Lightlike simultaneity, comoving observers and distances in general relativity

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Abstract

We state a condition for an observer to be comoving with another observer in general relativity, based on the concept of lightlike simultaneity. Taking into account this condition, we study relative velocities, Doppler effect and light aberration. We obtain that comoving observers observe the same light ray with the same frequency and direction, and so gravitational redshift effect is a particular case of Doppler effect. We also define a distance between an observer and the events that it observes, that coincides with the known *affine distance*. We show that affine distance is a particular case of radar distance in the Minkowski spacetime and generalizes the proper radial distance in the Schwarzschild spacetime. Finally, we show that affine distance gives us a new concept of distance in Robertson-Walker spacetimes, according to Hubble law.

1 Introduction

In general relativity it is often difficult to interpret when an observer β is comoving with another observer β', in the sense that β moves "like" β'. For example, given a particular coordinate system it is usual to suppose that stationary observers (i.e. with constant spatial coordinates) are comoving each one with respect to the other. But this criterion is coordinate-dependent: let us suppose that two observers are stationary using a particular coordinate system; then they are comoving each one with respect to the other. On the other hand, we can find another coordinate system in which one observer is stationary and the other one is not stationary; then they are not comoving each one with respect to the other. Since we want that the property "to be comoving with" was an intrinsic property of the observer (i.e. that an observer was able to decide if it is comoving with another observer or not, independently from the coordinate system), the "stationary criterion" is a bad criterion.

Given an observer β, there is a general method to check if it is comoving with another observer β', based on the concept of simultaneity. We have to build a simultaneity foliation associated with β [1], then parallelly transport the 4-velocity of β' to β, along geodesics joining β' with β in the leaves of the foliation, and finally compare it with the 4-velocity of β (see Figure 1).

There are a lot of kinds of simultaneities, but we are going to consider only two kinds of simultaneity foliations associated with a given observer β [1]: the Landau foliation $\mathcal{L}_\beta$, whose leaves are Landau submanifolds [2], also called Fermi surfaces (spacelike); and the past-pointing horismos foliation $\mathcal{E}^-_\beta$, whose leaves are past-pointing horismos submanifolds [3] (lightlike). We have to note that if we use Landau foliations, then the method to check if an observer is comoving with another one is symmetric; on the other hand, if we use past-pointing horismos foliations, then this method is not symmetric, i.e. one observer β can

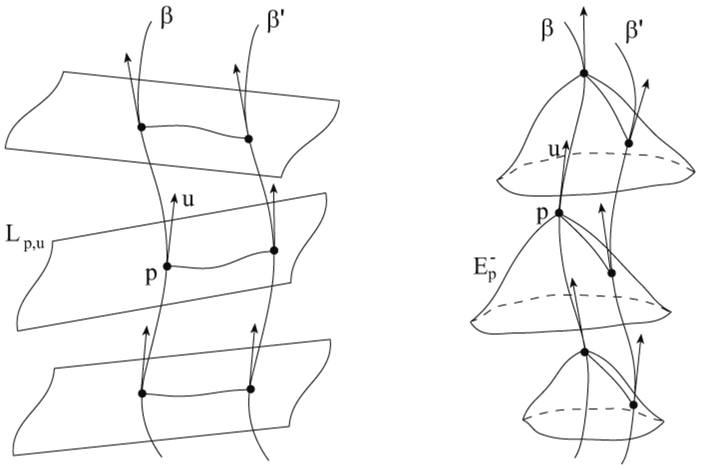

Figure 1: How to check if an observer β is comoving with another observer β', depending on the simultaneity foliation that we are using. Left: Landau foliation $\mathcal{L}_\beta$ (spacelike). Right: Past-pointing horismos foliation $\mathcal{E}_\beta^-$ (lightlike).

be comoving with another observer β', and β' being non comoving with β. But, since we are working in relativity, the non-symmetry is an acceptable property. So, the problem is to decide which simultaneity (spacelike or lightlike) is mathematically and physically more suitable for us:

(a) Mathematically: in a previous work [1] we proved that the Landau foliation $\mathcal{L}_\beta$ is not always defined in every convex normal neighborhood because its leaves can intersect themselves. For example, in a Minkowski spacetime if the observer β is not geodesic. Moreover, $\mathcal{L}_\beta$ is not necessarily spacelike at every point of a convex normal neighborhood. On the other hand, the past-pointing horismos foliation $\mathcal{E}_\beta^-$ is always well defined in every convex normal neighborhood and it is always lightlike.

(b) Physically: given an observer at an event p with 4-velocity u, the events of its Landau submanifold $L_{p,u}$ do not affect the observer at p in any way, since both electromagnetic and gravitational waves travel at the speed of light. On the other hand, the events of its past-pointing horismos submanifold E_p^- are precisely the events that affect and are observed by the observer at p, i.e. the events that *exist* for the observer at p.

Therefore, we are going to work in the framework of lightlike simultaneity. So, given an observer at an event p, we will say that the events of E_p^- are *lightlike-simultaneous* for this observer at p. In fact "to be lightlike-simultaneous for an observer" is the same as "to be observed simultaneously by an observer".

Hence, in Section 3, we define the *observers congruence comoving with a given observer*, according to the concept of lightlike simultaneity, and we give a method to measure relative velocities of observers in Section 3.1. Given a light ray, we study Doppler effect in Section 3.2, obtaining that the frequency of a light ray remains constant for comoving observers. This is apparently contradictory with *gravitational redshift* effect, stating that light rays gain or lose frequency in the presence of a gravitational field, and it is considered independent of Doppler effect. Gravitational redshift effect is completely explained in our formalism, showing that it is a particular case of a generalized Doppler effect. We also study light aberration effect in Section 3.3, obtaining that there is not light aberration between comoving observers.

The concept of distance is strongly bounded to the concept of simultaneity too. We are using lightlike simultaneity, so we have to measure distances between lightlike-simultaneous events, i.e. we need to measure lengths of light rays. In Section 4, we re-define a concept of distance (called *affine distance*) between an observer and the events that it observes, i.e. a distance between p and the events of E_p^-. In Section 5, we show that affine distance is a particular case of radar distance in the Minkowski spacetime and generalizes the proper radial distance in the Schwarzschild spacetime. Finally, we show that affine distance gives us a new concept of distance in Robertson-Walker spacetimes, according to Hubble law.

We work in a 4-dimensional lorentzian spacetime manifold $\mathcal{M}$, with $c = 1$ and ∇ the Levi-Civita connection, using the Landau-Lifshitz Spacelike Convention (LLSC). We suppose that $\mathcal{M}$ is a convex normal neighborhood [4]. Thus, given two events p and q in $\mathcal{M}$, there exists a unique geodesic joining p and q . The parallel transport from p to q along this geodesic will be denoted by τ_{pq}. If $\beta : I \to \mathcal{M}$ is a curve with $I \subset \mathbb{R}$ a real interval, we will identify β with the image βI (that is a subset in $\mathcal{M}$), in order to simplify the notation. If u is a vector, then $u^\perp$ denotes the orthogonal space of u. Moreover, if x is a spacelike vector, then $\|x\|$ denotes the module of x. Given a pair of vectors u, v, we use $g(u, v)$ instead of $u^\alpha v_\alpha$. If X is a vector field, X_p will denote the unique vector of X in $T_p\mathcal{M}$.

2 Preliminaries

An *observer* in the spacetime is determined by a timelike world line β, and the events of β are the *positions* of the observer. It is usual to identify an observer with its world line, and so β is an observer. The *4-velocity* of the observer is a future-pointing timelike unit vector field U defined in β and tangent to β. Given an event p, the 4-velocity of an observer at p is given by a future-pointing timelike unit vector u. It is also usual to identify an observer with its 4-velocity, since they are defined reciprocally. So, if u is the 4-velocity of an observer at p, we will say that u is an observer at p, in order to simplify the notation. To sum up, we will say that a timelike world line β is an observer, and a future-pointing timelike unit vector u in $T_p\mathcal{M}$ is an observer at p.

Given two observers u and u' at the same event p, there exists a unique vector $v \in u^\perp$ and a unique positive real number γ such that

$$u' = \gamma (u + v) . \tag{1}$$

As consequences, we have $0 \leq \|v\| < 1$ and $\gamma = -g(u', u) = \frac{1}{\sqrt{1-\|v\|^2}}$. We will say that v is the *relative velocity of u' observed by u*, and γ is the *gamma factor* corresponding to the velocity $\|v\|$.

A *free-falling test particle* is given by a timelike geodesic β (i.e. the world line of a geodesic observer) and a future-pointing timelike vector field M defined in β, tangent to β and parallelly transported along β (i.e. $\nabla_M M = 0$), called *mass vector field of β*. Given $p \in \beta$ and u an observer at p, the *mass of β observed by u* is given by $m := -g(M_p, u)$. The mass of a test particle does not modify the spacetime metric. The *3-moment of β observed by u* is given by $\rho := M_p - mu$. So, $\rho \in u^\perp$ and $M_p = mu + \rho$. Given a free-falling test particle β with 4-velocity U and mass vector field M, the *proper mass of β* (also known as *rest mass*) is given by $m^0 := -g(M_p, U_p)$, where $p \in \beta$. The proper mass is well defined, i.e. it does not depend on the point p. Moreover $M = m^0 U$.

A *light ray* is given by a lightlike geodesic λ and a future-pointing lightlike vector field F defined in λ, tangent to λ and parallelly transported along λ (i.e. $\nabla_F F = 0$), called *frequency vector field of λ*. Given $p \in \lambda$ and u an observer at p, there exists a unique vector $w \in u^\perp$ and a unique positive real number ν such that

$$F_p = \nu (u + w) . \tag{2}$$

As consequences, we have $\|w\| = 1$ and $\nu = -g(F_p, u)$. We will say that w is the *relative velocity of* λ *observed by* u, and ν is the *frequency of* λ *observed by* u. In other words, ν is the module of the projection of F_p onto $u^{\perp}$. A *light ray from* q *to* p is a light ray λ such that q, $p \in \lambda$ and $\exp_q^{-1} p$ is future-pointing.

Given two observers u and u' at the same event p of a light ray λ, using (2), the frequency vector F_p of λ is given by

$$F_p = \nu\,(u + w) = \nu'\,(u' + w')\,, \tag{3}$$

where ν, ν' are the frequencies of λ observed by u, u' respectively and w, w' are the relative velocities of λ observed by u, u' respectively. Applying (1), we obtain that

$$\nu' = \gamma\,(1 - g\,(v, w))\,\nu. \tag{4}$$

Expression (4) is the general expression of *Doppler effect*. For example, if $\frac{v}{\|v\|} = w$, i.e. the direction of the relative velocity of u' observed by u coincides with the direction of the relative velocity of λ observed by u, we have the usual redshift expression

$$\nu' = \sqrt{\frac{1 - \|v\|}{1 + \|v\|}}\,\nu.$$

On the other hand, taking into account (3) and (4), we have

$$w' = \frac{1}{\gamma\,(1 - g\,(v, w))}\,(u + w) - u'. \tag{5}$$

The fact that w' is different from w causes an *aberration* effect [5]. It is easy to prove that

$$\cos\theta = \frac{\cos\theta' - \|v\|}{1 - \|v\|\cos\theta'}, \tag{6}$$

where θ is the angle between $-w$ and v, and θ' is the angle between $-w'$ and the projection of v onto $u'^{\perp}$ (θ' is also the angle between $-w'$ and $-v'$, where v' is the relative velocity of u observed by u'). The expression (6) is the general expression of light aberration phenomenon [6], and the scalar function given by $\theta' - \theta$ is the *aberration angle of* u' *observed by* u *corresponding to* λ.

Let $p \in \mathcal{M}$ and $\varphi : \mathcal{M} \to \mathbb{R}$ defined by

$$\varphi\,(q) := g\left(\exp_p^{-1} q, \exp_p^{-1} q\right).$$

Then, it is a submersion and the set

$$E_p := \varphi^{-1}\,(0) - \{p\} \tag{7}$$

is a regular 3-dimensional submanifold, called *horismos submanifold of* p [3]. In other words, an event q in the spacetime is in E_p if and only if $q \neq p$ and there exists a lightlike geodesic joining p and q (i.e. a light ray from q to p). The submanifold E_p has two connected components, E_p^+ and E_p^- [7]; E_p^+ (respectively E_p^-) is the *future-pointing* (respectively *past-pointing*) *horismos submanifold of* p, and it is the connected component of (7) in which, for each event $q \in E_p^+$ (respectively $q \in E_p^-$), the preimage $\exp_p^{-1} q$ is a future-pointing (respectively past-pointing) lightlike vector.

We can construct horismos foliations in this way [1,8]: let β be an observer; then, we define $\mathcal{M}_\beta^+ := \cup_{p\in\beta} E_p^+$ and $\mathcal{M}_\beta^- := \cup_{p\in\beta} E_p^-$. So, there exists a foliation $\mathcal{E}_\beta^+$ (respectively $\mathcal{E}_\beta^-$) defined in $\mathcal{M}_\beta^+$ (respectively $\mathcal{M}_\beta^-$) whose leaves are future-pointing (respectively past-pointing) horismos submanifolds of events of β. The foliations $\mathcal{E}_\beta^+$ and $\mathcal{E}_\beta^-$ are called respectively *future-pointing* and *past-pointing horismos foliation generated by* β.

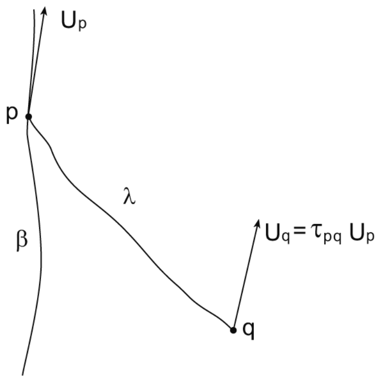

Figure 2: The extension of U at q is given by $\tau_{pq}U_p$, where $p \in \beta$ and there exists a light ray λ from q to p. So, we can build a reference frame from a single observer.

3 Comoving observers in the framework of lightlike simultaneity

As we discussed in the Introduction, we are going to work in the framework of lightlike simultaneity. So, for checking if an observer β *is comoving with* another observer β', we have to parallelly transport the 4-velocity of β' to β along lightlike geodesics joining β' with β in the leaves of the foliation $\mathcal{E}_\beta^-$, and finally compare it with the 4-velocity of β (see Figure 1-right). This is a non-symmetric method, i.e. if β is comoving with β' then β' is not necessarily comoving with β.

Given an observer β with 4-velocity U, we can construct an observers congruence extending U to $\mathcal{M}_\beta^-$ by means of parallel transports along light rays from events of $\mathcal{M}_\beta^-$ to events of β:

Definition 3.1. Let β be an observer with 4-velocity U. The *observers congruence associated with* β is the extension of U defined on $\mathcal{M}_\beta^- \cup \beta$ such that $U_q := \tau_{pq}U_p$, where $p \in \beta$, $q \in \mathcal{M}_\beta^-$, and there exists a light ray from q to p (see Figure 2).

Let β, β' be two observers. We will say that β *is comoving with* β' if β' is an observer of the observers congruence associated with β, i.e. β' is an integral curve of this vector field.

Since parallel transport conserves metric and causality, the observers congruence associated with a given observer β is actually an observers congruence, because it is a future-pointing timelike unit vector field defined in the open set $\mathcal{M}_\beta^- \cup \beta$. Moreover, β observes that its 4-velocity is the same as the 4-velocity of any observer of this congruence. So, they define a reference frame associated with the observer β in a natural way. According to this method, we state the next definition.

Definition 3.2. Let λ be a light ray from q to p and let u, u' be two observers at p, q respectively. We will say that *u is comoving with u'* if $\tau_{qp}u' = u$.

3.1 Relative velocity of an observer

We can generalize the concept of "relative velocity of an observer" (given in Section 2) for observers at two different events of the same light ray:

Definition 3.3. Let λ be a light ray from q to p and let u, u' be two observers at p, q respectively. The *relative velocity of u' observed by u* is the relative velocity of $\tau_{qp}u'$ observed by u, according to (1).

So, the relative velocity of u' observed by u is given by the unique vector $v \in u^{\perp}$ such that $\tau_{qp}u' = \gamma(u+v)$, where γ is the gamma factor corresponding to the velocity $\|v\|$. Note that $\tau_{qp}u'$ is the way u observers u', and so, it is the natural adaptation of u' at p.

We can generalize this definition for two observers β and β':

Definition 3.4. Let β, β' be two observers, and let U, U' be the 4-velocities of β, β' respectively. The *relative velocity of* β' *observed by* β is a vector field V defined on β such that V_p is the relative velocity of U'_q observed by U_p (in the sense of Definition 3.3), where p, q are events of β, β' respectively and there exists a light ray from q to p.

By Definitions 3.2 and 3.3, we have that u is comoving with u' if and only if the relative velocity of u' observed by u is zero. Analogously, by Definitions 3.1 and 3.4, we have that β is comoving with β' if and only if the relative velocity of β' observed by β is zero.

For example, in the Schwarzschild metric with spherical coordinates

$$ds^2 = -a^2(r)\,dt^2 + \frac{1}{a^2(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where $a(r) = \sqrt{1-\frac{2m}{r}}$ and $r > 2m$, we have that $\lambda : [r_1, +\infty) \to \mathcal{M}$ with $r_1 > 2m$ given by

$$\lambda(r) := \left(2m\ln\left(\frac{r-2m}{r_1-2m}\right) + r - r_1, r, \frac{\pi}{2}, 0\right) \tag{8}$$

is a radial light ray emitted from $q := \lambda(r_1) = (0, r_1, \pi/2, 0)$ and moving away from the event horizon $r = 2m$. Given a radius $r_2 > r_1$, let $p := \lambda(r_2)$ be an event of λ and let $u_1 = u_1^t \left.\frac{\partial}{\partial t}\right|_q + u_1^r \left.\frac{\partial}{\partial r}\right|_q + u_1^\theta \left.\frac{\partial}{\partial\theta}\right|_q + u_1^\varphi \left.\frac{\partial}{\partial\varphi}\right|_q$ be a vector in $T_q\mathcal{M}$. Taking into account the Christoffel symbols of the metric, it can be proved that

$$\begin{aligned}\tau_{qp}u_1 &= \frac{1}{2a_2^2}\left(\left(a_2^2+a_1^2\right)u_1^t + \left(1-\frac{a_2^2}{a_1^2}\right)u_1^r\right)\left.\frac{\partial}{\partial t}\right|_p \\ &\quad + \frac{1}{2}\left(\left(a_1^2-a_2^2\right)u_1^t + \left(1+\frac{a_2^2}{a_1^2}\right)u_1^r\right)\left.\frac{\partial}{\partial r}\right|_p + \frac{r_1}{r_2}u_1^\theta\left.\frac{\partial}{\partial\theta}\right|_p + \frac{r_1}{r_2}u_1^\varphi\left.\frac{\partial}{\partial\varphi}\right|_p,\end{aligned} \tag{9}$$

where $a_1 := a(r_1)$ and $a_2 := a(r_2)$.

- If u_1 is a stationary observer, then $u_1 = \frac{1}{a_1}\left.\frac{\partial}{\partial t}\right|_q$. Let u_2 be a stationary observer at p. By (9) and taking into account Definition 3.3, the relative velocity v of u_1 observed by u_2 is given by
$$v = a_2\frac{a_1^2-a_2^2}{a_1^2+a_2^2}\left.\frac{\partial}{\partial r}\right|_p, \tag{10}$$
and hence, $\|v\| = \frac{a_2^2-a_1^2}{a_2^2+a_1^2} < 1$. If $r_1 \to 2m$ then $\|v\| \to 1$. This accords with the fact that "a *particle* at rest in the space at $r = 2m$ would have to be a photon" [9].
- If u_1 is a radial free-falling observer, then $u_1 = \frac{E}{a_1^2}\left.\frac{\partial}{\partial t}\right|_q - \sqrt{E^2-a_1^2}\left.\frac{\partial}{\partial r}\right|_q$, where E is a constant of motion given by $E := \left(\frac{1-2m/r_0}{1-v_0^2}\right)^{1/2}$, r_0 is the radial coordinate at which the fall begins, and v_0 is the initial velocity [10]. Let u_2 be a stationary observer at p. So, by (9) and taking into account Definition 3.3, the relative velocity v of u_1 observed by u_2 is given by
$$v = -a_2\frac{\left(a_2^2+a_1^2\right)\sqrt{E^2-a_1^2} + E\left(a_2^2-a_1^2\right)}{\left(a_2^2-a_1^2\right)\sqrt{E^2-a_1^2} + E\left(a_2^2+a_1^2\right)}\left.\frac{\partial}{\partial r}\right|_p,$$
and hence, $\|v\| = \frac{\left(a_2^2+a_1^2\right)\sqrt{E^2-a_1^2}+E\left(a_2^2-a_1^2\right)}{\left(a_2^2-a_1^2\right)\sqrt{E^2-a_1^2}+E\left(a_2^2+a_1^2\right)} < 1$. If $r_1 \to 2m$ then $\|v\| \to 1$.

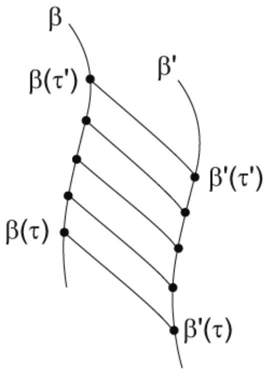

Figure 3: If β is comoving with β' then β observes that β' uses the same clock as its.

An observer with $r > 2m$ is unable to observe a free-falling particle crossing the event horizon, since light rays cannot escape from the zone $r \leq 2m$. Hence, it can never observe a free-falling particle reaching the speed of light. The only observer being able to observe a particle at $r = 2m$ is an observer which crosses the event horizon at the same time and at the same point as the particle. The relative velocity of the particle observed by this observer is smaller than the speed of light, as it is shown in [10].

3.2 Doppler effect and gravitational redshift

Taking into account Definition 3.3, we can generalize the expression of Doppler effect (4) for observers at different events of the same light ray:

Proposition 3.1. *Let λ be a light ray from q to p and let u, u' be two observers at p, q respectively. Then*

$$\nu' = \gamma \left(1 - g\left(v, w\right)\right) \nu, \tag{11}$$

where ν, ν' are the frequencies of λ observed by u, u' respectively, v is the relative velocity of u' observed by u, w is the relative velocity of λ observed by u and γ is the gamma factor corresponding to the velocity $\|v\|$.

Proof. Let F be the frequency vector field of λ. Then, $\nu' = -g\left(F_q, u'\right)$. Since parallel transport conserves metric, we have $\nu' = -g\left(\tau_{qp} F_q, \tau_{qp} u'\right) = -g\left(F_p, \tau_{qp} u'\right)$. So, the frequency of λ observed by $\tau_{qp} u'$ is also ν'. Taking into account (4) and Definition 3.3, expression (11) holds. □

Note that the proof of Proposition 3.1 assures that the frequency of λ observed by u' is the same as the frequency of λ observed by $\tau_{qp} u'$. Taking into account Definition 3.2, if u is comoving with u' then they observe λ with the same frequency. This result can be also obtained from expression (11), since the relative velocity v of u' observed by u is zero if u is comoving with u'.

So, given β an observer comoving with another observer β' and given λ a light ray from β' to β, we have that β and β' observe λ with the same frequency. Hence, within the framework of lightlike simultaneity, "*β is comoving with β'*" means "*β is spectroscopically comoving with β'*". This fact can be interpreted in this way: if β' emits n light rays in a unit of its proper time, then β observes also n light rays in a unit of its proper time. So, β observes that β' uses the "same clock" as its (see Figure 3).

Given two stationary observers (i.e. with constant spatial coordinates, for a given coordinate system) β, β', and a light ray λ from β' to β, the frequency of λ observed by β is, in general, different from the frequency observed by β'. This phenomenon is known as *gravitational redshift*. Since two stationary observers are in "rest" with respect to each other, they are supposed to be "comoving". Thus, gravitational redshift effect has been always considered independent from Doppler effect, arguing that photons lose or gain energy when rising or falling in a gravitational field. Nevertheless, in our formalism, stationary observers are not comoving in general. Hence, there appears a Doppler shift given by (11) that coincides with the known gravitational shift, explaining it in a natural way.

A clear example can be found in the Schwarzschild metric with spherical coordinates, considering the radial light ray λ given in (8). Let u_1 be a stationary observer at $q := \lambda(r_1)$, and let u_2 be another stationary observer at $p := \lambda(r_2)$, with $r_2 > r_1 > 2m$. Taking $a_1 := a(r_1)$ and $a_2 := a(r_2)$, we have that the relative velocity v of u_1 observed by u_2 is given by (10). Moreover, the relative velocity w of λ observed by u_2 is $a_2 \left.\frac{\partial}{\partial r}\right|_p$. Applying the general expression for Doppler effect (11), we have

$$\nu_1 = \frac{a_2}{a_1}\nu_2, \tag{12}$$

where ν_1, ν_2 are the frequencies of λ observed by u_1, u_2 respectively. This redshift is produced because u_2 is not comoving with u_1 in our formalism. Effectively, if we parallely transport u_1 to p along λ, we obtain the vector

$$\tau_{qp}u_1 = \frac{1}{2}\left(\frac{a_1}{a_2^2} + \frac{1}{a_1}\right)\left.\frac{\partial}{\partial t}\right|_p + \frac{1}{2}\left(a_1 - \frac{a_2^2}{a_1}\right)\left.\frac{\partial}{\partial r}\right|_p,$$

that it is obviously different from u_2.

Hence, given two equatorial stationary observers $\beta_1(\tau) := \left(\frac{1}{a_1}\tau, r_1, \pi/2, 0\right)$ and $\beta_2 := \left(\frac{1}{a_2}\tau, r_2, \pi/2, 0\right)$ with $\tau \in \mathbb{R}$, and a radial light ray λ from β_1 to β_2, equation (12) holds, where ν_1, ν_2 are the frequencies of λ observed by β_1, β_2 respectively. Equation (12) is the known expression for gravitational redshift in Schwarzschild metric, and so, it is a particular case of the generalized Doppler effect given by expression (11). Note that $\nu \to 0$ when $r_1 \to 2m$, according to the fact that $\|v\| \to 1$ when $r_1 \to 2m$ (see (10)).

Another example is the cosmological redshift produced by the expansion of the universe in the Robertson-Walker metric with spherical coordinates

$$ds^2 = -dt^2 + a^2(t)\left(\frac{1}{1-kr^2}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right),$$

where $a(t)$ is the *scale factor* and $k = -1, 0, 1$. Such redshift is too a particular case of Doppler effect because stationary observers (usually called "comoving", unfortunately for our formalism) are not comoving. This effect can be calculated using the Killing $(2,0)$-tensor $K(X,Y) := a^2(t)(g(X,Y) + g(X,U)g(Y,U))$, where X, Y are two vector fields and $U := \frac{\partial}{\partial t}$ is the 4-velocity vector field of the congruence of stationary observers. So, given X a geodesic vector field, we have that $K(X,X) = a^2(t)\left(g(X,X) + g(X,U)^2\right)$ is constant along its integral curves. Therefore, since the frequency vector field F of the light ray λ is geodesic and lightlike, we have that $a(t)g(F,U)$ is constant along λ. So, $a(t)\nu$ is constant too, where ν is the frequency of λ observed by a stationary observer of the congruence U. Hence, given two stationary observers β_1, β_2 and a light ray λ emitted by β_1 at coordinate time t_1 and observed by β_2 at coordinate time t_2, we have that the expression (11) for Doppler effect has the form

$$\nu_1 = \frac{a(t_2)}{a(t_1)}\nu_2, \tag{13}$$

where ν_1, ν_2 are the frequencies of λ observed by β_1 and β_2 respectively.

The functions $a(r)$ of (12) and $a(t)$ of (13) are responsible for the gravitational redshift in Schwarzschild and Robertson-Walker metrics. This functions are usually called *lapse functions.*

A discussion about these facts can also be found in [11].

3.3 Light aberration

Taking into account Definition 3.3, we can also generalize expressions (5) and (6) of light aberration effect for observers at different events of the same light ray:

Proposition 3.2. *Let λ be a light ray from q to p and let u, u' be two observers at p, q respectively. Then*

$$\tau_{qp}w' = \frac{1}{\gamma\left(1 - g\left(v, w\right)\right)}\left(u + w\right) - \tau_{qp}u', \tag{14}$$

where w, w' are the relative velocities of λ observed by u, u' respectively, v is the relative velocity of u' observed by u, and γ is the gamma factor corresponding to the velocity $\|v\|$. Moreover, if $\tau_{qp}w' \neq w$ then

$$\cos\theta = \frac{\cos\theta' - \|v\|}{1 - \|v\|\cos\theta'}, \tag{15}$$

where θ is the angle between $-w$ and v, and θ' is the angle between $-\tau_{qp}w'$ and the projection of v onto $\left(\tau_{qp}u'\right)^{\perp}$.

Proof. Let F be the frequency vector field of λ. Then, $F_p = \nu\left(u + w\right)$ and $F_q = \nu'\left(u' + w'\right)$. Since F is tangent to λ and geodesic, we have $F_p = \tau_{qp}F_q = \nu'\left(\tau_{qp}u' + \tau_{qp}w'\right)$. So, $\tau_{qp}w' = \frac{\nu}{\nu'}\left(u + w\right) - \tau_{qp}u'$. Applying Proposition 3.1, expression (14) holds. If $\tau_{qp}w' \neq w$ then expression (15) is obtained from (14) by simple algebraic manipulations. $\square$

If u is comoving with u', then $\tau_{qp}u' = u$, $v = 0$ and so, from (14), we have $\tau_{qp}w' = w$. Since $\tau_{qp}w'$ is the way u observes w', we can say that u and u' observes λ with the "same" relative velocity, and hence there is not light aberration between comoving observers.

4 Affine distance

To measure distances in our formalism we have to measure "lengths" of light rays, as we told in the Introduction. But light rays are lightlike curves and they have no length. To measure distances and angles, an observer has to project these light rays onto its physical space (i.e. the orthogonal space of its 4-velocity). This idea drives us to the next definition of distance.

Definition 4.1. Let λ be a light ray from q to p and let u be an observer at p. The *affine distance from q to p observed by u*, denoted as $d_u\left(q, p\right)$, is the module of the projection of $\exp_p^{-1} q$ onto $u^{\perp}$ (see Figure 4).

This concept of distance is defined according to the concept of lightlike simultaneity given by the past-pointing horismos submanifolds, because we measure distances between an event p and events that are observed simultaneously at p (i.e. events of E_p^-).

Taking into account Definition 4.1, we have $d_u\left(q, p\right) = -g\left(\exp_p^{-1} q, w\right)$, where w is the relative velocity of λ observed by u (see Figure 4). So, it is easy to prove that

$$d_u\left(q, p\right) = g\left(\exp_p^{-1} q, u\right). \tag{16}$$

In the tangent space $T_p\mathcal{M}$ we have that w and $\exp_p^{-1} q$ are proportional and opposite. Taking into account Definition 4.1, we have $\exp_p^{-1} q = -d_u\left(q, p\right)\left(u + w\right)$. Given another observer u'

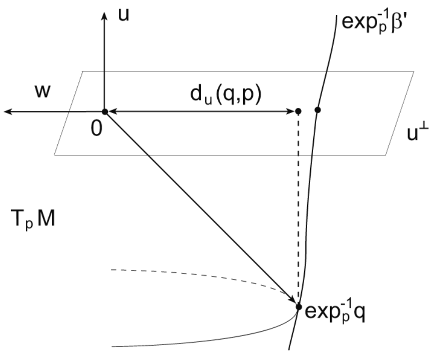

Figure 4: Scheme in $T_p\mathcal{M}$ of the affine distance from q to p observed by u, given in Definition 4.1. In this case, q is an event of a world line β'. Note that $d_u(q,p)$ does not depend on β'.

at p, we have $\exp_p^{-1} q = -d_{u'}(q,p)(u'+w')$, where w' is the relative velocity of λ observed by u'. Therefore, we obtain

$$d_{u'}(q,p) = \gamma(1 - g(v,w))\, d_u(q,p)\,, \tag{17}$$

where v is the relative velocity of u' observed by u and γ is the gamma factor corresponding to $\|v\|$.

If we compare (17) with (4), we realize that frequency and affine distance have the same behavior when a change of observer is done. Hence, if λ is a light ray from q to p and u, u' are two observers at p, we have

$$\frac{d_u(q,p)}{\nu} = \frac{d_{u'}(q,p)}{\nu'}, \tag{18}$$

where ν, ν' are the frequencies of λ observed by u, u' respectively.

The next proposition shows that the concept of distance given in Definition 4.1 coincides with the known concept of *affine distance* introduced in [12].

Proposition 4.1. *Let λ be a light ray from q to p, let u be an observer at p, and let w be the relative velocity of λ observed by u. If we parameterize λ affinely (i.e. $\nabla_{\dot\lambda(s)}\dot\lambda(s) = 0$) such that $\lambda(0) = p$ and $\dot\lambda(0) = -(u+w)$, then $\lambda(d_u(q,p)) = q$ (see Figure 5).*

Proof. By the properties of the exponential map (see [4]), we have $\lambda(s) = \exp_p(-s(u+w))$. So $\lambda(d_u(q,p)) = \exp_p(-d_u(q,p)(u+w)) = q$. ☐

Hence, given a light ray λ from q to p and an observer u at p, we can interpret the affine distance from q to p observed by u as the distance (or time) traveled by the light ray λ, measured by an observer at p with 4-velocity u. An equivalent result is given in the next corollary.

Corollary 4.1. *Let λ be a light ray from q to p, let u be an observer at p, and let w be the relative velocity of λ observed by u. If we parameterize λ affinely such that $\lambda(0) = q$, $\lambda(d) = p$ and $\dot\lambda(d) = u+w$, then d is the affine distance from q to p observed by u.*

Now, we are going to generalize Definition 4.1.

Definition 4.2. Let β, β' be two observers. The *affine distance from β' to β observed by β* is a real positive function d_β defined on β such that, given $p \in \beta$, $d_\beta(p)$ is the affine distance from q to p observed by u, where u is the 4-velocity of β at p, and q is the unique event of β' such that there exists a light ray from q to p.

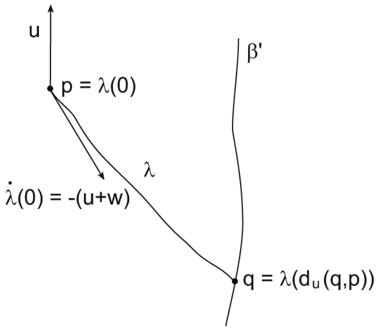

Figure 5: Scheme of Proposition 4.1, where q is an event of a world line β'. Note that $d_u\left(q,p\right)$ does not depend on β'.

Note that even if β is comoving with β', the affine distance d_β from β' to β observed by β is not necessarily constant. Inversely, if d_β is constant then β is not necessarily comoving with β', as we will see in Section 5.2. Only in some special cases we have that d_β is constant if and only if β is comoving with β'. For example, in the Minkowski spacetime if the observers β and β' are geodesic.

Finally, we can define a distance on E_p^- extending the concept of affine distance given in Definition 4.1, using the idea that an observer has to project light rays onto its physical space:

Definition 4.3. Let u be an observer at p, and $q,q'\in E_p^-\cup\{p\}$. The *affine distance from q to q' observed by u*, denoted as $d_u\left(q,q'\right)$, is the module of $\pi_{u^\perp}\left(\exp_p^{-1}q\right)-\pi_{u^\perp}\left(\exp_p^{-1}q'\right)$, where $\pi_{u^\perp}$ is the map "projection onto $u^\perp$".

It can be easily proved that

$$\begin{aligned} d_u\left(q,q'\right) &= \Big(g\left(\exp_p^{-1}q-\exp_p^{-1}q',\exp_p^{-1}q-\exp_p^{-1}q'\right) \\ &\quad +g\left(u,\exp_p^{-1}q-\exp_p^{-1}q'\right)^2\Big)^{1/2}. \end{aligned} \tag{19}$$

Moreover, expression (19) generalizes expression (16) in the sense that if we substitute q' by p in (19), we obtain (16).

The affine distance given in Definition 4.3 is symmetric, positive-definite and satisfies the triangular inequality. So, it has all the properties that must verify a topological distance defined on $E_p^-\cup\{p\}$.

5 Some examples of affine distance

In this Section we are going to show that affine distance is a particular case of *radar distance* in the Minkowski spacetime (concretely, for geodesic observers), and generalizes the *proper radial distance* in the Schwarzschild spacetime. Finally, we show that affine distance gives us a new concept of distance in Robertson-Walker spacetimes, according to Hubble law.

5.1 Minkowski

In the Minkowski metric with rectangular coordinates $ds^2=-dt^2+dx^2+dy^2+dz^2$, let us consider an event $q=\left(t_1,x_1,y_1,z_1\right)$ observed at $p=\left(t_2,x_2,y_2,z_2\right)$ by an observer $u=\gamma\left(\left.\frac{\partial}{\partial t}\right|_p+v^x\left.\frac{\partial}{\partial x}\right|_p+v^y\left.\frac{\partial}{\partial y}\right|_p+v^z\left.\frac{\partial}{\partial z}\right|_p\right)$, where γ is the gamma factor $\frac{1}{\sqrt{1-(v^x)^2-(v^y)^2-(v^z)^2}}$.

Then, using (16), we have the general expression for the affine distance from q to p observed by u:

$$\begin{aligned} d_u(q,p) &= g(q-p,u) \\ &= \gamma\left((t_2-t_1)+v^x(x_1-x_2)+v^y(y_1-y_2)+v^z(z_1-z_2)\right). \end{aligned} \tag{20}$$

Note that $(t_2-t_1)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$ because there is a light ray from q to p.

There exists a known method to measure distances between an observer β (that we can suppose parameterized by its proper time τ) and an observed event q, called "radar method", consisting on emitting a light ray from $\beta(\tau_1)$ to q, that bounces and arrives at $p=\beta(\tau_2)$. The *radar distance* between β and q observed by β is given by $\frac{1}{2}(\tau_2-\tau_1)$ [13]. So, considering a geodesic observer β passing through p with 4-velocity at p given by $u=\gamma\left(\frac{\partial}{\partial t}\big|_p+v^x\,\frac{\partial}{\partial x}\big|_p+v^y\,\frac{\partial}{\partial y}\Big|_p+v^z\,\frac{\partial}{\partial z}\big|_p\right)$, we have that

$$\beta(\tau)=\left(\gamma(\tau-\tau_2)+t_2,\gamma v^x(\tau-\tau_2)+x_2,\gamma v^y(\tau-\tau_2)+y_2,\gamma v^z(\tau-\tau_2)+z_2\right) \tag{21}$$

is the parameterization by its proper time. Setting out that $q-\beta(\tau_1)$ is lightlike and $\tau_2-\tau_1\neq 0$, from (21) we obtain

$$\frac{1}{2}(\tau_2-\tau_1)=\gamma\left((t_2-t_1)+v^x(x_1-x_2)+v^y(y_1-y_2)+v^z(z_1-z_2)\right). \tag{22}$$

Comparing (22) with (20), we state that affine distance coincides with radar distance for geodesic observers in Minkowski spacetime.

The radar distance between a non geodesic observer β and an observed event q depends on the world line β between $\beta(\tau_1)$ and $\beta(\tau_2)$. On the other hand, the affine distance only depends on the 4-velocity of the observer at $p=\beta(\tau_2)$, i.e. at the instant when the light ray arrives from q. So, it is easier to calculate and it has more physical sense.

5.2 Schwarzschild

In Schwarzschild metric with spherical coordinates, let β_1 and β_2 be two stationary observers like in Section 3.2. We are going to calculate the affine distance d from β_1 to β_2 observed by β_2. Since

$$\lambda(s)=\left(-a(r_2)\,s+2m\ln\left(1-\frac{s}{r_2\,a(r_2)}\right),r_2-a(r_2)\,s,\pi/2,0\right)$$

is a light ray parameterized as in the hypotheses of Proposition 4.1, with $p:=\lambda(0)\in\beta_2$ and $q:=\lambda\left(\frac{r_2-r_1}{a(r_2)}\right)\in\beta_1$, we have that the affine distance from q to p observed by u (where u is the 4-velocity of β_2 at p) is given by $d_u(q,p)=\frac{r_2-r_1}{a(r_2)}$. This expression only depends on r_1 and r_2, i.e. the events q and p can be any events of β_1 and β_2 respectively, such that there exists a light ray from q to p. Hence the affine distance d from β_1 to β_2 observed by β_2 is given by

$$d=\frac{r_2-r_1}{a(r_2)}. \tag{23}$$

So, d is constant, but β_2 is not comoving with β_1.

Expression (23) is precisely a known expression for the *proper radial distance* between spheres of radius r_1 and r_2 (see [13]). So, the affine distance generalizes the proper radial distance given in Schwarzschild metric.

5.3 Robertson-Walker

In Robertson-Walker metric with spherical coordinates, let β_1 and β_0 be two stationary observers at $r = r_1 > 0$ and $r = 0$ respectively. Let us suppose that β_1 emits a light ray λ at $t = t_1$ that arrives at β_0 at $t = t_0$. For studying distances in cosmology it is usual to consider the scale factor in the form

$$a(t) = a(t_0)\left(1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2\right) + \mathcal{O}\left(H_0^3 (t - t_0)^3\right) \tag{24}$$

where $a(t_0) > 0$, $H(t) = \dot{a}(t)/a(t)$ is the Hubble "constant", $H_0 = H(t_0) > 0$, $q(t) = -a(t)\ddot{a}(t)/\dot{a}(t)^2$ is the deceleration coefficient, and $q_0 = q(t_0) > 0$, with $|H_0(t - t_0)| \ll 1$ [13]. This corresponds to a universe in decelerated expansion and the time scales that we are going to use are relatively small.

The *proper distance*, d_{proper}, between two stationary observers at a given instant t is defined as the coordinate distance multiplied by the scale factor $a(t)$ (see [13]). The proper distance between β_1 and β_0 at $t = t_0$ is given by $d_{\text{proper}} := r_1 a(t_0)$. Obviously, this distance is not the same as the affine distance (which we are going to denote d_{affine}). We define the redshift parameter $z := \frac{a(t_0)}{a(t_1)} - 1$, obtaining that

$$d_{\text{proper}} = \frac{z}{H_0}\left(1 - \frac{1}{2}(1 + q_0) z\right) + \mathcal{O}\left(z^3\right). \tag{25}$$

Moreover, the *luminosity distance*, $d_{\text{luminosity}}$, between a stationary observer and a stationary light source at a given instant t is defined as $d_{\text{luminosity}} := \sqrt{\frac{L}{4\pi A}}$, where L is the absolute luminosity and A is the apparent luminosity (see [13]). Applied to β_0 and β_1 at $t = t_0$, we have

$$d_{\text{luminosity}} = \frac{z}{H_0}\left(1 + \frac{1}{2}(1 - q_0) z\right) + \mathcal{O}\left(z^3\right). \tag{26}$$

Comparing (26) with (25), we obtain that $d_{\text{proper}} < d_{\text{luminosity}}$ for $z \ll 1$. This distance is related to the geodesic deviation method, and it is studied in [14].

Finally, we are going to calculate the affine distance d_{affine} from β_1 to β_0 observed by β_0 at $t = t_0$. It can be interpreted as the distance traveled by the light ray λ measured by the observer β_0, and it will satisfy $r_1 a(t_1) < d_{\text{affine}} < r_1 a(t_0) = d_{\text{proper}}$. The vector field $-\frac{1}{a}\frac{\partial}{\partial t} + \frac{\sqrt{1-kr^2}}{a^2}\frac{\partial}{\partial r}$ is geodesic, lightlike and its integral curves are radial light rays that arrive at $r = 0$ (i.e. at β_0). So, to parameterize λ like in Proposition 4.1, we have to set out the system

$$\begin{cases} \dot{\lambda}^t(s) = \dfrac{-a(t_0)}{a(\lambda^t(s))} \\ \dot{\lambda}^r(s) = \dfrac{a(t_0)\sqrt{1 - k\lambda^r(s)^2}}{a^2(\lambda^t(s))} \\ \lambda^t(0) = t_0;\ \lambda^r(0) = 0 \end{cases}, \tag{27}$$

where λ^t and λ^r are the temporal and radial components of λ respectively. Using (24) and taking into account that $\lambda^t(d_{\text{affine}}) = t_1$ (by Proposition 4.1), from the integration of the first equation of (27) we obtain that

$$d_{\text{affine}} = (t_0 - t_1) - \frac{1}{2} H_0 (t_0 - t_1)^2 - \frac{1}{6} q_0 H_0^2 (t_0 - t_1)^3 + \mathcal{O}\left(H_0^3 (t_0 - t_1)^3\right). \tag{28}$$

Since $H_0(t_0 - t_1) = z - \left(1 + \frac{1}{2} q_0\right) z^2 + \mathcal{O}\left(z^3\right)$, from (28) we have

$$d_{\text{affine}} = \frac{z}{H_0}\left(1 - \frac{1}{2}(3 + q_0) z\right) + \mathcal{O}\left(z^3\right), \tag{29}$$

that is consistent with the Hubble law (for z of first order approximation). If we compare (29) with (25) we obtain that, effectively, $d_{\text{affine}} < d_{\text{proper}}$ for $z \ll 1$.

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