Metric geometry and energy integrals for convex bodies

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Metric spaces arising from Euclidean spaces by a change of metric: some history

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Classic results

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\[(X, d) \xrightarrow{f} (X, f(d)), \text{ where } f(d)(x, y) := f(d(x, y)).\]
Metric spaces arising from Euclidean spaces by a change of metric: some history

For the metric space \((\mathbb{R}, | \cdot |)\), Wilson considered the function \(f(t) = t^{1/2}\).

Denote \(d_{1/2} := f(| \cdot |) \Rightarrow d_{1/2}(x,y) = |x - y|^{1/2}\).

He showed that \((\mathbb{R}, d_{1/2})\) may be imbedded in a separable Hilbert space. In other words, he proved that there exist a distance preserving mapping (isometry) \(j: (\mathbb{R}, d_{1/2}) \rightarrow (\ell_2, \| \cdot \|_{\ell_2})\). That is, \(\|j(x) - j(y)\|_{\ell_2} = d_{1/2}(x,y) = |x - y|^{1/2}, \forall x, y \in \mathbb{R}\).
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Just for the record...

$$f^2(t) = \int_0^\infty \frac{\sin^2(st)}{s^2} dG(s),$$

where $G(s)$ is a non-decreasing function and $\int_1^\infty \frac{1}{s^2} dG(s)$ exists.
Metric spaces arising from Euclidean spaces by a change of metric: some history

They proved that, for $0 < \alpha < 1$, $f(t) = t^\alpha$ becomes a suitable metric transformation (fulfills the previous criterium!!). Denote $d_\alpha := f(| \cdot |) \leadsto d_\alpha(x, y) = |x - y|^\alpha$.

- The metric space $(\mathbb{R}, d_\alpha)$ is also isometrically imbeddable in a separable Hilbert space.
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**Question**

*Can we generalize this result to higher dimensions?*
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**Question**

*Can we generalize this result to higher dimensions?*

*In other words, is the metric space $(\mathbb{R}^n, d_\alpha)$ isometrically imbeddable in $\ell_2$, where $d_\alpha(x, y) = \|x - y\|^\alpha$?*
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**Classic results**

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**Theorem (Schoenberg)**

For $0 < \alpha < 1$, the metric space $\left( \mathbb{R}^n, d_\alpha \right)$ is imbeddable in $\ell_2$. 
Moreover, by combining Schoenberg’s proof and a classic result of Menger, we have that for every compact set $K \subset \mathbb{R}^n$ the metric space $(K, d_\alpha)$ may be *imbeddable in the surface of a Hilbert sphere.*
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**Theorem**

*For every compact set \( K \subset \mathbb{R}^n \), there exist a positive number \( r \) and a distance preserving mapping*

\[ j : (K, d_\alpha) \rightarrow (rS_{\ell_2}, \| \cdot \|_{\ell_2}) \]
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It is natural to define,

$$\rho_\alpha(K) := \inf r \leadsto \text{least possible radius.}$$
A connection with another area

All these results can be framed within a vast area called "metric geometry".
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Link

Metric Geometry ↔ Potential Theory
Let $K \subset \mathbb{R}^n$ be a compact set and $\mu$ a signed Borel measure supported on $K$ of total mass one (i.e., $\mu(K) = 1$.)
Energy Integrals: some definitions

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For a real number $p$ (for us, $0 < p < 2$), we define

$$I_p(\mu; K) := \int_K \int_K \|x-y\|^p d\mu(x)d\mu(y)$$
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And define,

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The connection!


**Theorem (Alexander-Stolarsky)**

Let $K \subset \mathbb{R}^n$ be a compact set. Then,

$$\rho_\alpha(K) = \sqrt{\frac{M_{2\alpha}(K)}{2}}$$

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We will be focused on computing the value of $M_{2\alpha}(K)$. 
Denote by $B_n$ the unit ball in $\mathbb{R}^n$.

- $M_1(B_1) = M_1([−1, 1]) = 1$ (Alexander-Stolarsky, Trans. AMS. ’74)

$M_1(B_3) = 2$ (Alexander, Proc. AMS. '77)

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**Theorem (Hinrichs, Nickolas and Wolf)**

\[ M_1(B_n) = \frac{\pi^{1/2} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \]

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The number $\frac{\pi^{1/2} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$ is exactly $\pi_1(id : \ell^n_2 \to \ell^n_2)$. 
Theorem (Carando, G., Pinasco)

\[ M_p(B_n) = M_p([-1,1]) \frac{\pi^{1/2} \Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{n}{2}\right)} \]
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**Theorem (Carando, G., Pinasco)**

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M_p(B_n) = M_p([-1, 1]) \frac{\pi^{1/2}}{\Gamma(p+1/2)} \frac{\Gamma(n+p)}{\Gamma(n/2)} = M_p([-1, 1]) \pi_p(id : \ell_2^n \to \ell_2^n)^p
\]

Using \(\lim_{m \to \infty} \frac{\Gamma(m+c)}{\Gamma(m)m^c} = 1\), and the previous result we get:

**Corollary**

\[
\rho_\alpha(B_n) \asymp n^{\frac{\alpha}{2}}.
\]
How to compute the value $M_p(B_n)$?

Using the rotation-invariance of the Gaussian measure + spherical coordinates, we have:

**Lemma**

For every $x \in \mathbb{R}^n$, we have

$$\|x\|_p^p = b_p(n) \int_{S^{n-1}} |\langle x, t \rangle| \, d\lambda(t),$$

where $S^{n-1}$ is the unit sphere, $\lambda$ its normalized Haar measure and

$$b_p(n) = \pi^{\frac{p}{2}} \frac{\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{n}{2}\right)} = \pi^p \left(\text{id}: \ell^2_n \to \ell^p_n\right).$$
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b_p(n) = \frac{\pi^{1/2} \Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{n}{2}\right)} = \pi_p(id : \ell_2^n \to \ell_2^n)^p.
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How to compute the value $M_p(B_n)$?

Let $\mu$ be a signed borel measure on $B_n$ of total mass one.

\[ I_p(\mu; B_n) := \int_{B_n} \int_{B_n} \|x - y\|^p d\mu(x) d\mu(y) \]
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For $t \in S^{n-1}$, let $\pi_t : B_n \to D_t$ the orthogonal projection of $B_n$ in its diameter $D_t$ (in the direction of $t$), then

$|\langle x - y, t \rangle|^p = \|\pi_t(x) - \pi_t(y)\|^p$. 
Then,

\[ I_p(\mu; B_n) = b_p(n) \int_{S^{n-1}} \left[ \int_{B_n} \int_{B_n} \| \pi_t(x) - \pi_t(y) \|^p d\mu(x) d\mu(y) \right] d\lambda(t) \]
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Now \( \mu\pi_t^{-1} \) is also a measure on \( D_t \) of total mass one. Since \( D_t \equiv [-1, 1] \) we have

\[ \int_{D_t} \int_{D_t} |u - v|^p d\mu\pi_t^{-1}(u) d\mu\pi_t^{-1}(v) = I_p(\mu\pi_t^{-1}, D_t) \leq M_p([-1, 1]). \]
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Therefore,

\[ I_p(\mu; B_n) \leq b_p(n) \int_{S^{n-1}} \left[ M_p([-1, 1]) \right] d\lambda(t) = b_p(n)M_p([-1, 1]). \]
Then,

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\[ M_p(B_n) \leq b_p(n)M_p([-1, 1]) \]
How to get equality? In other words, how can we prove that

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We found a sequence \((\mu_k)_{k \in \mathbb{N}}\) of signed measures of total mass one \(B_n\) such that, for every direction \(t\),

\[ \lim_{k \to \infty} \int_{D_t} \int_{D_t} |u - v|^p d\mu_k \pi_t^{-1}(u) d\mu_k \pi_t^{-1}(v) = M_p([-1, 1]). \]
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\[ I_p(\mu; B_n) = b_p(n) \int_{S^{n-1}} \left[ \int_{D_t} \int_{D_t} |u - v|^p d\mu_t^{-1}(u)d\mu_t^{-1}(v) \right] d\lambda(t) \]

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\[ \lim_{k \to \infty} \int_{D_t} \int_{D_t} |u - v|^p d\mu_k^{-1}(u)d\mu_k^{-1}(v) = M_p([-1, 1]). \]

Therefore, \(I_p(\mu_k; B_n) \to b_p(n)M_p([-1, 1]).\)
Bounds for other convex bodies

Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body, then $K$ is just the unit ball of an $n$-dimensional Banach space $(E, \| \cdot \|_E)$.
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**Question**

*How can we estimate the value of $\rho_\alpha(B_E)$, $0 < \alpha < 1$?*
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**Question**

*How can we estimate the value of $\rho_\alpha(B_E)$, $0 < \alpha < 1$? Or, equivalently, how can we compute $M_p(B_E)$, $0 < p < 2$?*
Theorem

\[ M_p(B_E) \leq M_p([-1, 1]) \frac{\pi^{1/2} \Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{n}{2}\right)} \int_{S^{n-1}} \|t\|_E^p d\lambda(t). \]
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This bound is expressed in terms of the mean width of \( B_E \), and is good enough in many cases!
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### Remark:

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Absolutely summing operators

An operator $T \in \mathcal{L}(X; Y)$ is absolutely $p$-summing if there is a constant $C$ and a probability Borel-Radon measure $\nu$ on $B_{X'}$ such that

$$\|Tx\|_Y^p \leq C^p \int_{B_{X'}} |\langle x, x' \rangle|^p d\nu(x') \quad \forall x \in X.$$
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Note that if $T = id : E \to \ell_2^n$ we have, for every $x \in \mathbb{R}^n$

$$\|x\|^p \leq \pi_p(id : E \to \ell_2^n)^p \int_{B_{E'}} |\langle x, x' \rangle|^p d\nu(x').$$
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In particular, if \( E = \ell_1^n \), then

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In particular, if $E = \ell_1^n$, then

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Therefore, $M_p(B_{\ell_1^n}) \leq \widetilde{C}_p$, for every $n$.

Theorem (Carando, G., Pinasco)

For $0 < \alpha < 1$ there exist a constant $R_\alpha$ such that for every $n$, there exist an isometric imbedding

$$j : (B_{\ell_1^n}, d_\alpha) \rightarrow (R_\alpha S_{\ell_2}, \| \cdot \|_{\ell_2}).$$
Several open questions

- What is the asymptotic behavior of $\rho_\alpha(B_{\ell_q})$, for $2 \leq q \leq \infty$?
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Conjecture: $\rho_\alpha(B_{\ell_q}) \asymp n^{q^\frac{\alpha}{q}}$. 
Several open questions

- What is the asymptotic behavior of $\rho_\alpha(B_{\ell_q^n})$, for $2 \leq q \leq \infty$?
  Conjecture: $\rho_\alpha(B_{\ell_q^n}) \asymp n^{\frac{\alpha}{q'}}$.
- What is the exact value of $M_p([-1, 1])$?
Thank you!!!!