Dynamics of differentiation and integration operators on weighted spaces of entire functions

María José Beltrán Meneu

José Bonet and Carmen Fernández

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Aim of the talk

To study the dynamics of the operators:

Differentiation: \( Df := f' \)

Integration: \( Jf(z) := \int_0^z f(\xi) d\xi, \; z \in \mathbb{C} \)

Hardy operator: \( Hf(z) := \frac{1}{z} \int_0^z f(\xi) d\xi, \; z \in \mathbb{C} \)

on weighted Banach spaces of entire functions.

- \( D, J \) and \( H \) are continuous on \((H(\mathbb{C}), co)\), where \( co \) denotes the compact-open topology.
- \( DJf = f \) and \( JDf(z) = f(z) - f(0) \; \forall f \in H(\mathbb{C}), \; z \in \mathbb{C} \).
Dynamics on operators

Given a Banach space $X$,

$$\mathcal{L}(X) := \{ T : X \to X \text{ linear and continuous } \}.$$  

Given $T \in \mathcal{L}(X)$, the pair $(X, T)$ is a linear dynamical system.

**Definitions**

- Let $x \in X$. The **orbit of** $x$ **under** $T$ **is** the set
  
  $$\text{Orb}(x, T) := \{ x, Tx, T^2x, \ldots \} = \{ T^nx : n \geq 0 \}.$$  

- $x \in X$ is a **periodic point** if $\exists n \in \mathbb{N}$ such that $T^nx = x$. 

Given a Banach space $X$ and $T \in \mathcal{L}(X)$, it is said that:

**Definitions**

- $T$ topologically mixing $\iff \forall U, V \neq \emptyset$ open, $\exists n_0 : T^n U \cap V \neq \emptyset$ $\forall n \geq n_0$.
- $T$ hypercyclic $\iff \exists x \in X$, $\text{Orb}(T, x) := \{x, Tx, T^2x, \ldots\}$ is dense in $X \Rightarrow X$ SEPARABLE!!

**Definition (Godefroy, Shapiro, 1991)**

$T$ is chaotic if

- $T$ has a dense set of periodic points,
- $T$ is hypercyclic.
Dynamics on operators

Given a Banach space $X$ and $T \in \mathcal{L}(X)$, it is said that:

**Definitions**

- $T$ power bounded $\iff \sup_n \|T^n\| < \infty$
- $T$ Cesàro power bounded $\iff \sup_n \|\frac{1}{n} \sum_{k=1}^{n} T^k\| < \infty$
- $T$ mean ergodic $\iff$

$$\forall x \in X, \exists P x := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} T^k x \in X$$

- $T$ uniformly mean ergodic $\iff$

$$\left\{ \frac{1}{n} \sum_{k=1}^{n} T^k \right\}_n$$

converges in the operator norm.
Classical results

Mac Lane (1952)

\[ D : H(\mathbb{C}) \to H(\mathbb{C}) \text{ is hypercyclic, i.e.,} \]

\[ \exists f_0 \in H(\mathbb{C}) : \forall f \in H(\mathbb{C}), \ \exists (n_k)_k \subseteq \mathbb{N} \text{ such that} \]

\[ f_0^{(n_k)} \to f \text{ uniformly on compact sets.} \]

Proposition

The integration operator \( J : H(\mathbb{C}) \to H(\mathbb{C}) \) is not hypercyclic. In fact, for each \( f \in H(\mathbb{C}) \), the sequence \( (J^nf)_n \) converges to 0 in \( H(\mathbb{C}) \).
Weights

A weight $\nu$ on $\mathbb{C}$ is a strictly positive continuous function on $\mathbb{C}$ which is radial, i.e. $\nu(z) = \nu(|z|)$, $z \in \mathbb{C}$, $\nu(r)$ is non-increasing on $[0, \infty]$ and rapidly decreasing, that is, it satisfies $\lim_{r \to \infty} r^n \nu(r) = 0$ for each $n \in \mathbb{N}$.

For $r \geq 0$ and $f \in H(\mathbb{C})$, consider

$$M_p(f, r) := \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p}$$

for $1 \leq p < \infty$

and

$$M_\infty(f, r) := \sup_{|z|=r} |f(z)|, \ r \geq 0.$$

Note that for each $1 \leq p < \infty$ and each $n \in \mathbb{N}$, we have

$$M_p(z^n, r) = M_\infty(z^n, r) \text{ for each } r > 0.$$
Weighted spaces of entire functions

Given a weight $\nu$, $1 \leq p \leq \infty$, and $1 \leq q < \infty$,

$$B_{p,q}(\nu) := \left\{ f \in H(\mathbb{C}) : \|f\|_{p,q,\nu} := \left(2\pi \int_0^\infty r\nu(r)^q M_p(f,r)^q dr\right)^{1/q} < \infty \right\}$$

$$B_{p,\infty}(\nu) := \left\{ f \in H(\mathbb{C}) : \|f\|_{p,\infty,\nu} := \sup_{r>0} \nu(r) M_p(f,r) < \infty \right\}$$

$$B_{p,0}(\nu) := \left\{ f \in H(\mathbb{C}) : \lim_{r \to \infty} \nu(r) M_p(f,r) = 0 \right\}.$$

For $1 \leq p, p_1, p_2 \leq \infty$, $1 \leq q, q_1, q_2 \leq \infty$, $p_1 \leq p_2$, $q_1 \leq q_2 \neq \infty$,

$$B_{p,q_1}(\nu) \subseteq B_{p,q_2}(\nu) \subseteq B_{p,0}(\nu) \subseteq H(\mathbb{C})$$

$$B_{p_2,q}(\nu) \subseteq B_{p_1,q}(\nu)$$

with continuous inclusions.
Definition

In the case $p = \infty$ and $q \in \{0, \infty\}$, we get the weighted Banach spaces of entire functions:

\[ H^\infty_\nu := \{ f \in H(\mathbb{C}) : \|f\|_\nu := \sup_{z \in \mathbb{C}} \nu(z)|f(z)| < \infty \} \]
\[ H^0_\nu := \{ f \in H(\mathbb{C}) : \lim_{|z| \to \infty} \nu(z)|f(z)| = 0 \}. \]

Given $a \in \mathbb{R}$, $\alpha > 0$, consider $\nu_{a,\alpha}(z) := |z|^a e^{-\alpha |z|}$, for $|z| \geq r_0$, and the spaces $B_{p,q}(a,\alpha)$, $H^\infty_{a,\alpha}$ and $H^0_{a,\alpha}$. For $a = 0$, we omit the $a$.

- $f \in H^\infty_{\alpha} \iff \exists C > 0 : |f(z)| \leq Ce^{\alpha |z|} \ \forall z \in \mathbb{C}$.
- $H^\infty_{\alpha} \cong \ell_\infty$ and $H^0_{\alpha} \cong c_0$ (Lusky).
- $P$ are dense in $B_{p,q}(a,\alpha)$, $1 \leq q < \infty$, $q = 0$, but the monomials are not a Schauder basis in general if $p \in \{1, \infty\}$ (Lusky).
Continuity, norms and spectrum

For every $1 \leq p \leq \infty$ the bidual of $B_{p,0}(\nu)$ is isometrically isomorphic to $B_{p,\infty}(\nu)$ (Lusky).

**Lemma**

Assume $T : (H(\mathbb{C}), co) \rightarrow (H(\mathbb{C}), co)$ continuous and $T(\mathcal{P}) \subseteq \mathcal{P}$, let $\nu$ be a weight and $1 \leq p \leq \infty$. The following conditions are equivalent:

(i) $T(B_{p,\infty}(\nu)) \subset B_{p,\infty}(\nu)$,

(ii) $T : B_{p,\infty}(\nu) \rightarrow B_{p,\infty}(\nu)$ is continuous,

(iii) $T(B_{p,0}(\nu)) \subset B_{p,0}(\nu)$,

(iv) $T : B_{p,0}(\nu) \rightarrow B_{p,0}(\nu)$ is continuous.

If (i)-(iv) hold, then $\|T\|_{\mathcal{L}(B_{p,\infty}(\nu))} = \|T\|_{\mathcal{L}(B_{p,0}(\nu))}$ and $\sigma_{B_{p,\infty}(\nu)}(T) = \sigma_{B_{p,0}(\nu)}(T)$. 
The continuity of $D$ and $J$ on $H^\infty_\nu(\mathbb{C})$ is determined by the growth or decline of $\nu(r)e^{\alpha r}$ for some $\alpha > 0$ in an interval $[r_0, \infty[$.

**Proposition.**

Let $\nu$ be a weight function such that $\sup_{r>0} \frac{\nu(r)}{\nu(r+1)} < \infty$ and let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $q = 0$. Then the differentiation operator $D : B_{p,q}(\nu) \to B_{p,q}(\nu)$ is continuous.
Continuity, norms and spectrum of $D$

If $\nu(r) = r^a e^{-\alpha r}$ ($\alpha > 0$, $a \in \mathbb{R}$) for $r \geq r_0$, $1 \leq q < \infty$:

$$\|z^n\|_{p,q,a,\alpha} \approx \left( \frac{(n+a)q+1}{e\alpha q} \right)^{a+n+\frac{3}{2q}}$$

with equality for $a = 0$.

**Proposition**

For $n > |a|$,

$$\|D^n\|_{p,q,a,\alpha} = O\left( n! \left( \frac{e\alpha}{n-|a|} \right)^{n-|a|} \right).$$

If $1 \leq q < \infty$,

$$n! \left( \frac{e\alpha q}{(a+n)q+1} \right)^{n+a+\frac{3}{2q}} = O(\|D^n\|_{p,q,a,\alpha})$$

if $q = \infty$,

$$n! \left( \frac{e\alpha}{a+n} \right)^{n+a} = O(\|D^n\|_{p,\infty,a,\alpha}).$$
Continuity, norms and spectrum of $D$

**Proposition**

For every $\alpha > 0$ and $a \in \mathbb{R}$, the spectrum $\sigma_{a,\alpha}(D) = \alpha \mathbb{D}$.

**Proposition**

Let $\nu$ be a weight such that $D$ is continuous on $B_{p,q}(\nu)$, $1 \leq p \leq \infty$, $q \in \{0, p, \infty\}$, and that $\nu(r)e^{\alpha r}$ is non increasing for some $\alpha > 0$. If $|\lambda| < \alpha$, the operator $D - \lambda I$ is surjective and it even has a continuous linear right inverse

$$K_{\lambda}f(z) := e^{\lambda z} \int_{0}^{z} e^{-\lambda \xi} f(\xi)d\xi, \ z \in \mathbb{C}$$

In particular, this is satisfied by the weight $\nu_{a,\alpha}(r) = r^{a}e^{-\alpha r}$ for $r$ big enough (proved by Atzmon, Brive (2006), in the case $a = 0$).
Proposition

For the weight $\nu(r) = r^a e^{-\alpha r}$ ($\alpha > 0$, $a \in \mathbb{R}$) for $r$ big enough, we have:

- $\|J^n\|_{p,q,a,\alpha} \cong 1/\alpha^n$, with equality for $a = 0$ if $q \in \{0, \infty\}$, and $1/\alpha^n \lesssim \|J^n\|_{p,p,a,\alpha} \lesssim \left(\frac{p}{\alpha p - 1}\right)^n$ if $1 \leq p < \infty$, $p > \frac{1}{\alpha}$, $n \in \mathbb{N}$.

- $\sigma_{a,\alpha}(J) = (1/\alpha)\overline{D}$, if $q \in \{0, \infty\}$, and $(1/\alpha)\overline{D} \subseteq \sigma(J) \subseteq \frac{p}{\alpha p - 1} \overline{D}$ for $1 \leq p < \infty$, $p > \frac{1}{\alpha}$, $p = q$. 
Continuity, norms and spectrum of $H$

**Theorem**

Given a weight $\nu$, the Hardy operator $H : B_{p,q}(\nu) \to B_{p,q}(\nu)$, $Hf(z) = \frac{1}{z} \int_0^z f(\zeta) d\zeta$, $z \in \mathbb{C}$, is well defined and continuous with norm $\|H\| = 1$. Moreover, $H^2$ is compact and $H^2(B_{p,\infty}(\nu)) \subseteq B_{p,0}(\nu)$. In particular, its spectrum is $\sigma(H) = \{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\}$. If the integration operator $J : B_{p,q}(\nu) \to B_{p,q}(\nu)$ is continuous, then $H$ is compact and $H(B_{p,\infty}(\nu)) \subseteq B_{p,0}(\nu)$.

**Remark**

For the weight $\nu(r) = \exp(-(\log r)^2)$:

- $J$ is not continuous on $H^\infty_{\nu}(\mathbb{C})$ (Harutyunyan, Lusky)
- $H : H^\infty_{\nu}(\mathbb{C}) \to H^0_{\nu}(\mathbb{C})$, $H : H^0_{\nu}(\mathbb{C}) \to H^0_{\nu}(\mathbb{C})$, are compact (Lusky).
Hypercyclicity and chaos

Theorem

Assume $D : B_{p,q}(\nu) \to B_{p,q}(\nu)$ continuous, $q \neq \infty$. TFAE:

(i) $D$ is topologically mixing.

(ii) $\lim_{n \to \infty} \frac{\|z^n\|_{p,q,\nu}}{n!} = 0$.

Theorem

Assume $D : B_{p,q}(\nu) \to B_{p,q}(\nu)$ continuous, $q \neq \infty$. TFAE:

(i) $D$ is chaotic.

(ii) $D$ has a periodic point different from 0.

(iii) $\lim_{r \to \infty} \nu(r) \frac{e^r}{r^{\frac{1}{q}-\frac{1}{2p}}} = 0$ if $q = 0$ and $r^{\frac{1}{q}-\frac{1}{2p}} e^r \in L_{\nu}^q([r_0, \infty[)$ for some $r_0 > 0$, if $1 \leq q < \infty$. 

Corollary

Consider the weight $\nu_{a,\alpha}$.

(a) $0 < \alpha < 1 \implies D$ is neither hypercyclic nor chaotic on $B_{p,q}(\nu)$.

(b) $\alpha > 1 \implies$ then $D$ is topologically mixing and chaotic on $B_{p,q}(\nu)$.

(c) $\alpha = 1 \implies D$ is hypercyclic (even topologically mixing) if and only if $a < \frac{1}{2} - \frac{3}{2q}$ and $D$ is chaotic if and only if $a < \frac{1}{2p} - \frac{2}{q}$. For $p = \infty$ we set $1/p := 0$ and for $q = 0$ we set $1/q := 0$.

Remark

$J$ and $H$ are never hypercyclic on $B_{p,q}(\nu)$. 
Mean ergodicity

Remark

\[ T \in \mathcal{L}(X) \text{ Cesàro bounded and } P(d) = 0 \text{ for every } d \in D, \ D \subseteq X \text{ dense } \implies T \text{ mean ergodic.} \]

Proposition

Let \( T = D \) or \( T = J \) and assume that \( T \) is continuous on \( B_{p,\infty}(\nu) \), and equivalently, on \( B_{p,0}(\nu) \). TFAE:

(i) \( T : B_{p,\infty}(\nu) \to B_{p,\infty}(\nu) \) is uniformly mean ergodic,
(ii) \( T : B_{p,0}(\nu) \to B_{p,0}(\nu) \) is uniformly mean ergodic,
(iii) \( \lim_{m \to \infty} \frac{||T + \cdots + T^m||_{p,\nu}}{m} = 0. \)

Moreover, if \( 1 \in \sigma_{\nu}(T) \), then \( T \) is not uniformly mean ergodic.
Mean ergodicity. Two useful results

**Theorem (Lin)**

Let $T \in \mathcal{L}(X)$ such that $\|T^n/n\| \to 0$. Then,

$$T \text{ uniformly mean ergodic } \iff (I - T)X \text{ is closed}.$$  

**Theorem (Lotz)**

Let $T \in \mathcal{L}(H_\alpha^\infty)$ such that $\|T^n/n\| \to 0$. Then,

$$T \text{ mean ergodic } \iff T \text{ uniformly mean ergodic}.$$  

$H_\alpha^\infty$ is a Grothendieck Banach space with the Dunford-Pettis property, since it is isomorphic to $\ell_\infty$ by a result due to Galbis.
Mean ergodicity of the differentiation operator.

**Theorem.**

Let $\nu(r) = e^{-\alpha r}$, $r \geq 0$.

- $D$ is power bounded on $H^\infty_\alpha(\mathbb{C})$ or $H^0_\alpha(\mathbb{C})$ if and only if $\alpha < 1$.
- $D$ is uniformly mean ergodic on $H^\infty_\alpha(\mathbb{C})$ and $H^0_\alpha(\mathbb{C})$ if $\alpha < 1$.
- $D$ not mean ergodic if $\alpha > 1$, and
- $D$ is not mean ergodic on $H^\infty_1(\mathbb{C})$ and not uniformly mean ergodic on $H^0_1(\mathbb{C})$. 
Let \( \nu(r) = e^{-\alpha r}, \ r \geq 0. \)

- \( J \) is never hypercyclic.
- \( J \) is power bounded on \( H^\infty_\alpha(\mathbb{C}) \) or \( H^0_\alpha(\mathbb{C}) \) if and only if \( \alpha \geq 1. \)
- If \( \alpha > 1 \), \( J \) is uniformly mean ergodic on \( H^\infty_\alpha(\mathbb{C}) \) and \( H^0_\alpha(\mathbb{C}) \).
- \( J \) is not mean ergodic on these spaces if \( \alpha < 1. \)
- If \( \alpha = 1 \), then \( J \) is not mean ergodic on \( H^\infty_1(\mathbb{C}) \), and mean ergodic but not uniformly mean ergodic on \( H^0_1(\mathbb{C}) \).

For every weight \( \nu \), \( H \) is power bounded, not hypercyclic and uniformly mean ergodic on \( B_{p,q}(\nu) \).
## Summary

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Open problems

(1) Is the operator of differentiation $D$ mean ergodic on $H^0_1(\mathbb{C})$?

In other words:
Assume that $f \in H(\mathbb{C})$ satisfies $\lim_{|z| \to \infty} |f(z)| \exp(-|z|) = 0$. Does it follow that

$$\lim_{n \to \infty} \frac{1}{n} \sup_{z \in \mathbb{C}} |f'(z) + \cdots + f^{(n)}(z)| \exp(-|z|) = 0?$$
(2) Are there mean ergodic operators on a separable Banach space that are hypercyclic?

It is clear that no power bounded operator can be hypercyclic. However, there are examples of mean ergodic operators $T$ on a Banach space such that the sequence $(||T^n||)_n$ tends to infinity. Classical examples are due to Hille in 1945. A general construction was presented by Tomilov and Zemanek in 2004.
References


References


