List of contributions

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Generalization of a motion law for foliations in general relativity

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Abstract

It is known that if the integral submanifolds of a foliation are totally geodesic then this foliation satisfies a motion law in the space-time. In this work, we show how this motion law can be extended by introducing some stability conditions between foliations. A particular case of stability (called regular stability) is studied and characterized in terms of the Riemann curvature tensor. This allows us to interprete regular self-stability as a motion law for flat foliations. We proof the existence of regularly self-stable foliations of dimension greater than 1 in *pp*-wave spacetimes, but we show that there are not any foliation of this kind in Schwarzschild and Robertson-Walker space-times.

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1. Introduction

We work on a n-dimensional space-time manifold $\mathcal{M}$ and we denote the Levi-Civita connection by ∇. We use the convention that span $(X_1, \ldots, X_p)$ denotes the subbundle spanned by the vector fields $X_1, \ldots, X_p$, and it is called *distribution*. Usually, a distribution of dimension p is called a p-distribution. All bases of distributions are local. A distribution that has an integral submanifold (leaf) in every point is a *foliation*. We say that a foliation is a *totally geodesic foliation* if its leaves are totally geodesic submanifolds.

Let Ω be a foliation, X a vector field of Ω, c a maximal integral curve of X and

$$\tau_t^c : T_{c(0)}\mathcal{M} \longrightarrow T_{c(t)}\mathcal{M}$$

the parallel transport along $c(t)$, for all $t \in I$, where I is the domain of c. Then, Ω verifies a *motion law* (see [1, 2]) if

$$\tau_t^c \Omega(c(0)) = \Omega(c(t)), \quad t \in I.$$

This motion law is equivalent to say that Ω is a totally geodesic foliation, i.e. $\nabla_Y X \in \Omega$ for all vector fields $X, Y \in \Omega$.

2. Stability

Given two distributions Ω, Ω' we will say that Ω is stable with respect to Ω', and it will be denoted by

$$\nabla_{\Omega'}\Omega \subset \Omega,$$

if $\nabla_Y X \in \Omega$ for all vector fields $X \in \Omega$, $Y \in \Omega'$.

It is easy to prove that, given Ω, Ω' two distributions, $\nabla_{\Omega'}\Omega \subset \Omega$ if and only if $\nabla_{\Omega'}\Omega^{\perp} \subset \Omega^{\perp}$. It is known that a distribution Ω is univoquely determined by its orthogonal distribution $\Omega^{\perp}$. In addition, this property says that Ω and $\Omega^{\perp}$ have the same stability properties.

Let Ω be a distribution. We will say that Ω is self-stable if it is stable with respect to itself. Clearly, Ω is self-stable if and only if it is a totally geodesic foliation.

3. Regular stability

Given two distributions Ω, Ω' we will say that Ω is regularly stable with respect to Ω', and it will be denoted by

$$\nabla_{\Omega'}\Omega = 0,$$

if there exists a basis $\{X_i\}_{i=1}^p$ of Ω such that $\nabla_Y X_i = 0$, for $i = 1, .., p$ and for all vector field $Y \in \Omega'$. In this case, we will say that the basis $\{X_i\}_{i=1}^p$ is a regularly stable basis of Ω with respect to Ω'. Only some special bases of Ω are regularly stable with respect to a given distribution Ω'.

Let Ω be a distribution. We will say that Ω is regularly self-stable if it is regularly stable with respect to itself.

Given a regularly stable basis of Ω with respect to Ω', we can build all the regularly stable bases by means of linear combinations with constant functions for Ω':

Proposition 1 *Let Ω, Ω' be two distributions such that $\nabla_{\Omega'}\Omega = 0$, and let $\{X_i\}_{i=1}^p$ be a regularly stable basis of Ω with respect to Ω'. Then, $\{\overline{X}_i\}_{i=1}^p$ is another regularly stable basis of Ω with respect to Ω' if and only if there exists a family of functions $\left\{\alpha_i^j\right\}_{i,j=1}^p$ such that*

- $\det \alpha_i^j \neq 0$,
- $\overline{X}_i = \alpha_i^j X_j$ *for all* $i = 1, \ldots, p$,
- $Y\left(\alpha_i^j\right) = 0$, *for all* $i, j = 1, ..., p$, *and for all* $Y \in \Omega'$ *(i.e.* $\left\{\alpha_i^j\right\}_{i,j=1}^p$ *is a family of constant functions for* Ω'*).*

The next theorem is the main result of the work. It is very useful to study stability by means regular stability.

Theorem 2 *Let Ω and Ω' be a p-distribution and a q-foliation respectively such that $\nabla_{\Omega'}\Omega \subset \Omega$. Then, $\nabla_{\Omega'}\Omega = 0$ if and only if $R(Y, Z)X = 0$ for all $X \in \Omega$ and for all $Y, Z \in \Omega'$, where R is the Riemann curvature tensor.*

A proof of the Theorem can be found in [3]. There are several important corollaries of Theorem 2:

Corollary 3 *In a flat space-time (Minkowski), given Ω a distribution and Ω' a foliation, we have that $\nabla_{\Omega'}\Omega \subset \Omega$ if and only if $\nabla_{\Omega'}\Omega = 0$.*

Corollary 4 *Let Λ be a 1-foliation and let Ω be a distribution. Then $\nabla_\Lambda\Omega \subset \Omega$ if and only if $\nabla_\Lambda\Omega = 0$.*

Corollary 5 *Let Ω be a self-stable foliation. Then Ω is regularly self-stable if and only if $R(Y, Z)X = 0$, for all $X, Y, Z \in \Omega$.*

Corollary 5 allows us to give a geometric interpretation of regular self-stability: let Ω be a regularly self-stable foliation; on one hand, Ω is self-stable, so each leaf has a differentiable submanifold structure with the induced metric; on the other hand, it is satisfied that $R(Y, Z)X = 0$, for all $X, Y, Z \in \Omega$. So, each leaf is a flat submanifold with the induced metric. So, regular self-stability is a motion law for flat foliations.

4. Stable submanifolds

Theory of Stability of Distributions can be extended to Submanifolds. In this way, there appear new concepts of Stability:

- (Regularly) Stable Distributions with respect to Submanifolds.
- (Regularly) Stable Submanifolds with respect to Distributions in the tangent bundle of the submanifold.
- (Regularly) Self-stable Submanifolds.

5. Stability relations

Let Ω be a lightlike 3-foliation in a 4-dimensional space-time, and let U be a future-pointing timelike unit vector field (i.e. the 4-velocities of a congruence of observers). There exists a basis of Ω in the form $\{X_1, X_2, N + U\}$, where $X_1, X_2, N \in U^\perp$ and they are linearly independent (see [4]).

- $\Omega = \text{span}\,(X_1, X_2, N + U)$ is a foliation of wave-fronts of a congruence of massless particles. So, N is the relative direction of propagation of Ω observed by U.
- $\Omega^\perp = \text{span}\,(N + U)$ is the orthogonal foliation of Ω. Its integral curves are the trajectories associated with these massless particles.
- $U^\perp = \text{span}\,(X_1, X_2, N)$ is the physical spaces distribution of U.
- $\Omega \cap U^\perp = \text{span}\,(X_1, X_2)$ is the spacelike wave-fronts distribution of Ω observed by U.
- $\left(\Omega \cap U^\perp\right) \oplus \text{span}(U) = \text{span}\,(X_1, X_2, U)$ is the timelike wave-fronts distribution of Ω observed by U.
- $\Omega_U^- := \text{span}\,(X_1, X_2, -N + U)$ is the opposite distribution of Ω observed by U.

We have a lot of results of stability with respect to another given distribution Ω' involving these distributions. For example:

- If $U^\perp$ and Ω are stable, then $\Omega \cap U^\perp$ is also stable. Moreover, if $U^\perp$ or Ω is regularly stable, then $\Omega \cap U^\perp$ is also regularly stable.

- If $U^{\perp}$ and $\left(\Omega \cap U^{\perp}\right) \oplus \operatorname{span}(U)$ are stable, then $\Omega \cap U^{\perp}$ is also stable. Moreover, if $U^{\perp}$ or $\left(\Omega \cap U^{\perp}\right) \oplus \operatorname{span}(U)$ is regularly stable, then $\Omega \cap U^{\perp}$ is also regularly stable.

- If $U^{\perp}$ and Ω are stable, then Ω_U^- is stable if and only if $\left(\Omega \cap U^{\perp}\right) \oplus \operatorname{span}(U)$ is stable.

All these properties can be applied to self-stability too. Moreover, there are some specific self-stability results:

- Let U be a timelike vector field, then $\operatorname{span}(U)$ is self-stable (i.e. U is geodesic) if and only if its physical space distribution $U^{\perp}$ is stable with respect to $\operatorname{span}(U)$.

- Let Ω be a self-stable lightlike 3-distribution. Then $\Omega^{\perp}$ (that is a lightlike 1-distribution) is also self-stable. Nevertheless, the reciprocal does not hold in general.

6. Stability of p-foliations with respect to vector fields (1-foliations)

Let Ω be a p-foliation, and let U be a vector field. Taking into account Corollary 4, we have that $\nabla_{\operatorname{span}(U)}\Omega \subset \Omega$ if and only if $\nabla_{\operatorname{span}(U)}\Omega = 0$, i.e. there exist regularly stable bases $\{X_i\}_{i=1}^{p}$ of Ω with respect to $\operatorname{span}(U)$. These bases are parallelly transported along the integral curves of U, and so, we can reconstruct the entire foliation Ω from only one leaf, by means of parallel transports of a regularly stable basis along the integral curves of U.

If the dimension or the codimension of Ω is 1, then we can assure that there exist a vector field U such that $\nabla_{\operatorname{span}(U)}\Omega = 0$.

An important case is the study of the stability of lightlike 3-foliations with respect to observers congruences (timelike 1-foliations) in a 4-dimensional space-time. If Ω is a lightlike 3-foliation, it represents the wave fronts of a congruence of massless particles. Each observer, at each instant, observes these wave fronts always *in the same way*.

7. Examples

Using Corollary 4 it is easy to give examples of (regularly) stable foliations with respect to congruences of observers in Schwarzschild and Robertson-Walker space-times, and examples of regularly self-stable foliations by Corollary 5. They can be found in [3] and [5].

In these space-times it can be proved that there are not any distribution Ω of dimension greater than 1 satisfying $R(Y,Z)X = 0$ for all $X, Y, Z \in \Omega$. So, by the Theorem 2 there are not any regularly self-stable distribution of dimension greater than 1.

In the *pp*-wave metric $ds^2 = dy^2 + dz^2 - 2Hdu^2 - 2dudv$, where $H = H(u, y, z)$, let us consider the lightlike 3-foliation

$$\Omega := \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

which leaves have $u = constant$ (see [6]). It is self-stable and satisfies $R(Y,Z)X = 0$ for all $X, Y, Z \in \Omega$. So, it is regularly self-stable. Effectively, a regularly self-stable basis is given by $\left\{ \frac{\partial}{\partial v}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$. Therefore, the subfoliations $\left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial y} \right\rangle$, $\left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial z} \right\rangle$, $\left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ are regularly self-stable 2-foliations (timelike, timelike, and spacelike respectively). Moreover, the timelike 2-foliation $\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle$ is regularly self-stable too. A regularly self-stable basis is given by $\left\{ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\}$.

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