Comoving Observers and Distances in the Framework of Lightlike Simultaneity

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Based on the concept of lightlike simultaneity, we state a condition for an observer to be comoving to another observer, in general relativity. As a consequence of this condition, we show that gravitational redshift effect is a particular case of Doppler effect. We also define a distance between an observer and the events that it observes, according to lightlike simultaneity, obtaining geometrical properties. Finally we show that it generalizes some known distances in Minkowski and Schwarzschild space-times, and gives us a new concept of distance in Robertson-Walker space-times.

1 Introduction

In General Relativity it is often difficult to interpret when two disjoint observers are comoving. For example, given a particular coordinate system it is usual to suppose that stationary observers (i.e. with constant spatial coordinates) are comoving. But if an observer is not stationary, we are unable to check whether another observer is comoving to it or not.

Given an observer $\beta$, there is a method to check if it is comoving to another observer $\beta'$, based on the concept of simultaneity. We have to build a simultaneity foliation associated to $\beta$, then parallelly transport the 4-velocity of $\beta'$ to $\beta$, along geodesics joining $\beta'$ with $\beta$ in the leaves of the foliation, and finally compare it with the 4-velocity of $\beta$ (see Fig. 1).

There are two kinds of simultaneity foliations associated to a given observer $\beta$ [1]: the Landau foliation $L_\beta$, whose leaves are Landau submanifolds [2] (spacelike); and the past-pointing horismos foliation $E^-_\beta$, whose leaves are past-pointing horismos submanifolds [3] (lightlike). So, the problem is to decide which simultaneity is mathematically and physically more suitable.

Mathematically: In a previous work [1] we proved that the Landau foliation $L_\beta$ is not always defined in every convex normal neighborhood. Its leaves can intersect themselves, for example, in the Minkowski space-time if the observer $\beta$ is not geodesic. Moreover, $L_\beta$ is not necessarily spacelike at every point of a convex normal neighborhood. On the other hand, the past-pointing horismos foliation $E^-_\beta$ is always well defined in every convex normal neighborhood and it is always lightlike.

Physically: Given an observer at a point $p$ with 4-velocity $u$, the events of its Landau submanifold $L_{p,u}$ do not affect the observer at $p$ in any way, since both electromagnetic and gravitational waves travel at the speed of light, $c$. So, the events of its past-pointing horismos submanifold $E^-_p$ are precisely the events that affect and are observed by the observer at $p$, i.e. the events that exist for the observer at $p$. Therefore, it is logical to accept that the events that are simultaneous with an observer at a given instant were precisely the events that affect and are observed by the observer at that very instant.
Fig. 1. How to check if an observer $\beta$ is comoving to another observer $\beta'$, depending on the simultaneity foliation that we are using. Left: Landau foliation $\mathcal{L}_\beta$ (spacelike). Right: Past-pointing horismos foliation $\mathcal{E}^-_\beta$ (lightlike).

We suppose that the space-time manifold $\mathcal{M}$ is a convex normal neighborhood. Thus, given two points $p$ and $q$ in $\mathcal{M}$, there exists a unique geodesic joining $p$ and $q$. The parallel transport along this geodesic will be denoted by $\tau_{pq}$. Moreover, we are going to identify an observer with its 4-velocity. So, an observer at $p$ is a future-pointing timelike unit vector $u$ in $T_p\mathcal{M}$.

2 Comoving observers

According to Section 1, given two events $p, q$ in a lightlike geodesic $\lambda$, we will say that an observer $u$ at $p$ is comoving to another observer $u'$ at $q$ if and only if $\tau_{qp}u' = u$.

Using this criteria, we can construct the observers congruence comoving to a given observer $\beta$ translating the 4-velocity of $\beta$ along future-pointing lightlike geodesics in the leaves of the future-pointing horismos foliation $\mathcal{E}^+_\beta$ [1]. The result is a vector field which integral curves are all the observers that are comoving to $\beta$.

2.1 Relative velocity of $u'$ observed by $u$

Given two observers $u, u'$ at the same event $p$, we have that $u'$ can be decomposed in the form

$$u' = \gamma (u + v),$$

where $v \in u^\perp$, $0 \leq \|v\| < 1$ and $\gamma := -g(u, u') = \frac{1}{\sqrt{1-\|v\|^2}}$. This decomposition is unique.

So, we will say that $v$ is the relative velocity of $u'$ observed by $u$, and $\gamma$ is the gamma factor corresponding to $\|v\|$.

This definitions can be generalized for observers at different events of the same lightlike geodesic, using our concept of comoving observers. Given two observers $p, q$ in a lightlike geodesic $\lambda$, and two observers $u, u'$ at $p, q$ respectively, we have that $\tau_{qp}u'$ is comoving to $u'$ and so, it is the natural adaptation of $u'$ at $p$. Analogously, $\tau_{qp}u'$ can be decomposed in the form

$$\tau_{qp}u' = \gamma (u + v),$$

where $v \in u^\perp$, $0 \leq \|v\| < 1$ and $\gamma := -g(u, \tau_{qp}u') = \frac{1}{\sqrt{1-\|v\|^2}}$. This decomposition is unique.

So, we will say that $v$ is the relative velocity of $u'$ observed by $u$, and $\gamma$ is the gamma factor corresponding to $\|v\|$.
2.2 Doppler effect and gravitational redshift

A light ray is given by a lightlike geodesic $\lambda$ and a future-pointing lightlike vector field $F$ defined in $\lambda$, such that $\nabla_F F = 0$ (i.e. it is geodesic and so, tangent to $\lambda$), called frequency vector field of $\lambda$. Given an event $p$ of $\lambda$ and an observer $u$ at $p$, the frequency of $\lambda$ observed by $u$ is given by $\nu := -g(u, F_p)$. So, we have that $F_p$ can be decomposed in the form

$$F_p = \nu (u + s),$$

where $s \in u^\perp, \|s\| = 1$ and $\nu := -g(u, F_p)$. This decomposition is unique. So, we will say that $s$ is the relative velocity of $\lambda$ observed by $u$.

Given two observers $u, u'$ at the same event $p$ of a light ray $\lambda$, we have that

$$\nu' = \gamma (1 - g(v, s)) \nu, \quad (1)$$

and

$$s' = \frac{1}{\gamma (1 - g(v, s))} (u + s) - u,$$

where $v$ is the relative velocity of $u'$ observed by $u$, $s, s'$ are the relative velocities of $\lambda$ observed by $u, u'$ respectively, and $\nu, \nu'$ are the frequencies of $\lambda$ observed by $u, u'$ respectively. If $\nu$ and $\nu'$ are different, we have a Doppler effect. In the same way, if $s$ and $s'$ are different, we have an aberration effect.

We can generalize the expression of Doppler effect for observers at different events of the same light ray, taking into account the generalized definitions of Section 2.1. Given two observers $u, u'$ at two different events $p, q$ resp. of the same light ray $\lambda$, we have that $\nu' = \gamma (1 - g(v, s)) \nu$, where $v$ is the relative velocity of $u'$ observed by $u$, $s$ is the relative velocity of $\lambda$ observed by $u$, and $\nu, \nu'$ are the frequencies of $\lambda$ observed by $u, u'$ respectively.

Moreover, $\nu'$ is the frequency of $\lambda$ observed by $\tau_{qp} u'$. So, if an observer is comoving to another observer, they observe the same light ray with the same frequency. This fact is apparently contradictory with gravitational redshift effect, that it is produced when, given a coordinate system, stationary observers (i.e. with constant spatial coordinates) observe the same light ray with different frequencies. It is usual to consider stationary observers as comoving and then it is usual to say that comoving observers observe the same light ray with different frequencies. But in our formalism, stationary observers are not comoving in general, and so gravitational redshift is just a particular case of Doppler effect.

3 Distances

To study distances in the framework of lightlike simultaneity, we have to measure distances between simultaneous events, and so, we have to measure lengths of light rays. Given two events $p, q$ of the same light ray $\lambda$ (with $\exp_p^{-1} q$ past-pointing), and an observer $u$ at $p$, we define the lightlike distance from $p$ to $q$ observed by $u$ as the module of the projection of $\exp_p^{-1} q$ onto $u^\perp$ (see Fig. 2-Left), i.e.

$$d := g (\exp_p^{-1} q, u). \quad (2)$$

Given another observer $u'$ at $p$, we have

$$d' = \gamma (1 - g(v, s)) d, \quad (3)$$

where $v$ is the relative velocity of $u'$ observed by $u$, $s$ is the relative velocity of $\lambda$ observed by $u$, and $d, d'$ are the lightlike distances from $p$ to $q$ observed by $u, u'$ respectively. Note that expressions (1) and (3) are analogous.

To interpret lightlike distance, we have the next result:
Proposition 1. Given two events \( p, q \) of the same light ray \( \lambda \) (with \( \exp^{-1} q \) past-pointing), and an observer \( u \) at \( p \), if we parametrize \( \lambda \) such that \( \lambda(0) = p \) and \( \dot{\lambda}(0) = -(u + s) \), where \( s \) is the relative velocity of \( \lambda \) observed by \( u \), then

\[
\lambda(d) = q,
\]

where \( d \) is the lightlike distance from \( p \) to \( q \) observed by \( u \) (see Fig. 2-Right).

So, we can interpret the lightlike distance as the length (or time) parameter of a light ray, given an observer.

Note 1. Another result equivalent to Proposition 1 is obtained if we parametrize \( \lambda \) such that \( \lambda(0) = q \), \( \lambda(d) = p \), and \( \dot{\lambda}(d) = u + s \). In this case, we have that \( d \) is the distance from \( p \) to \( q \) observed by \( u \). This result is very useful to calculate lightlike distances.

3.1 Minkowski

Lightlike distance coincides with the known “radar distance” for geodesic observers in Minkowski space-time:

Given an observer \( \beta \), an event \( p \) of \( \beta \) and another event \( q \) such that there exists a light ray from \( q \) to \( p \), the radar distance between \( p \) and \( q \) observed by \( \beta \) is defined as

\[
d_{\text{radar}} := (\tau_2 - \tau_1)/2,
\]

where \( p = \beta(\tau_2) \) and there exists a light ray from \( \beta(\tau_1) \) to \( q \) (see Fig. 3).

A good distance should only depend on \( p, q \) and \( u \). So, in general, radar distance is not a good distance because it depends on the world line of the observer between \( \beta(\tau_1) \) and \( \beta(\tau_2) \). But if the observer is geodesic, then it only depends on the observer at \( p \) and the observed event \( q \), obtaining

\[
d_{\text{radar}} = \gamma \left( (t_2 - t_1) + v^x (x_1 - x_2) + v^y (y_1 - y_2) + v^z (z_1 - z_2) \right),
\]

where \( q = (t_1, x_1, y_1, z_1) \), \( p = (t_2, x_2, y_2, z_2) \) and \( u = \gamma \left( \frac{\partial}{\partial t} \big|_p + v^x \frac{\partial}{\partial x} \big|_p + v^y \frac{\partial}{\partial y} \big|_p + v^z \frac{\partial}{\partial z} \big|_p \right) \) is the observer at \( p \) with \( \gamma := \left( 1 - (v^x)^2 - (v^y)^2 - (v^z)^2 \right)^{-1/2} \). Expression (4) is precisely \( g(q - p, u) \), i.e. the lightlike distance from \( p \) to \( q \) observed by \( u \) (see (2)).
\[ d_{\text{lightlike}} = \frac{r_2 - r_1}{a(r_2)}. \]
distance between $\beta_1$ and $\beta_0$ at $t = t_0$ is given by $d_{\text{proper}} := r_1 a(t_0)$. Obviously, this distance is not the same as the lightlike distance (which we are going to denote $d_{\text{lightlike}}$). We define the redshift parameter $z := \frac{a(t_0)}{a(t)} - 1$, obtaining that

$$d_{\text{proper}} = \frac{z}{H_0} \left( 1 - \frac{1}{2} (1 + q_0) z \right) + \mathcal{O}(z^3).$$

Moreover, the luminosity distance, $d_{\text{luminosity}}$, between a stationary observer and a stationary light source at a given instant $t$ is defined as $d_{\text{luminosity}} := \sqrt{\frac{L}{4\pi A}}$, where $L$ is the absolute luminosity and $A$ is the apparent luminosity (see [4]). Applied to $\beta_0$ and $\beta_1$ at $t = t_0$, we have

$$d_{\text{luminosity}} = \frac{z}{H_0} \left( 1 + \frac{1}{2} (1 - q_0) z \right) + \mathcal{O}(z^3).$$

Comparing (8) with (7), we obtain that $d_{\text{proper}} < d_{\text{luminosity}}$ for $z \ll 1$. This distance is related to the geodesic deviation method, and it is studied in [5].

Finally, we are going to calculate the lightlike distance $d_{\text{lightlike}}$ from $\beta_1$ to $\beta_0$ observed by $\beta_0$ at $t = t_0$. It can be interpreted as the distance travelled by the light ray $\lambda$ measured by the observer $\beta_0$, and it will satisfy $r_1 a(t_1) < d_{\text{lightlike}} < r_1 a(t_0) = d_{\text{proper}}$. The vector field $\frac{1}{a} \frac{\partial}{\partial t} + \frac{\sqrt{-k}}{a} \frac{\partial}{\partial r}$ is geodesic, lightlike and its integral curves are radial light rays that arrive at $r = 0$ (i.e. at $\beta_0$). So, to parameterize $\lambda$ like in Proposition 1, we have to set out the system

$$
\begin{cases}
\lambda^t(s) = \frac{-a(t_0)}{a(\lambda^t(s))} \\
\lambda^r(s) = \frac{a(t_0)\sqrt{1 - k\lambda^r(s)^2}}{a^2(t_0)} \\
\lambda^t(0) = t_0; \lambda^r(0) = 0
\end{cases}
$$

where $\lambda^t$ and $\lambda^r$ are the temporal and radial components of $\lambda$ respectively. Using (6) and taking into account that $\lambda^t(d_{\text{lightlike}}) = t_1$ (by Proposition 1), from the integration of the first equation of (9) we obtain that

$$d_{\text{lightlike}} = (t_0 - t_1) - \frac{1}{2} H_0 (t_0 - t_1)^2 - \frac{1}{6} q_0 H_0^2 (t_0 - t_1)^3$$

$$+ \mathcal{O}(H_0^3 (t_0 - t_1)^3).$$

Since $H_0(t_0 - t_1) = z - (1 + \frac{1}{2} q_0) z^2 + \mathcal{O}(z^3)$, from (10) we have

$$d_{\text{lightlike}} = \frac{z}{H_0} \left( 1 - \frac{1}{2} (3 + q_0) z \right) + \mathcal{O}(z^3),$$

that is consistent with the Hubble law (for $z$ of first order approximation). If we compare (11) with (7) we obtain that, effectively, $d_{\text{lightlike}} < d_{\text{proper}}$ for $z \ll 1$.

References