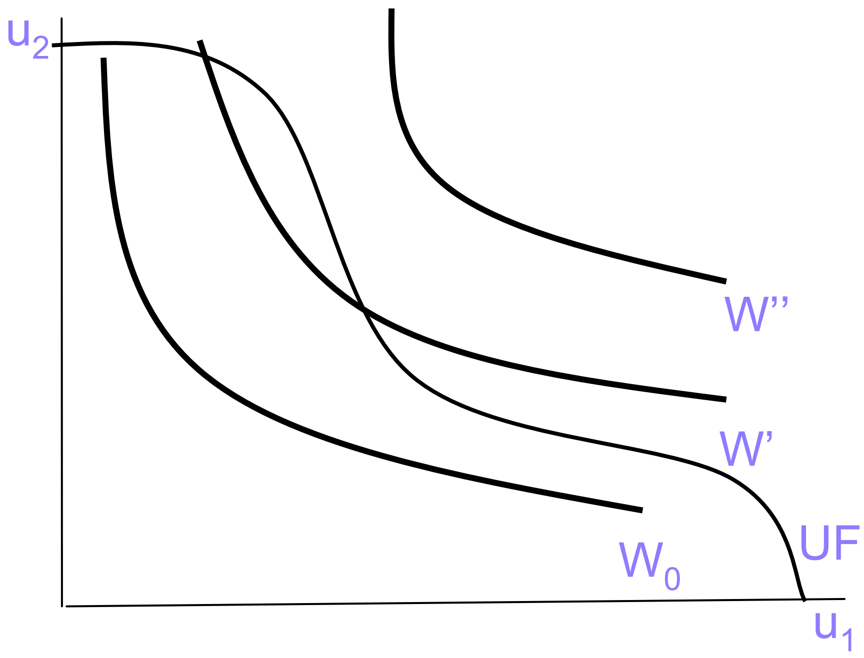
- ***Consequences of benevolence*** (or monotonicity). Assume two agents. The swf is $W(u_1,u_2)$
- 1. The indiference curves of welfare or isowelfare curves more far away from the origin represent higher welfare levels. .
- 2. Isowelfare curves have negative slope. Let $W(u_1,u_2)=W_0$ be an isowelfare curve.
- $dW=W_1\, du_1+W_2\, du_2=0$

$$\frac{du_2}{du_1} = -\frac{W_1}{W_2} < 0$$

- since $W_1>0$ y $W_2>0$

Social Welfare functions and Social Optima.

- Isowelfare curves in the utility space.



Social optima.

- A ***social*** optimum maximizes the paretian swf over the set of feasible allocations.
- As the set of feasible allocations can be expressed as the set of utility possiblities, the maximization can be written as:
- Max $W(u_1,u_2,\ldots u_n)$ s.a. $F(u_1, u_2,\ldots,u_n)=0$.
- For two agents: Max $W(u_1,u_2)$, s.a. $F(u_1, u_2)=0$
- *Associated Lagrangian:* $L(u_1,u_2, \lambda)= W(u_1,u_2)- \lambda F(u_1,u_2)$

$$\frac{\delta L}{\delta u_1} = \frac{\delta W}{\delta u_1} - \lambda \frac{\delta F}{\delta u_1} = 0$$

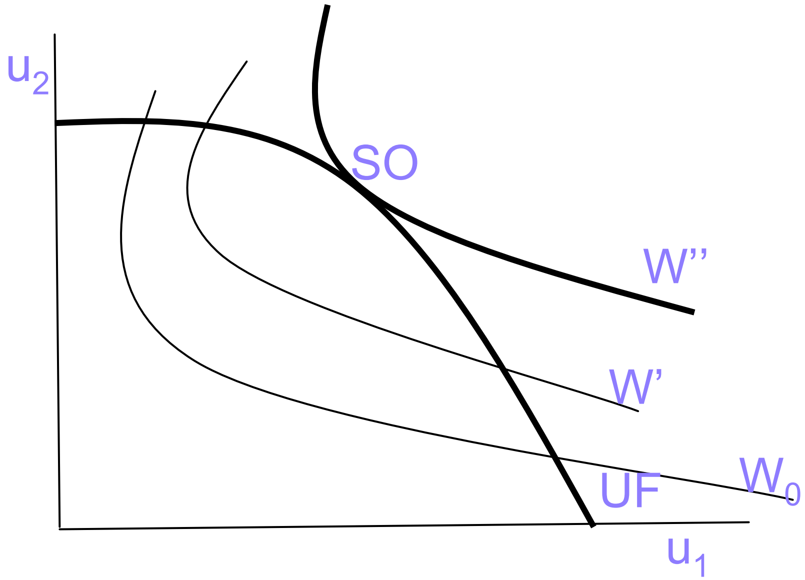
$$\frac{\delta L}{\delta u_2} = \frac{\delta W}{\delta u_2} - \lambda \frac{\delta F}{\delta u_2} = 0$$

$$\frac{\delta L}{\delta \lambda} = F(u_1, u_2) = 0$$

Social Optima.

- The F.O.C. imply (for interior solutions):

$$\frac{\frac{\delta W}{\delta u_1}}{\frac{\delta W}{\delta u_2}} = \frac{\frac{\delta F}{\delta u_1}}{\frac{\delta F}{\delta u_2}} \rightarrow \frac{du_2}{du_1} \text{en W} = \frac{du_2}{du_1} \text{en F} \rightarrow RMS_W = RMS_F$$



At the SO, the isowelfare W'' is tangent to the UF: $MRS_W = MRS_F$

Social optima.

Let (u_1^{SO}, u_2^{SO}), be the pair of utilities at a SO. These utilities are associated to an allocation $x^*=(x_1^*,x_2^*)$ in the Egeworth's box that maximizes a swf. Then

Proposition: If x^* maximizes a swf, then x^* will be Pareto efficient.

Proof: Trivial, by monotonicity of W. Suppose, on the contrary, that x^* is not Pareto efficient. Then, it will exist another feasible allocation x' such that:

$u_i(x_i')> u_i(x_i)$ for all $i=1,2,..,n$, and by monotonicity of W

$W(u_1(x_1'),u_2(x_2'),\ldots, u_n(x_n'))> W(u_i(x_1^*),u_2(x_2^*),\ldots, u_n(x_n^*))$, that contradicts that x^* maximizes the swf $W(u_1,u_2,\ldots, u_n)$.

Conclusion: Pareto efficiency is necessary for the SO.

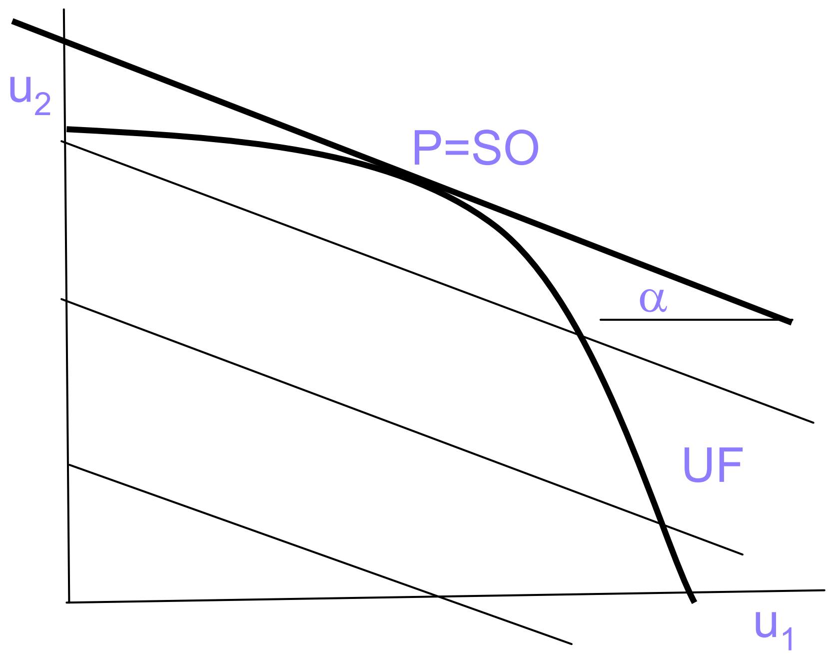
Social Optima.

- Consequences: The welfare maximizing allocation are PO.
- Question: Is any PO achievable as the maximum of a swf? NO in general, but:
- ***Proposición***: Let x^* be a PO allocation, with $x_i^* >> 0$, $i=1,2,..,n$. The individual utility functions u_i, $i=1,2,..,n$, are concave, continuous and monotone. Then, there exists achoice of parameters a_i^* such that x^* maximizes $\sum_i a_i^* u_i(x_i)$ subject to the feasibility constraint.

- *Proof:* For two agents the proof is very simple and can be graphically explained.
- Construct the set of U of utility possibilities. Since the u_i are concave, U is a convex set.

Social Optima.

Isowelfares: $W_0=a_1u_1+a_2u_2$, o $u_2=W_0/a_2-(a_1/a_2)u_1$, $du_2/du_1=-(a_1/a_2)$ o $MRS_W=a_1/a_2$



Let *P* be the *PO* allocation to be achieved as a *SO,* and let $MRS_{F\ en\ P}=\alpha$.
Then, as at the *SO*: $MRS_W=MRS_F$, then choose a ratio $a_1/a_2=\alpha$, and the tangency of the isowelfare curve with the UF is in *P*.
Hence *P=OS.*

Social Choice: Arrow's Impossibility Theorem

- *Problem*: With the Pareto's criterion there are many efficient allocations associated with different *distributions of welfare* among individuals.
- *Question*: How to choose a socially optimal allocation?

- *Change of approach:* From *individual preferences*, some *social preferences* could be defined such that they *order* the set of feasible allocations.

- Let $\succcurlyeq_i$ be the *preference* relationship of individual *i* defined over set A: set of all feasible allocations. Notice that $\succcurlyeq_i$ is now defined over allocations instead of over individual consumption baskets:
  $u_1(x_1,x_2,\ldots,x_n)$, $u_2(x_1,x_2,\ldots,x_n)$,…, $u_n(x_1,x_2,\ldots,x_n)$.

Social Choice: Arrow's Impossibility Theorem

- Is there any mechanism to obtain from the so defined individual preferences $\{\succcurlyeq_i\}$ a social preference relationship $\succcurlyeq$, guaranteeing that the social orderings satisfy some desiderable properties?

- If such a mechanism exists, then it will be called a ***Social Welfare Function*** *(SWF)*.

- Thus, a *SWF* is a mechanism or aggregation rule of individual preferences, obtaining a social ordering of the distinct allocations.

Social Choice: Arrow's Impossibility Theorem

- *Clarification*:
- A *Bergsonian social welfare function (swf)* is a function on the set of utility possibilities and associates (in some precise way) a real number to each vector of utilities belonging to it, thus generating an ordering of this set.
- A *Social Welfare Function (SWF)* is a function on the set of individual preferences over the set of the possible social states, and associates a social preference to each possible configuration of individual preferences.
- The *SWF* concept is more general than that of *swf*. Changes in the individual preferences for a given *SWF* will change the social preferences and hence the *swf*. And, a different *SWF* on a given set of individual preferences will produce a different social ordering and hence a distinct swf as well.

Social Choice: Arrow's Impossibility Theorem

- Let *A* be a set of alternatives or social states, and $\{\succcurlyeq_i\}$ be the individual preferences over A. Consider a criterion or aggregation rule of $\{\succcurlyeq_i\}$, generating some social preferences $\succcurlyeq$.
- <u>*Desiderable properties of the aggregation criterion*</u>.
- 1. *Completeness,*
- 2. *Reflexivity*,
- 3. *Transitivity*,

The rule generates an order

- 4. *Universality or condition of unrestricted domain*:
  For all $\{\succcurlyeq_i\}$, there exists *a* social preference
  $\succcurlyeq\ = \Sigma(\succcurlyeq_1, \succcurlyeq_2, \ldots, \succcurlyeq_n)$,

where Σ denotes the aggregation rule or mechanism of individual preferences.

Social Choice: Arrow's Impossibility Theorem

- *Universality or condition of unrestricted domain* (cont): This property says that from any set of individual preferences $\{\succcurlyeq_i\}$, a social preference relationship $\succcurlyeq$ can be derived.
- This property has a clear logical content since it is a *completeness* property with regards the obtention of $\succcurlyeq$. It also has a clear *political content*: the aggregation mechanism is permissive enough to admit any system of values and/or rules of individual behaviors.
- We will see next that this property is not generally satisfied:
- Example:The *Vote-Paradox*

The Vote-Paradox

- **Example:The *<u>Vote-Paradox:</u>***
- Suppose three agents and three social states (or alternatives): *{a,b,c}*
- Aggregation rule: *<u>mayority rule</u>*: state "a" is socially preferred to state "b", if "a" is preferred by the mayority of the individuals.
- Individual preferences over social states:
- $(a,b,c)_1$, $(b,c,a)_2$, $(c,a,b)_3$

The Vote-Paradox

- *The Vote-Paradox (cont):*
- Social preferences: (pair-comparisons)

$$(a,b)\ \begin{cases} a: 2 \text{ votes} \\ b: 1 \text{ vote} \end{cases} \mapsto a \succ b \quad \text{or} \quad [a,b]$$

$$(b,c)\ \begin{cases} b: 2 \text{ votes} \\ c: 1 \text{ vote} \end{cases} \mapsto b \succ c \quad \text{or} \quad [b,c]$$

Transitivity implies that $a \succ c$, or [a,c].

Let us compare now the pair (a,c)

$$(a,c)\ \begin{cases} a: 1 \text{ vote} \\ c: 2 \text{ votes} \end{cases} \mapsto c \succ a \quad \text{or} \quad [c,a]$$

The rule is not transitive!

The Vote-Paradox

- *<u>The Vote-Paradox (cont):</u>* then, this mechanism may create a problem: it can generate social orderings that are *not transitive.*
- The mayority rule fails for this particular individual preferences, but it would produce a transitive social ordering for identical preferences, for example, $(a,b,c)_i$, i=1,2,3.
- Another example: $(a,c,b)_1$, $(b,a,c)_2$, $(a,b,c)_3$
- What is desired is that the social choice rule works out for *<u>any</u>* type of individual preferences.
- Types of possible preferences for three social states:
- $(a,b,c)_i$, $(a,c,b)_i$, $(b,a,c)_i$, $(b,c,a)_i$, $(c,a,b)_i$, $(c,b,a)_i$
- Each one of them can be combined with each of all the others, so that there are: 6^3=216 possible types of preferences.
- The social choice rule has to be valid for all of them: the *domain* of the function transforming a set of individual preferences in a social ordering is not restricted.

Social Choice: Arrow's Impossibility Theorem

5. *Unanimity or Pareto rule:*
   For any pair **a** and **b** in A, if a $\succ_i$ b for all i, then a $\succ$ b.

- This condition is either too weak or too strong
- *Too weak* in the sense that any social preference must consider **a** better than **b** if **all** the individual so consider it.
- *Too strong* in the sense that to consider unanimity as the unique criterion of the social rule implies that the orderig relationship is going to almost never order.

Social Choice: Arrow's Impossibility Theorem

6. *Independence of the irrelevant alternatives*: For any pair **a** and **b** in A_0 ($A_0 \subseteqq A$), if $a \succcurlyeq_i b$ and $a \succcurlyeq_i^* b$, for all i, then $a \succcurlyeq b$ and $a \succcurlyeq^* b$, for all A_0.

- A_0 is any subset of A, and $\{\succcurlyeq_i\}$ y $\{\succcurlyeq_i^*\}$ are two sets of individual preferences.
- If individual preferences change but leave unchanged each i's individual preferences between **a** y **b**, then social preferences must keep that **a** is socially preferred to **b**.

Social Choice: Arrow's Impossibility Theorem

Indep. irrelevant alternatives (Cont) →Implications:

Consider three alternatives (a,b,c) and two individuals. A change only in the position of **c** in the individual orderings (a change of the preferences) does not affect the social ordering between **a** and **b** (**c** is the irrelevant alternative in the choice between **a** and **b**).

Example: $\begin{cases} \text{Individual one: } (a,b,c)_1 \rightarrow a \succ_1 b \\ \text{individual two: } (b,c,a)_2 \rightarrow b \succ_2 a \end{cases}$

Consider the aggregation rule: ***Voting through orderings***: an integer (a number of points) is assigned to each alternative with the property that the more preferred alternatives are assigned the smaller integers. Points are aggregated to compare any pair of alternatives and the alternative with less points is the social winner.

Social Choice: Arrow's Impossibility Theorem

Example (cont): Then $(a=1,b=2,c=3)_1$ and $(b=1,c=2,a=3)_2$.

Comparing the pair of alternatives (a,b): a=1+3=4 and b=2+1=3 $\rightarrow$

$b \succ a$ (**b** is socially preferred to **a**).

Consider a change of the individual preferences:

$$\begin{cases} \text{Individual one: } (a,c,b)^*_1 \rightarrow a \succ_1^* b \\ \text{individual two: } (b,a,c)^*_2 \rightarrow b \succ_2^* a \end{cases}$$

Individual preferences between a and b remain the same

Then: $(a=1,c=2,b=3)_1$ and $(b=1,a=2,c=3)_2$

Socially choosing between (a,b): a=1+2=3 and b=3+1=4 $\rightarrow$

$a \succ b$ (**a** is now socially preferred to **b**).

The individual preferences between **a** and **b** have not changed but the social preferences have changed.

The aggregation rule: ***Voting through orderings*** is not independent of the irrelevant alternatives.

Social Choice: Arrow's Impossibility Theorem

<u>Another example</u> showing that the aggregation rule of *Voting through orderings* is not independent of the irrelevant alternatives:

Let A=(a,b,c) and consider again that:

$\begin{cases} \text{Individual one: } (a,b,c)_1 \rightarrow a \succ_1 b \\ \text{individual two: } (b,c,a)_2 \rightarrow b \succ_2 a \end{cases}$

and remember that since $(a=1,b=2,c=3)_1$ and $(b=1,c=2,a=3)_2$, such an aggregation rule produces, when comparing between (a,b): a=1+3=4 y b=2+1=3 $\rightarrow b \succ a$ (**b** is soc. preferred to **a**).

<u>Consider</u> the subset of A, A_0=(a,b), then $(a,b)_1$ and $(b,a)_2$, and this rule says that since $(a=1,b=2)_1$ and $(b=1,a=2)_2$, then a=1+2=3 and b=2+1=3, and socially: $a \backsim b$.

Social Choice: Arrow's Impossibility Theorem

7. *No dictatorship:*

   There is no individual i* such that for all **a** and **b** in A: $a \succcurlyeq_{i^*} b$, implies that $a \succcurlyeq b$.

This property avoids that an individual is fundamental (decisive) in all choices, regardless of the other individual preferences.

Social Choice: Arrow's Impossibility Theorem

- ***Arrow's Impossibility Theorem:***
- *If a mechanism of social choice generates a social ordering satisfying properties 1-6, then it will be a dictatorship: all social orderings are those of a unique individual*

- The Theorem shows that the desiderable properties of a social ordering coming from a social choice rule are not compatible with democracy: there is no "perfect system" to take social decisions.
- If we use some system, then we will loose some of the properties defined in 1-7.

Social Choice: Arrow's Impossibility Theorem

How the Social Choice Theory keeps developing in spite of the non-existence result?

1. To relax the condition of universality or not restricted domain
2. To ask only for no-cyclicity, instead of transitivity.

Etc.