The Bishop-Phelps-Bollobás property for numerical radius in $L_1$.

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X Banach, $f \in X^*$

$$\|f\| = \sup_{x \in B_X} |f(x)| = f(x_0) \quad \text{for some } x_0 \in B_X.$$ 

**Theorem (Bishop-Phelps, 1961 - Bollobás, 1970)**

$x \in B_X$, $f \in X^*$, $\|f\| = 1$, if $f(x) > 1 - \epsilon^2/2$
then $\exists (y, g) \in \Pi_1(X)$ with

$$\|f - g\| \leq \epsilon, \quad \|x - y\| \leq \epsilon.$$ 

To be more specific, if we denote by $\Pi_1(X) =: \{(y, g) \in B_X \times B_{X^*} : \|g\| = \|y\| = |g(y)|\}$ we can interpret the Bollobás theorem, roughly speaking as, any ordered pair that "almost belong" to $\Pi_1(X)$ can be approximated in the product norm by elements of $\Pi_1(X)$. 
**OPERATORS?**

**Definition (BPBp for Operators, Acosta-Aron-García-Maestre, 2008)**

Given $X$, $Y$ Banach spaces. We say the couple $(X; Y)$ satisfy the Bishop-Phelps-Bolloás property for operators, $BPBp$ if $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$, $\beta(\epsilon) > 0$ with $\lim_{\epsilon \to 0} \beta(\epsilon) = 0$ s.t. $\forall T \in S_{\mathcal{L}(X,Y)}$, $x \in S_X$ with $\|Tx\| > 1 - \delta(\epsilon)$ then $\exists y \in S_X$ and $\exists G \in S_{\mathcal{L}(X,Y)}$ that satisfy

$$\|G\| = |G(y)| = 1, \quad \|x - y\| < \beta(\epsilon) \quad \& \quad \|G - T\| < \epsilon.$$ 

**Theorem (Acosta-Aron-García-Maestre, 2008)**

The couple $(\ell_1, Y)$ has the BPBp if and only if $Y$ has the AHSP.
**Definition (Acosta-Aron-García-Maestre, 2008)**

Given $X$ Banach. We say that $X$ has the approximating hyperplane series property, $AHSP$, if $\forall \epsilon > 0$, exists $\delta, 0 < \delta < \epsilon$ s.t. for every sequence $(x_n)_{1}^{\infty} \subseteq S_X$ and for every convex series $\sum_{1}^{\infty} \alpha_n$ with $\|\sum_{1}^{\infty} \alpha_n x_n\| > 1 - \delta$ there exists a subset $A \subseteq \mathbb{N}$ and $\{z_k\}_{k \in A} \subseteq S_X$ satisfy

1A $\|z_k - x_k\| < \epsilon$ for all $k \in A$.

1B $\|g(z_k)\| = 1$ for some $g \in S_{X^*}$ and for all $k \in A$.

2 $\sum_{k \in A} \alpha_k > 1 - \delta$.

**Theorem (Choi-Kim, 2011)**

If $Y$ has the Radon-Nikodym property, then

$$(L_1(\mu), Y) \text{ has the BPBp if and only if } Y \text{ has the AHSP}.$$
Given $T \in \mathcal{L}(X)$, the numerical radius is defined by

$$\nu(T) := \sup\{|g(Ty)| : (y, g) \in \Pi_1(X)\}$$

We are interested in the case of numerical index 1 a.e. $\nu(T) = \|T\|$ for all $T \in \mathcal{L}(X)$.

**Definition (BPB $\nu$, Guirao-Kozhushkina)**

Given $X$ Banach. We say that $X$ has the Bishop-Phelps-Bolloás property for numerical radius, BPB $\nu$, if there exists a function $\delta : (0, 1) \mapsto (0, 1)$ s.t. $\forall T \in \mathcal{L}(X)$, with $\nu(T) = 1$, $\forall (x, f) \in \Pi_1(X)$ with $|f(Tx)| > 1 - \delta(\epsilon)$ there exists $G \in \mathcal{L}(X)$ with $\nu(G) = 1$ and exists $(y, g) \in \Pi_1(X)$ s.t.

$$|g(Gy)| = 1, \quad \|x - y\|, \|f - g\| \leq \epsilon \quad \& \quad \nu(G - T) \leq \epsilon.$$
**Lemma (Guirao-Kozhushkina)**

If \((x, f) \in B_{\ell_1} \times B_{\ell_\infty}\) and \(f(x) > 1 - \epsilon\) then \(\exists (y, g) \in \Pi_1(\ell_1)\) with 
\[
\|x - y\|, \|f - g\| \leq \epsilon \quad \text{and}
\]
\[
g = f \chi(A_1 \cup A_2)^c + \chi A_1 - \chi A_2
\]

where \(A_1 = \{n \in \mathbb{N} : f_n > 1 - \epsilon\}\), \(A_2 = \{n \in \mathbb{N} : f_n < -1 + \epsilon\}\).

**Theorem (Guirao-Kozhushkina)**

\(\ell_1\) has the BPB\(\nu\).
Lemma

Given a set $A \subseteq \mathbb{R}$ and an operator $T \in \mathcal{L}(L_1)$. The operator $\hat{T} : \mathcal{L}(L_1) \mapsto \mathcal{L}(L_1)$ defined by $\hat{T}(h) = T(h\chi_A - h\chi_{A^c})\chi_A - T(h\chi_A - h\chi_{A^c})\chi_{A^c}$ is an isometry.

In particular, for every point $x \in L_1$ and every linear form $f \in L_\infty$, if we denote by $\hat{f} = f\chi_A - f\chi_{A^c}$, then $< \hat{x}, \hat{f} > = < x, f >$ and $\hat{x} = x\chi_A - x\chi_{A^c}$, $< \hat{T}(\hat{x}), \hat{f} > = < T(x), f >$.

Theorem

$L_1(\mu)$ has the $BPB - \nu$. 
Introduction

BPB property for operators

BPB property for numerical radius

BPB – ν for $L_1$

BIBLIOGRAPHY I


J.A. Guirao and O. Kozhushkina. The bishop-phelps-bollobás property for numerical radius in $l_1(\mathbb{C})$. *Preprint*.

Yun Sung Choi, Sun Kwang Kim, Han Ju Lee, and Miguel Martín. The Bishop-Phelps-Bollobás theorem for operators on $L_1(\mu)$. *Preprint*. 